PROXIMITY CONVERGENCE STRUCTURES

ELLEN ELIZABETH REED
PROXIMITY CONVERGENCE STRUCTURES

ELLEN E. REED

In this paper the notion of proximity convergence structures is introduced. These constitute a layer between Cauchy structures and uniform convergence structures (in the sense of Cook and Fischer [1]). They are a natural generalization of proximity structures. A study of the relations among these various structures constitutes §§2 and 3. In §4, compact extensions for a special class of proximity convergence spaces are constructed, and a characterization of these is obtained. They satisfy a mapping property with respect to compact \( T_2 \) proximity convergence spaces which satisfy a strong regularity condition. One problem left open is the obtaining of a more reasonable definition of regularity for these spaces.

1. Proximity convergence structures. A proximity convergence structure is the natural analogue, in the context of convergence spaces, of a proximity structure. Here convergence space is used in the sense of Fischer [3], and proximity in the sense of Efremovič and Smirnov. A proximity convergence structure is a filter of proximity-like orders on a set \( X \), and satisfies a composition property. If the filter is principal then it corresponds to an ordinary proximity.

The notation used is largely that in Cook and Fischer [1]. By \( \mathcal{O}(X) \) is meant the set of all symmetric topogenous orders on \( X \).

So a relation \( < \) on the subsets of \( X \) is in \( \mathcal{O}(X) \) iff it satisfies the following:

(ST 1) \( \emptyset < A < X \) for \( A \subseteq X \);
(ST 2) \( A < B \Rightarrow A \subseteq B \);
(ST 3) if \( A' \subseteq A < B \subseteq B' \) then \( A' < B' \);
(ST 4) if \( A < C \) and \( B < C \) then \( A \cup B < C \); also if \( C < A \) and \( C < B \) then \( C < A \cap B \);
(ST 5) \( A < B \) then \( X \setminus B < X \setminus A \).

DEFINITION 1. A proximity convergence structure on a set \( X \) is a family \( \mathcal{P} \subseteq \mathcal{O}(X) \) satisfying

(P 1) if \( <_1, <_2 \in \mathcal{P} \) then \( <_1 \cap <_2 \in \mathcal{P} \);
(P 2) if \( < \in \mathcal{P} \) then \( < \circ < \in \mathcal{P} \);
(P 3) if \( < \in \mathcal{P} \) and \( < \subseteq <' \in \mathcal{O}(X) \) then \( <' \in \mathcal{P} \).

We will call \( (X, \mathcal{P}) \) a proximity convergence space. Both concepts will be abbreviated by p.c.s.

REMARK AND DEFINITION 2. We say one p.c.s. on \( X \) is less than
another if it contains it. Under this ordering the set of all p.c.s.'s on $X$ is a complete lattice. The largest member is $\{\subseteq\}$ and corresponds to the discrete topology on $X$. The smallest is $\mathcal{O}(X)$, which yields the indiscrete topology. (See Definition 30.) The intersection of any family of p.c.s.'s on $X$ is also a p.c.s., so that suprema are easily described.

**Definition 3.** If $G$ is a nonempty subset of $\mathcal{O}(X)$ then clearly there is a smallest p.c.s. $[G]$ containing $G$. We will call $G$ a base, provided $[G]$ consists of refinements of orders in $G$. In case $[G]$ consists of refinements of finite intersections of orders in $G$, we call $G$ a subbase for $[G]$.

As in the uniform case, ordinary proximity relations on $X$ correspond to “principal” p.c.s's.; i.e., those which have a single element as base.

**Theorem 4.** Let $\ll \in \mathcal{O}(X)$. Then $\ll$ is a proximity on $X$ iff $\{\ll\}$ is a base for a p.c.s. on $X$.

**Proof.** Let $\mathcal{I}$ denote the set of refinements (in $\mathcal{O}(X)$) of $\ll$. If $\ll$ is a proximity relation then $\ll = \ll \circ \ll$ and hence $\mathcal{I}$ satisfies (P2). The other properties are clearly satisfied. Conversely, if $\mathcal{I}$ is a p.c.s. then $\ll \circ \ll \in \mathcal{I}$ and so $\ll$ is dense. Clearly then $\ll$ is a proximity relation.

**Definition 5.** If $\subset \subset$ is a proximity on $X$ we will call $[\subset \subset]$ a proximity structure.

2. Relation with uniform convergence structure. As with ordinary proximities, each uniform convergence structure (abbreviated u.c.s.) gives rise to a p.c.s. This allows us to divide the uniform convergence structures into proximity classes. Each class contains a smallest member, which is strongly bounded. This last is a condition stronger than total boundedness, and more satisfying in that every proximity class contains a unique strongly bounded member. (A class can contain more than one totally bounded member.) Moreover if the p.c.s. is a proximity structure than the strongly bounded member in its class is a uniform structure; the other totally bounded uniform convergence structures in the class will not be uniform structures.

**Definition 6.** A standard filter on $X \times X$ is a symmetric filter $\Phi \subseteq [\mathcal{A}]$, the filter generated by the diagonal on $X$. For $\Phi$ a standard filter we define
A <φ B iff H(A) ⊆ B for some H ∈ Φ.

This imitates the usual way a proximity is obtained from a uniformity. Notice that if Φ is standard, <φ ∈ ℂ(X).

If ℐ is a uniform convergence structure (abbreviated u.c.s.) we define

\[ \mathcal{B}_ℐ = \{ <φ : \Phi \text{ is a standard filter in } ℐ \} . \]

It turns out that \( \mathcal{B}_ℐ \) is a base for a p.c.s. \( ℙ_ℐ \) on X.

**Lemma 7.** Let \( \Phi \) and \( \Psi \) be standard filters on \( X \times X \).

(i) If \( \Theta = \Phi \cap \Psi \) then \( <\Theta = <\Phi \cap <\Psi \)

(ii) If \( \mathcal{A} = \Phi \circ \Phi \) then \( <\mathcal{A} = <\Phi \circ <\Phi \).

**Proof.** Straightforward.

**Theorem 8.** If \( ℐ \) is a u.c.s. on X then \( \mathcal{B}_ℐ \) is a base for a p.c.s. \( ℙ_ℐ \) on X. If \( ℐ \) is generated by a uniformity \( ℳ \) then \( ℙ_ℐ \) is a proximity structure generated by \( <\mathcal{A} \).

**Proof.** From the preceding lemma it is clear that \( \mathcal{B}_ℐ \) is a base for a p.c.s. on X. Suppose \( ℳ \) is a uniformity which generates \( ℐ \). Then for \( <\in ℙ_ℐ \) we have \( <\mathcal{A} \subseteq <\). Hence \{<\mathcal{A}\} is a base for \( ℙ_ℐ \).

**Definition 9.** If \( ℾ \) is a cover of \( X \) we define \( H_ℾ = \bigcup \{ C \times C : C \in ℾ \} \). If \( ℾ \) is finite then any entourage which contains \( H_ℾ \) is said to be strongly bounded. A filter \( \Phi \) on \( X \times X \) is strongly bounded iff it consists of strongly bounded entourages. A u.c.s. is strongly bounded iff it has a base of strongly bounded filters.

**Remark 10.** Notice that for uniform structures strongly bounded is equivalent to totally bounded. However, in the case of a u.c.s. total-boundedness is a weaker condition.

**Theorem 11.** Every strongly bounded u.c.s. is totally bounded.

**Proof.** Let \( ℐ \) be a strongly bounded u.c.s. on X, and let \( ℰ \) be an ultrafilter on X. Let \( \Phi \) be any strongly bounded filter in \( ℐ \). We claim that \( \Phi \subseteq ℰ \times ℰ \).

Let \( H \in \Phi \), and let \( ℾ \) be a finite cover of \( X \) such that \( H_ℾ \subseteq H \). Since \( ℰ \) is an ultrafilter, \( ℰ \cap ℾ \neq \emptyset \). But if \( C \in ℰ \cap ℾ \) then \( H \supseteq C \times C \in ℰ \times ℰ \).

**Theorem 12.** Let \( ℐ \) be a u.c.s. on X. The following conditions
are equivalent:
(i) $\mathcal{F}$ is strongly bounded;
(ii) $\mathcal{F}$ has at least one strongly bounded member;
(iii) The filter $[\mathcal{A}]^*$ of all strongly bounded entourages is in $\mathcal{F}$.

Proof. Since $\mathcal{F} \neq \emptyset$, clearly (i) $\implies$ (ii). Now suppose (ii) holds. Note $[\mathcal{A}]^*$ is a filter. If $\Phi$ is any strongly bounded filter in $\mathcal{F}$ then $\Phi \subseteq [\mathcal{A}]^*$.

Finally, assume $[\mathcal{A}]^* \in \mathcal{F}$. If $\Phi \in \mathcal{F}$ then $\Phi \cap [\mathcal{A}]^*$ is a strongly bounded filter in $\mathcal{F}$, and is contained in $\Phi$.

LEMMA 13. A strongly bounded u.c.s. is the smallest member of its proximity class.

Proof. Let $\mathcal{F}$ and $\mathcal{K}$ be u.c.s.'s on $X$ and suppose $\mathcal{F}$ is strongly bounded, with $\mathcal{P}_\mathcal{F} = \mathcal{P}_\mathcal{K}$. We wish to show $\mathcal{K} \subseteq \mathcal{F}$. Let $\Phi \in \mathcal{K}$ and let $\Psi = \Phi \cap \Phi^{-1} \cap [\mathcal{A}]$. Then $\Psi \in \mathcal{P}_\mathcal{X} = \mathcal{P}_\mathcal{F}$, so we can choose $\theta \in \mathcal{F}$ so that $\Psi \subseteq \theta$. Let $\theta^* = \theta \cap [\mathcal{A}]^*$. We claim that $\theta^* \circ \theta^* \subseteq \Phi$.

Let $H \in \theta^*$, and let $\mathcal{C}$ be a finite cover of $X$ with $H \subseteq H$. Then for $C \in \mathcal{C}$ we have $C \subseteq H(C)$. Let $K_c \in \Psi$ such that $K_c(C) \subseteq H(C)$, and define $K$ to be the intersection of the $K_c$'s. Then $K \in \Phi$. We claim $K \subseteq H \circ H$.

Let $(x, y) \in K$. Choose $C \in \mathcal{C}$ so $x \in C$. Then $y \in K_c(C) \subseteq H(C)$. Set $c \in C$ with $(c, y) \in H$. Then $(x, c) \in C \times C \subseteq H$. Hence $(x, y) \in H \circ H$.

THEOREM 14. Let $\mathcal{P}$ be a p.c.s. on $X$, and define

$$\mathcal{B}_\mathcal{P} = \{ \Phi : \Phi \text{ is standard and } <_\mathcal{P} \in \mathcal{P} \}. $$

Then $\mathcal{B}_\mathcal{P}$ is a base for a strongly bounded u.c.s. $\mathcal{J}_\mathcal{P}$ in the proximity class of $\mathcal{P}$.

Proof. If $\Phi = [\mathcal{A}]$ then $<_\mathcal{A} = \subseteq$, so $[\mathcal{A}] \in \mathcal{B}_\mathcal{P}$. From Lemma 7 it is clear that $\mathcal{B}_\mathcal{P}$ is a base for a u.c.s. $\mathcal{J}_\mathcal{P}$.

(1) $\mathcal{J}_\mathcal{P}$ is strongly bounded.

Let $\theta = [\mathcal{A}]^*$. We will show $<_\theta = \subseteq$, so that $\theta \in \mathcal{B}_\mathcal{P}$. Let $A \subseteq B$, and define $\mathcal{C} = \{ B, X \setminus A \}$. Then $H_\theta \in [\mathcal{A}]^*$ and $H_\theta(A) \subseteq B$. Thus $A <_\theta B$.

(2) $\mathcal{J}_\mathcal{P}$ is in the proximity class of $\mathcal{P}$.

Clearly the p.c.s. determined by $\mathcal{J}_\mathcal{P}$ is contained in $\mathcal{P}$. Now let $<_\mathcal{P}$. We define

$$\mathcal{A} = \{ H \subseteq X \times X : A < H(A) \text{ if } A \subseteq X \}$$

$$\mathcal{B} = \{ H_\mathcal{P} : \exists A, B \subseteq X \text{ with } A <^2 B \text{ and } \mathcal{C} = \{ B, X \setminus A \} \}. $$
Notice $\mathcal{B} \subseteq \mathcal{A}$, so $\mathcal{B}$ is a subbase for a proper filter $\Phi$ on $X \times X$. Since each member of $\mathcal{B}$ is symmetric, clearly $\Phi$ is symmetric; hence $\Phi$ is standard. We will show $<^2 \subseteq <_o \subseteq <$. If this holds, then $\Phi \in \mathcal{I}$ and hence $<$ is in the p.c.s. induced by $\mathcal{I}$.

If $A <^2 B$ we define $\mathcal{C} = \{B, X \setminus A\}$. Then $H_\mathcal{C} \in \Phi$, and $H_\mathcal{C}(A) \subseteq B$. Thus $<^2 \subseteq <_o$. To show that $<_o \subseteq <$ it is sufficient to establish that $\Phi \subseteq A$.

Let $H_i \in \mathcal{B}$ for $1 \leq i \leq n$ and suppose $\bigcap_i H_i \subseteq H$. For each $i$, let $A_i <^2 B_i$ such that $H_i = H_{\mathcal{C}_i}$, where $\mathcal{C}_i = \{B_i, X \setminus A_i\}$. Choose $D_i$ so $A_i < D_i < B_i$, and define $\mathcal{D}_i = \{D_i, X \setminus D_i\}$. Set $\mathcal{K} = \prod_i \mathcal{D}_i$, and for $k \in \mathcal{K}$ let $C_k = \bigcap_i k(i)$. Note the $C_k$'s cover $X$.

Now let $E \subseteq X$. We must show $E < H(E)$. This holds, provided $E \cap C_k < H(E)$ for $k \in \mathcal{K}$. We will actually show that if $E \cap C_k \neq \emptyset$ then $C_k < H(E)$.

Let $k \in \mathcal{K}$, with $E \cap C_k \neq \emptyset$. Define $h(i)$ to be $B_i$ if $k(i) = D_i$, and $X \setminus A_i$ otherwise. Then $k(i) < h(i)$ for $1 \leq i \leq n$, and so $C_k < \bigcap_i h(i)$. We claim $\bigcap_i h(i) \subseteq H(E)$.

Let $x \in \bigcap_i h(i)$, and pick $x_0 \in E \cap C_k$. We will show $(x_0, x) \in H$. Choose $i$, and suppose $k(i) = D_i$. Then $h(i) = B_i$, and so $(x_0, x) \in D_i \times B_i \subseteq H_i$. Similarly if $k(i) = X \setminus D_i$ then $x_0$ and $x$ are both in $X \setminus A_i$, and hence $(x_0, x) \in H$.

**Theorem 15.** If $\mathcal{P}$ is a proximity structure then $\mathcal{I}_\mathcal{P}$ is a uniform structure.

**Proof.** Suppose $\ll < \mathcal{P}$ generates $\mathcal{P}$. Let $\Phi \in \mathcal{I}_\mathcal{P}$ so $<_o \subseteq \ll$ and $\Phi$ is strongly bounded. We claim $\Phi^2 \subseteq \mathcal{I}_\mathcal{P}$.

Let $\mathcal{V} \in \mathcal{I}_\mathcal{P}$ and assume $\mathcal{V}$ is standard. Then $<_\mathcal{V} \subseteq \ll$, so $<_o \subseteq <\mathcal{V}$. Let $H \in \Phi$. Then we can choose $\mathcal{C}$ a finite cover of $X$ such that $H_{\mathcal{C}} \subseteq H$. For $C \in \mathcal{C}$ we have $C < H(C)$. Pick $K \in \mathcal{V}$ so $K(C) \subseteq H(C)$ for all $C$ in $\mathcal{C}$. Then $K \subseteq H^2$, so $H^2 \in \mathcal{V}$. This establishes that $\Phi^2 \subseteq \mathcal{V}$.

**Example 16.** We conclude this section with an example to show that a totally bounded u.c.s. need not be strongly bounded. Let $\tau$ be a compact $T_2$ convergence structure on a set $X$, and suppose that every finite intersection of convergent filters has a member with an infinite complement. For example, we would let $\tau$ be the usual topology on the closed unit interval. Let $\mathcal{I}$ be the u.c.s. generated by $\{\mathcal{F} \times \mathcal{F} : \mathcal{F}$ is convergent$\}$. Clearly $\mathcal{I}$ is totally bounded. We claim it is not strongly bounded.

Let $\Phi \in \mathcal{I}$. We will exhibit a member of $\Phi$ which is not strongly bounded. Let $\mathcal{F}_1, \cdots, \mathcal{F}_n$ be convergent filters with $(\bigcap_i \mathcal{F}_i \times \mathcal{F}_i) \cap [A] \subseteq \emptyset$. Pick $F \in \bigcap_i \mathcal{F}_i$ so that $X \setminus F$ is infinite. Define $H = (F \times X \setminus F) \cap [A]$.


Now let $C$ be any cover $X$ with $H^c \subseteq H$. For $x \in X \setminus F$ let $C_x \in C$ such that $x \in C_x$. Since $C_x \times C_x \subseteq H$, clearly $C_x = \{x\}$ for $x \in F$. Thus $C$ is infinite, and $H$ is not strongly bounded.

3. Relation with Cauchy structures. In contrast to the classical case, a totally bounded Cauchy structure $\mathcal{C}$ can be induced by several different p.c.s.'s. However there always exist a smallest and a largest p.c.s. which induce $\mathcal{C}$. If $\mathcal{C}$ is uniform, the smallest p.c.s. associated with it is a proximity structure, but the largest need not be. We call the smallest p.c.s. yielding $\mathcal{C}$ a saturated p.c.s.

**Definition 17.** A Cauchy structure on $X$ is a family $\mathcal{C}$ of proper filters on $X$ such that

(C1) if $x \in X$ then $\mathcal{C}$ contains a member of $\mathcal{C}$
(C2) if $\mathcal{F}$ is a proper filter which contains a member of $\mathcal{C}$ then $\mathcal{F} \subseteq \mathcal{C}$;
(C3) if $\mathcal{F}, \mathcal{C} \in \mathcal{C}$ with $\mathcal{F} \vee \mathcal{C} \neq \mathcal{C}$ then $\mathcal{F} \cap \mathcal{C} \subseteq \mathcal{C}$.

Keller [4] has shown that $\mathcal{C}$ is a Cauchy structure on $X$ iff it is the set of Cauchy filters for some u.c.s. on $X$. If $\mathcal{C}$ is induced by a uniformity we call $\mathcal{C}$ a uniform Cauchy structure. We say $\mathcal{C}$ is totally bounded iff every ultrafilter on $X$ is in $\mathcal{C}$.

**Definition 18.** For $\mathcal{F}$ a filter on $X$ we define a relation $<_\mathcal{F}$ on $X$ by $A <^\mathcal{F} B$ iff $A \subseteq B$ and $B$ or $X \setminus A$ is in $\mathcal{F}$.

**Remark 19.** Notice that $<_\mathcal{F}$ is in $\mathcal{C}(X)$. Also if $\Phi = (\mathcal{F} \times \mathcal{F}) \cap [\mathcal{A}]$ then $<_\Phi = <^\mathcal{F}$.

**Theorem 20.** Let $\mathcal{C}_\mathcal{F} = \{\mathcal{F} : <^\mathcal{F} \in \mathcal{P}\}$, where $\mathcal{P}$ is a p.c.s. on $X$. If $\mathcal{F}$ is any totally bounded u.c.s. in the proximity class of $\mathcal{P}$ then $\mathcal{C}_\mathcal{F}$ is the set of $\mathcal{F}$-Cauchy filters.

**Proof.** Let $\mathcal{F}$ be a filter on $X$ and define $\Phi = (\mathcal{F} \times \mathcal{F}) \cap [\mathcal{A}]$. If $\mathcal{F}$ is $\mathcal{F}$-Cauchy then $\Phi \in \mathcal{F}$, and so $<_\mathcal{F} = <^\mathcal{F} \in \mathcal{P}$. Hence $\mathcal{F} \subseteq \mathcal{C}_\mathcal{F}$.

Conversely, suppose $\mathcal{F} \subseteq \mathcal{C}_\mathcal{F}$. Then $<_\mathcal{F} = <^\mathcal{F} \subseteq \mathcal{P} = \mathcal{P}_\mathcal{F}$ and so we can choose $\mathcal{P} \in \mathcal{F}$ with $<_\mathcal{P} \subseteq <^\mathcal{F}$. Let $\mathcal{U}$ be an ultrafilter containing $\mathcal{F}$. Then $\mathcal{U}$ is $\mathcal{F}$-Cauchy, and therefore $\mathcal{P}(\mathcal{U})$ is also $\mathcal{F}$-Cauchy. (By $\mathcal{P}(\mathcal{U})$ is meant the filter generated by all sets of the form $H(U)$, where $H \in \mathcal{P}$ and $U \in \mathcal{U}$. It is easy to check that $[\mathcal{P} \cap (\mathcal{U} \times \mathcal{U})]^\mathcal{F} \subseteq \mathcal{P}(\mathcal{U}) \times \mathcal{P}(\mathcal{U})$.)

We claim that $\mathcal{P}(\mathcal{U}) \subseteq \mathcal{F}$. Let $H \in \mathcal{P}$ and $U \in \mathcal{U}$. Then $U <^\mathcal{F} H(U)$,
and since \( <_\gamma \subseteq <_\phi \) we can choose \( K \in \Phi \) with \( K(U) \subseteq H(U) \). Now pick \( F \in \mathcal{F} \) so \( F \times F \subseteq K \). Then since \( \mathcal{F} \subseteq \mathcal{U} \) we have \( F \cap U \neq \emptyset \), and so \( F \subseteq K(U) \). This establishes that \( H(U) \in \mathcal{F} \), as desired.

**Remark 21.** This theorem tells us that the totally bounded u.c.s.'s in the same proximity class all induce the same Cauchy structure.

**Definition 22.** A p.c.s. \( \mathcal{P} \) is compatible with a totally bounded Cauchy structure \( \mathcal{C} \) iff \( \mathcal{C} = \mathcal{C}_\mathcal{P} \).

**Notation 23.** Let \( \mathcal{F} \) be a filter on \( X \) and let \( \varphi \in \mathcal{O}(X) \). Then

\[
r_{<_\varphi}(\mathcal{F}) = \{ A : F < A \text{ for some } F \in \mathcal{F} \}.
\]

Notice \( r_{<_\varphi}(\mathcal{F}) \) is a filter contained in \( \mathcal{F} \).

**Definition 24.** Let \( \mathcal{C} \) be a totally bounded Cauchy structure on \( C \).

1. \( \mathcal{P}_1(\mathcal{C}) = \{ <_{<_\varphi} : \mathcal{F} \in \mathcal{C} \} \);
2. \( \mathcal{P}_2(\mathcal{C}) = \{ < \in \mathcal{O}(X) : \mathcal{F} \in \mathcal{C} \implies r_{<_\varphi}(\mathcal{F}) \in \mathcal{C} \} \).

**Theorem 25.** If \( \mathcal{C} \) is a totally bounded Cauchy structure on \( X \) then \( \mathcal{P}_1(\mathcal{C}) \) is the largest p.c.s on \( X \) compatible with \( \mathcal{C} \), and \( \mathcal{P}_2(\mathcal{C}) \) is the smallest. Moreover, \( \mathcal{I} = \{ <_{<_\varphi} : \mathcal{F} \in \mathcal{C} \} \) is a subbase for \( \mathcal{P}_1(\mathcal{C}) \).

**Proof.**

1. \( \mathcal{I} \) is a subbase for \( \mathcal{P}_1(\mathcal{C}) \).

Let \( \mathcal{R} \) be the set of refinements of finite intersections of orders in \( \mathcal{I} \). We need \( \mathcal{R} = \mathcal{P}_1(\mathcal{C}) \). It is sufficient to show that \( \mathcal{R} \) is a p.c.s. Clearly \( \mathcal{R} \) satisfies (P1) and (P 3).

Let \( \mathcal{F}_i, \cdots, \mathcal{F}_n \in \mathcal{C} \) with \( <_{i_0} \subseteq <_{i} \). Suppose \( \bigcap_i <_{i_0} \subseteq <_{\epsilon} \mathcal{O}(X) \). We wish to show \( <_{\epsilon} \in \mathcal{R} \). We may assume the \( \mathcal{F}_i \)'s are pairwise disjoint; i.e., \( \mathcal{F}_i \cap \mathcal{F}_j = \{ \emptyset \} \) for \( i \neq j \). This follows by induction from (C 3), since if \( \mathcal{F}_i \cap \mathcal{F}_j \neq \{ \emptyset \} \) we replace \( <_{i} \cap <_{j} \) by \( <_{\epsilon} \), where \( \mathcal{F} = \mathcal{F}_i \cap \mathcal{F}_j \). Choose \( F_i \in \mathcal{F}_i \) so that the \( F_i \)'s are pairwise disjoint.

Suppose now that \( A <_{i} B \) for \( 1 \leq i \leq n \). We will show \( A <_{\epsilon} B \). For each \( i \), define

\[
D_i = \begin{cases} 
F_i \cap B & \text{if } B \in \mathcal{F}_i \\
F_i \backslash A & \text{if } B \notin \mathcal{F}_i .
\end{cases}
\]

Note \( D_i \in \mathcal{F}_i \) for all \( i \). Let \( H = (\bigcup_i D_i \times D_i) \cup A \). We claim \( A < H(A) < B \).

Clearly \( A \subseteq H(A) \). To see that \( H(A) \subseteq B \), let \( a \in A \) with \( (a, x) \in H \). If \( x = a \) then \( x \in B \). If \( x \neq a \) then for some \( i \), \( a \) and \( x \) are both in \( D_i \). Since \( a \notin \mathcal{F}_i \backslash A \), clearly \( D_i = F_i \cap B \), and so \( x \in B \).
Now fix \( i \). We wish to show \( A <_i H(A) <_i B \). It is sufficient to show that either \( H(A) \) or \( X \setminus H(A) \) or \( B \setminus A \) is in \( \mathcal{F} \). If \( D_i = F_i \setminus A \) it is not difficult to prove that \( D_i \cap H(A) = \emptyset \), so that \( X \setminus H(A) \in \mathcal{F} \).

(Recall the \( D_j \)'s are pairwise disjoint.) If \( D_i = F_i \cap B \) and \( D_i \cap A = \emptyset \) then clearly \( B \setminus A \in \mathcal{F} \). If \( D_i \cap A \neq \emptyset \) then \( D_i \subseteq H(A) \) and so \( H(A) \in \mathcal{F} \).

(2) \( \mathcal{P}_s(\mathcal{C}) \) is a p.c.s.

If \( < = <_1 \cap <_2 \) and \( \mathcal{F} \in \mathcal{C} \) then \( r_<(\mathcal{F}) = r_<_1(\mathcal{F}) \cap r_<_2(\mathcal{F}) \).

Using this and (C3), we conclude \( \mathcal{P}_s(\mathcal{C}) \) is closed under finite intersections. Similarly \( r_<(\mathcal{F}) = r_<(r_<(\mathcal{F})) \), so \( \mathcal{P}_s(\mathcal{C}) \) is closed under “squaring”. Since \( r_<(\mathcal{F}) \subseteq r_<(r_<(\mathcal{F})) \) whenever \( < \subseteq <' \) clearly (P3) holds.

(3) \( \mathcal{P}_L(\mathcal{C}) \subseteq \mathcal{P}_s(\mathcal{C}) \).

It is sufficient to show that \( <_x \in \mathcal{P}_s(\mathcal{C}) \) for \( \mathcal{F} \in \mathcal{C} \). Let \( < = <_x \) and let \( \mathcal{G} \in \mathcal{C} \). If \( \mathcal{G} \cap \mathcal{F} = \emptyset \) then \( r_<(\mathcal{F}) = \mathcal{G} \); and if \( \mathcal{G} \cap \mathcal{F} \neq \emptyset \) then \( r_<(\mathcal{F}) \supseteq \mathcal{G} \cap \mathcal{F} \). Thus in either case \( r_<(\mathcal{F}) \in \mathcal{C} \).

(4) \( \mathcal{P}_s(\mathcal{C}) \) and \( \mathcal{P}_L(\mathcal{C}) \) are both compatible with \( \mathcal{C} \). Let \( \mathcal{G} \) denote the Cauchy structure induced by \( \mathcal{P}_s(\mathcal{C}) \); and similarly for \( \mathcal{G}_L \). Suppose \( \mathcal{F} \in \mathcal{C} \). Then by definition of \( \mathcal{P}_L(\mathcal{C}) \) we have \( <_x \in \mathcal{P}_L(\mathcal{C}) \) and hence \( \mathcal{F} \in \mathcal{G}_L \). Therefore \( \mathcal{C} \subseteq \mathcal{G}_L \subseteq \mathcal{G}_s \).

Now suppose \( \mathcal{G} \in \mathcal{G}_s \). Then \( <_x \in \mathcal{P}_s(\mathcal{C}) \). Let \( < = <_x \) and let \( \mathcal{U} \) be an ultralimit containing \( \mathcal{F} \). Then \( \mathcal{U} \in \mathcal{C} \), and so by definition of \( \mathcal{P}_s(\mathcal{C}) \) we have \( r_<(\mathcal{U}) \in \mathcal{C} \). But \( r_<(\mathcal{U}) \subseteq \mathcal{G} \), and so \( \mathcal{C} \subseteq \mathcal{C} \).

(5) If \( \mathcal{P} \) is a p.c.s. compatible with \( \mathcal{C} \) then \( \mathcal{P}_s(\mathcal{C}) \subseteq \mathcal{P} \subseteq \mathcal{P}_L(\mathcal{C}) \).

For \( \mathcal{F} \in \mathcal{C} = \mathcal{C}_s \) we have \( <_x \in \mathcal{P} \). Thus \( \mathcal{P}_L(\mathcal{C}) \subseteq \mathcal{P} \). Now let \( < \in \mathcal{P} \) and choose \( \mathcal{F} \in \mathcal{C} \). Let \( \mathcal{G} = r_<(\mathcal{F}) \). We must show \( \mathcal{G} \in \mathcal{C} \); i.e., \( <_x \in \mathcal{P} \). It is straightforward to check that \( <_x \cap < \) is a base for \( \mathcal{P} \).

Remark 26. This theorem tells us that each totally bounded Cauchy structure has a largest and smallest p.c.s. compatible with it. Since an intersection of proximity convergence structures is also a p.c.s., we see that the set of proximity convergence structures compatible with a given totally bounded Cauchy structure is a complete lattice.

Theorem 27. If \( \mathcal{C} \) is a totally bounded Cauchy structure and \( \mathcal{P} \) is a proximity structure compatible with \( \mathcal{C} \) then \( \mathcal{P} = \mathcal{P}_s(\mathcal{C}) \).

Proof. Let \( \mathcal{P} \) be a p.c.s. compatible with \( \mathcal{C} \) and suppose \( \{<\} \) is a base for \( \mathcal{P} \). We will show \( \mathcal{P}_s(\mathcal{C}) \subseteq \mathcal{P} \).

Let \( < \in \mathcal{P}_s(\mathcal{C}) \) and suppose \( A < B \). We wish to show \( A <_x < B \). For this it is sufficient to produce a filter \( \mathcal{F} \) in \( \mathcal{C} \) with \( A <_x B \).

(Recall if \( \mathcal{F} \in \mathcal{C} \) then \( <_x \in \mathcal{P} \) and so \( \subseteq \subseteq <_x \).)
Set $\mathcal{S} = \{D: A < D\} \cup \{X|E: E < B\}$. Then since $A < B$, $\mathcal{S}$ has the finite intersection property. Let $\mathcal{U}$ be an ultrafilter containing $\mathcal{S}$. Then $\mathcal{U} \in \mathcal{C}$. Since $< \circ \mathcal{S}(\mathcal{C})$ we have $r_<(\mathcal{U}) \in \mathcal{C}$. Clearly neither $B$ nor $X\setminus A$ is in $r_(\mathcal{U})$.

**Remark and Definition 28.** From this theorem it follows easily that if $\mathcal{C}$ is uniform (and totally bounded) then $\mathcal{P}_s(\mathcal{C})$ is the unique proximity structure compatible with $\mathcal{C}$. We will call $\mathcal{P}_s(\mathcal{C})$ a saturated p.c.s. (whether or not $\mathcal{C}$ is uniform). Obviously then every proximity structure is saturated.

**Example 29.** Even if $\mathcal{C}$ is uniform, $\mathcal{P}_l(\mathcal{C})$ need not be a proximity structure. For example let $\mathcal{C}$ be a totally bounded uniformity with Cauchy family $\mathcal{C}$. Assume that no finite intersection of Cauchy filters equals $\{X\}$. This is the case as long as $\mathcal{C} \neq \{X \times X\}$, but the proof is somewhat involved and will not be given. Certainly it is true for the usual uniformity on the closed unit interval. Assume also that if $A <_x A$ then $A = \emptyset$ or $X$. This is true if the associated topology is connected, for example.

Suppose $<_x \in \mathcal{P}_l(\mathcal{C})$. By Theorem 25, there are Cauchy filters $\mathcal{F}_1, \ldots, \mathcal{F}_n$ such that $\cap_i <_x \subseteq <_x$. Therefore if $F \in \cap_i \mathcal{F}_i$ then $F <_x F$, and so $F = X$. Hence $\cap_i \mathcal{F}_i = \{X\}$, which is impossible. Therefore $<_x \notin \mathcal{P}_l(\mathcal{C})$, and so $\mathcal{P}_l(\mathcal{C}) \neq \mathcal{P}_s(\mathcal{C})$. By Theorem 27, $\mathcal{P}_l(\mathcal{C})$ is not a proximity structure.

4. The $\Sigma$-compactification. A p.c.s. is compact, provided the associated convergence structure is compact. A compactification of a p.c.s. is a compact p.c.s. in which the given space can be densely embedded. In general a p.c.s has many compactifications. We will confine ourselves to one, called the $\Sigma$-compactification. This works at least for relatively round spaces, and has a nice characterization. Using it we can obtain a generalization of the classical one-to-one correspondence between proximity structures and $T_2$ compactifications of a given topological space.

Continuous maps to compact $T_2$ spaces can be extended to this compactification, provided the range spaces satisfy a strong regularity condition. We leave open the problem of obtaining the “right” definition of regularity for a p.c.s.

**Definition 30.** Let $\mathcal{P}$ be a p.c.s. on $X$. For $x \in X$ we define $\tau_<(x)$ to be the intersection ideal generated by the filters of the form $r_<(\mathcal{U})$, where $< \in \mathcal{P}$.

**Theorem 31.** If $\mathcal{I}$ is in the proximity class of $\mathcal{P}$ then $\tau_{\mathcal{I}} = \tau_{\mathcal{P}}$. 
Proof. Notice that \( \{ r(x) : \tau_x \in \mathcal{P} \} \) is a base for \( \tau_x \). Thus if \( F \in \tau_x \) then for some \( \tau_x \in \mathcal{P} \) we have \( r(x) \subseteq F \). Let \( Y \in \tau_x \) with \( \tau_x \subseteq \tau_x \). Now, \( F \subseteq \tau \times \tau(x) \) so \( \tau(x) \in \tau_{\tau_x} \). But \( \tau(x) \subseteq r(x) \), since for \( H \in \mathcal{T} \) we have \( \{ x \} \subseteq H(x) \).

Now suppose \( F \in \tau_x \). Let \( G = F \cap \tau \) and let \( \Phi = G \times G \cap [4] \). Then \( \Phi \in \mathcal{J} \) and so \( \tau_x \in \mathcal{P} \). Set \( \tau_x = \tau_x = \tau \). Then \( r(x) \subseteq F \).

Remark 32. We can also describe \( \tau_x \) as follows: \( F \in \tau_x \) iff for some \( Y \in \mathcal{F} \cap \tau \) we have \( \tau_x \in \mathcal{P} \).

Next we will describe the construction of the \( \Sigma \)-extension of a p.c.s.

Definition 33. Let \( \mathcal{C} \) be a Cauchy structure on \( X \). Two filters in \( \mathcal{C} \) are equivalent iff their intersection is in \( \mathcal{C} \). We denote the associated partition by \( X^*(\mathcal{C}) \), or just \( X^* \). The map which assigns to a point \( x \in X \) the equivalence class of \( x \) is denoted by \( j \). If \( (X, \mathcal{C}) \) is \( T_2 \) then \( j \) is an injection of \( X \) into \( X^* \).

We define \( \Sigma \) to be the set of all maps \( \sigma \) which assign to each equivalence class \( p \) in \( X^* \) a filter in \( p \); we further require for \( x \in X \) and \( \sigma \in \Sigma \) that \( \sigma(j(x)) = x \).

For each \( \sigma \) in \( \Sigma \) we obtain a map from \( \mathcal{P}(X) \) to \( \mathcal{P}(X^*) \); namely,

\[
A^\sigma = \{ p \in X^* : A \in \sigma(p) \}.
\]

This allows us to define a map from \( \mathcal{O}(X) \) to the set of relations on \( X^* \). For \( \tau \in \mathcal{O}(X) \) we define \( A \subseteq B \) iff there are subsets \( C \) and \( D \) of \( X \) with \( A \subseteq C^o \), \( D^o \subseteq B \), and \( C < D \).

Now suppose \( \mathcal{C} \) is totally bounded, and let \( \mathcal{P} \) be a compatible p.c.s. We define \( \mathcal{P}_2 = \{ \tau \in \mathcal{O}(X)^* : \sigma \in \Sigma, \exists \tau \in \mathcal{P} \text{ with } \tau^o \subseteq \tau \} \). It is easy to check that \( \mathcal{P}_2 \) is a p.c.s. on \( X \). We will call \( (j, (X^*, \mathcal{P}_2)) \) the \( \Sigma \)-extension of \( (X, \mathcal{P}) \). It is closely related to the Kowalsky completion of \( (X, \mathcal{C}) \), described in [5] and in [7].

Definition 34. Let \( k : (X, \mathcal{P}) \to (Y, \mathcal{O}) \). For \( \tau \in \mathcal{O}(X) \) we define \( k(\tau) \in \mathcal{O}(Y) \) by \( A k(\tau) B \) iff \( A \subseteq B \) and \( k^{-1}(A) < k^{-1}(B) \). We say \( k \) is a dense embedding of \( (X, \mathcal{P}) \) into \( (Y, \mathcal{O}) \), provided \( k \) is one-to-one and for \( \tau \in \mathcal{O}(X) \) we have \( \tau \in \mathcal{P} \) iff \( k(\tau) \in \mathcal{O} \).

Next we will establish that \( j \) is a dense embedding of \( (X, \mathcal{P}) \) into \( (X^*, \mathcal{P}_2) \).

Lemma 35. Let \( (X, \mathcal{P}) \) be \( T_2 \) and let \( \tau' \) denote the convergence structure induced by \( \mathcal{P}_2 \).

(i) If \( p \in X^* \) and \( F \in p \) then \( j(F) \in \tau'(p) \).

(ii) If \( G \in \tau'(p) \) and \( \sigma \in \Sigma \) then the filter \( C_\sigma = \{ A : A^\sigma \in G \} \) is in \( p \).
Proof. Suppose $\mathcal{F} \in \mathcal{P}$ and define $\mathcal{G} = j(\mathcal{F}) \cap \hat{\mathcal{P}}$. To show $j(\mathcal{F}) \rightarrow \mathcal{P}$ it is sufficient to establish $<_{\mathcal{G}}$ is in $\mathcal{P}_2$.

Pick $\sigma \in \Sigma$ and set $\mathcal{H} = \mathcal{F} \cap \sigma(p)$. Now $\mathcal{H}$ is Cauchy, and so $<_{\mathcal{H}} \in \mathcal{P}$. Observe that $\mathcal{H} \subseteq \mathcal{G}$, so that $<_{\mathcal{G}} \subseteq <_{\mathcal{H}}$.

Now assume $\mathcal{U} \in \tau'(p)$, and let $\sigma \in \Sigma$. Pick $< \in \mathcal{P}_2$ with $r_{<}(\hat{\mathcal{P}}) \subseteq \mathcal{U}$, and choose $<_{1} \in \mathcal{P}$ so that $<_{1} \subseteq <$. Then $r_{<_{1}}(\sigma(p)) \subseteq \mathcal{U}$. For if $A \in \sigma(p)$ and $A <_{1} B$ then $A_{\sigma} <_{1} B_{\sigma}$ and hence $A_{\sigma} < B_{\sigma}$. Since $p \in A_{\sigma}$ we have $B_{\sigma} \in r_{<}(\hat{\mathcal{P}}) \subseteq \mathcal{U}$.

Now $\sigma(p) \in \mathcal{P}$ and $<_{1} \in \mathcal{P}$. Therefore $r_{<_{1}}(\sigma(p)) \in \mathcal{P}$. (Use Theorem 25 and (C 3)).

THEOREM 36. Let $(X, \mathcal{P})$ be $T_2$. Then $(X^{*}, \mathcal{P}_2)$ is $T_2$ and $j$ is a dense embedding of $(X, \mathcal{P})$ into $(X^{*}, \mathcal{P}_2)$.

Proof. Suppose $\mathcal{G}$ converges to both $p$ and $q$. Let $\sigma \in \Sigma$. By the preceding lemma $\mathcal{G}_\sigma \in \mathcal{P} \cap \mathcal{Q}$. Thus $p = q$, and $\mathcal{P}_2$ is $T_2$.

Notice that for $\sigma \in \Sigma$ and $A \subseteq X$ we have $j^{-1}(A_{\sigma}) = A$. Here strong use is made of the fact that $\sigma(j(x)) = x$ for $x \in X$. From this it is easy to see that for $< \in \mathcal{P}$ and $\sigma \in \Sigma$ we have $<_{\sigma} \subseteq j(<)$. Thus $j(<) \in \mathcal{P}_2$.

Now suppose $< \in \mathcal{G}(X)$ and $j(<) \in \mathcal{P}_2$. Let $\sigma \in \Sigma$ and choose $<_{1} \in \mathcal{P}$ with $<_{1} \subseteq j(<)$. Using the same fact as before, we see that $<_{1} \subseteq <$. This establishes that $j$ is an embedding.

It is easy to check that $j(X)$ is dense in $X^{*}$, since for $\mathcal{F} \in \mathcal{P}$ we have $j(\mathcal{F}) \rightarrow \mathcal{P}$. (Lemma 35). Next we will give conditions under which the $\Sigma$-extension is actually a compactification.

DEFINITION 37. Let $(X, \mathcal{P})$ be a p.c.s. For $\sigma \in \Sigma$ we define

$$<_{\sigma} = \bigcap \{<_{\mathcal{F}}: \mathcal{F} = \sigma(p) \text{ for some } p \in X^{*}\}.$$ 

Then $\mathcal{P}$ is relatively round iff each $<_{\sigma}$ is in $\mathcal{P}$.

Notice that every proximity structure is relatively round. In fact if $\subset \subset$ is a proximity on $X$ then $\subset \subset = \bigcap \{<_{\mathcal{F}}: \mathcal{F} \in \mathcal{G}(\subset \subset)\}$.

THEOREM 38. If $(X, \mathcal{P})$ is relatively round and $T_2$ then $(j, (X^{*}, \mathcal{P}_2))$ is a compactification of $(X, \mathcal{P})$.

Proof. In view of Theorem 36, we need only establish that $\mathcal{P}_2$ is compact. Let $\mathcal{U}$ be an ultrafilter on $X^{*}$.

Notice that for $\sigma \in \Sigma$, if $A <_{\sigma} B$ then $(X^{*}\setminus B_{\sigma}) \subseteq (X\setminus A)_{\sigma}$; thus either $B_{\sigma}$ or $(X\setminus A)_{\sigma}$ is in $\mathcal{U}$. This yields $<_{\sigma} \subseteq <_{\mathcal{U}_{\sigma}}$. Since $\mathcal{P}$ is relatively round, we conclude $\mathcal{U}_{\sigma}$ is Cauchy for $\sigma \in \Sigma$.

Moreover, the $\mathcal{U}_{\sigma}$'s are all in the same equivalence class. To see
this, suppose $\sigma$ and $\mu$ are in $\Sigma$ and let $\eta(p) = \sigma(p) \cap \mu(p)$ for $p \in X^*$. Then $\eta \in \Sigma$, and also $U_\eta \subseteq U_\sigma \cap U_\mu$. Thus $U_\sigma$ and $U_\mu$ are equivalent.

Let $q$ be the equivalence class of the $U_i$'s. We claim $U \rightarrow q$.

Next we wish to characterize the $\Sigma$-compactification of $(X, \mathcal{P})$ as its unique relatively round $T_\delta$ compactification. This will be done by using the corresponding fact for uniform convergence spaces, established in [7].

**Definition 39.** Let $f: (X, \mathcal{P}) \rightarrow (Y, \mathcal{D})$. Then $f$ is $p$-continuous iff $f(<) \in \mathcal{D}$ whenever $< \in \mathcal{P}$.

**Lemma 40.** Let $f: (X, \mathcal{P}) \rightarrow (Y, \mathcal{D})$

(i) $f$ is $p$-continuous iff it is uniformly continuous with respect to $\mathcal{I}_\sigma$ and $\mathcal{I}_\sigma$.

(ii) $f$ is an embedding of $(X, \mathcal{P})$ into $(Y, \mathcal{D})$ iff it embeds $(X, \mathcal{I}_\sigma)$ in $(Y, \mathcal{I}_\sigma)$.

**Proof.** Notice that if $\Phi$ is a standard filter on $X \times X$ and $\Psi = (f \times f)(\Phi) \cap [\Delta]$ then $<_\Psi = f(<_\Phi)$. Clearly then (i) holds. Also if $\Psi \in \mathcal{I}_\sigma$ and $f$ is a $p$-embedding then $\Phi \in \mathcal{I}_\sigma$. Therefore every $p$-embedding is a uniform embedding.

Now assume $f$ is a uniform embedding. Suppose $< \in \mathcal{O}(X)$ with $f(<) \in \mathcal{D}$. Pick $\theta \in \mathcal{I}_\sigma$ with $<_\theta \subseteq f(<)$. Set $\theta_1 = (f \times f)^{-1}(\theta)$. We claim $\theta_1 \in \mathcal{I}_\sigma$ and $<_\theta \subseteq f(<)$.

Since $<_\theta$ is defined, $\theta$ is standard; therefore $\theta_1$ is standard, and in particular it is proper. Note $\theta \subseteq (f \times f)(\theta_1)$, so that $\theta_1 \in \mathcal{I}_\sigma$. Now if $A <_{\theta_1} B$ then $f(A) <_\theta Y \setminus f(X \setminus B)$. Since $<_\theta \subseteq f(<)$ we conclude $A < B$.

**Definition 41.** Let $f: (X, \mathcal{P}) \rightarrow (Y, \mathcal{D})$. By $\Sigma(f)$ we mean the set of all maps $\sigma$ which assign to each point $y$ in $Y$ a filter converging to $y$. We further require that for $y \in f(X)$ and $\sigma \in \Sigma(f)$ we have $\sigma(y) = \hat{y}$.

We define $(f, (Y, \mathcal{D}))$ to be relatively round provided $<_\sigma \in \mathcal{D}$ for each $\sigma$ in $\Sigma(f)$. We say $(f, (Y, \mathcal{I}_\sigma))$ is relatively round iff for $\sigma \in \Sigma(f)$ the filter $\bigcap \{ \sigma(y) \times \sigma(y): y \in Y \}$ is in $\mathcal{I}_\sigma$.

**Lemma 42.** If $(k, (Y, \mathcal{D}))$ is a relatively round compactification of $(X, \mathcal{P})$ then $(k, (Y, \mathcal{I}_\sigma))$ is a relatively round completion of $(X, \mathcal{I}_\sigma)$.

**Proof.** From the preceding lemma we know that $k$ is an embedding of $(X, \mathcal{I}_\sigma)$ into $(Y, \mathcal{I}_\sigma)$. Since $\mathcal{I}_\sigma$ and $\mathcal{D}$ induce the same
convergence structure \( \tau' \), clearly this embedding is dense. Since \( \tau \) is compact, \( \mathcal{J}_\sigma \) is complete.

Now let \( \sigma \in \Sigma(f) \). Then \( <_\sigma \in \mathcal{P} \). Set \( \theta = \bigcap \{ \sigma(y) \times \sigma(y) : y \in Y \} \).
We claim \( <_\theta = <_\sigma \), so that \( \theta \in \mathcal{J}_\sigma \). To see that \( <_\sigma \subseteq <_\theta \) notice that if \( A <_\sigma B \) then \( (B \times B) \cap (X \setminus A \times X \setminus A) \in \theta \).

**Theorem 43.** If \( (X, \mathcal{P}) \) is relatively round and \( T_2 \) then \( (j, (X^*, \mathcal{P}_2)) \) is the unique relatively round \( T_2 \) compactification of \( (X, \mathcal{P}) \).

**Proof.** In [7], Theorem 19, it was shown that any two relatively round \( T_2 \) completions of a u.c.s. are equivalent. From this, and from the two preceding lemmas, it follows that \( (X, \mathcal{P}) \) can have at most one relatively round \( T_2 \) compactification.

By Theorem 38 we know \( (j, (X^*, \mathcal{P}_2)) \) is a compactification of \( (X, \mathcal{P}) \). To see that it is relatively round pick \( \sigma \in \Sigma(j) \) and set \( \mu \in \Sigma \). Let \( \eta(p) = \sigma(p)^{\mu} \) for \( p \in X^* \). By Lemma 35, \( \eta(p) \in \mathcal{P} \) for \( p \in X^* \). It is easy to check that if \( p = j(x) \) then \( \eta(p) = \hat{x} \). Thus \( \eta \in \Sigma \), and \( <_\eta \in \mathcal{P} \). Notice that \( <^\eta \subseteq <_\sigma \), so that \( <_\sigma \in \mathcal{P}_2 \).

**Theorem 44.** If \( (X, \mathcal{P}) \) is a relatively round saturated \( T_2 \) p.c.s. then \( (X^*, \mathcal{P}_2) \) is saturated.

**Proof.** Suppose \( <' \in \mathcal{O}(X^*) \), and \( r_<,(\mathcal{F}) \) is Cauchy whenever \( \mathcal{F} \) is. Let \( \sigma \in \Sigma \) and define

\[
A < B \text{ iff } X^* \setminus (X \setminus A)^{\sigma} <' B^\sigma.
\]

Then \( < \in \mathcal{O}(X) \) and \( <^\sigma \subseteq <' \). We claim \( < \in \mathcal{P} \).

Let \( \mathcal{F} \in \mathcal{C}_{\cdot} \), and let \( p \) be its equivalence class. Then \( j(\mathcal{F}) \to p \) (Lemma 35). Define \( \mu \in \Sigma(j) \) by \( q \to j(\sigma(q)) \cap \hat{q} \). Since \( \mathcal{P} \) is relatively round, so is \( (j, (X^*, \mathcal{P}_2)) \) (Theorem 43). Thus \( <_\mu \in \mathcal{P}_2 \), and \( \mathcal{C} = r_<,(\mathcal{F}_1) \) converges to \( p \). Let \( \mathcal{G} = r'_<(\mathcal{F}_1) \). Then \( \mathcal{G} \to p \), and so \( \mathcal{C} \subseteq \mathcal{P}_2 \).

It is not difficult to check that \( \mathcal{C} \subseteq r_<,(\mathcal{F}) \) so that \( r_<,(\mathcal{F}) \) is Cauchy. Since \( \mathcal{P} \) is saturated we conclude \( < \in \mathcal{P} \), and \( <' \in \mathcal{P}_2 \).

**Remark 45.** There is a one-to-one correspondence between certain \( T_2 \) compactifications of a given \( T_2 \) convergence space \( (X, \tau) \) and certain of its compatible p.c.s.'s. If \( \mathcal{P} \) is relatively round then \( (j, (X^*, \tau(\mathcal{P}_2))) \) is a \( T_2 \) compactification of \( (X, \tau) \). It is also a relatively round compactification meaning that if \( \mathcal{F} \to p \) and \( \sigma \in \Sigma(j) \) then \( r_<,(\mathcal{F}) \to p \). Thus the map \( \mathcal{P} \to (j, (X^*, \tau(\mathcal{P}_2))) \) takes relatively round p.c.s.'s on \( (X, \tau) \) to relatively round \( T_2 \) compactifications of \( (X, \tau) \).

This map is one-to-one, provided we limit ourselves to saturated structures. This follows from the preceding theorem and from the
fact that a homeomorphism is \( p \)-continuous with respect to the largest compatible saturated structures.

The above map is also a surjection. Given a relatively round \( T_2 \) compactification \((k, (Y, \tau'))\) we define \( \mathcal{P}' \) to be the (unique) compatible saturated p.c.s. Set \( \mathcal{P} = \{ <; k(<) \in \mathcal{P}' \} \). Then \( \mathcal{P} \) is relatively round, saturated and compatible with \( \tau \). Moreover, \((k, (Y, \mathcal{P}'))\) is a compactification of \((X, \mathcal{P})\). Using Theorem 43, we can establish that the given compactification is equivalent to \((j, (X^*, \tau(\mathcal{P}_2)))\).

If \( \mathcal{P} \geq \mathcal{P} \), then \( \kappa_i \geq \kappa_z \). (\( \kappa_i \) is the compactification associated with \( \mathcal{P}_i \).) However it is not clear the converse holds.

In the final part of this section we will show that a certain class of \( p \)-continuous functions on \((X, \mathcal{P})\) extend to its \( \Sigma \)-compactification.

**Definition 46.** For any convergence space \((X, \tau)\) we define an order \( <^c \) on \( X \) by \( A <^c B \) iff \( \overline{A} \subseteq B^c \). A compatible p.c.s. \( \mathcal{P} \) is \( c \)-regular iff \( <^c \in \mathcal{P} \). A compatible u.c.s. \( \mathcal{I} \) is \( c \)-regular iff it is regular in the sense of Pervin and Biesterfeldt [6]. In their notation, this means if \( \Phi \in \mathcal{I} \) then \( \Phi^c \in \mathcal{I} \).

**Remark 47.** Both these definitions of regularity seem too strong. If \( \mathcal{P} \) is \( c \)-regular then \( \tau_\mathcal{P} \) is a regular topological structure. The same is true if \( \mathcal{I} \) is \( c \)-regular and strongly bounded. Finding a better definition of regularity has proved unexpectedly difficult.

**Theorem 48.** Let \( \mathcal{I} \) be the strongly bounded u.c.s. in the proximity class of \( \mathcal{P} \). Then \( \mathcal{P} \) is \( c \)-regular iff \( \mathcal{I} \) is \( c \)-regular.

**Proof.** Let \( \Phi \) be a standard, strongly bounded member of \( \mathcal{I} \), and set \( \Psi = \Phi^c \cap (\Phi^c)^{-1} \). We will establish that \( <^c \in \mathcal{P} \) iff \( \Psi \in \mathcal{I} \).

Since the standard strongly bounded members of \( \mathcal{I} \) are a base for \( \mathcal{I} \) this is sufficient to establish the desired equivalence.

(1) \((<^c \cap <^c)^2 \subseteq <^c\).

This is established by the following observations.

(i) If \( H \subseteq X \times X \) then \( H^c(A) \subseteq H(A)^- \) for \( A \subseteq X \).

If \( (a, x) \in H^c \) with \( a \in A \) then \( x \in H(a)^- \subseteq H(A)^- \).

(ii) If \( H = H^{-1} \) then \( (H^c)^{-1}(A^c) \subseteq H(A) \). Let \( a \in A^c \) with \( (x, a) \in H^c \). Then \( a \in H(x)^- \) and so \( A \cap H(x) \neq \emptyset \). For \( z \in A \cap H(x) \) we have \( x \in H(z) \subseteq H(A) \).

(2) \(<^c \subseteq <^c\).

We will show first that if \( K \) is strongly bounded then \( A^- \subseteq K^c(A) \) for \( A \subseteq X \). Let \( \mathcal{C} \) be a finite cover of \( X \) such that \( H_\mathcal{C} \subseteq K \). Pick \( x \in A^- \). Then there is a set \( C \in \mathcal{C} \) with \( x \in C^- \) and \( C \cap A \neq \emptyset \). To see this, let \( \mathcal{F} \to x \) such that \( A \in \mathcal{F} \), and let \( \mathcal{U} \) be an ultrafilter
containing \( \mathcal{F} \). Then \( \mathcal{U} \cap \mathcal{C} \neq \emptyset \), and for \( C \) in \( \mathcal{U} \cap \mathcal{C} \) the desired conditions hold.

Now pick \( u \in C \cap A \). Then \( C \subseteq K(u) \) and so \( x \in K(u)^- \). This means \( (u, x) \in K^0 \) and thus \( x \in K^0(A) \).

From this it follows that if \( A < \gamma B \) then \( A^- \subseteq B^- \). Moreover, \( A \subseteq B \); note \( X \setminus B \sim \gamma X \setminus A \), so that \( (X \setminus B)^- \subseteq X \setminus A \). Therefore if \( A < \gamma B \) then \( A^- \subseteq B \).

**Theorem 49.** Let \((X, \mathcal{P})\) be \( T_2 \). Every \( p \)-continuous function from \((X, \mathcal{P})\) to a \( c \)-regular compact \( T_2 \) p.c.s. has a unique extension to \((X^*, \mathcal{P}^*_2)\).

**Proof.** Let \( f \) be a \( p \)-continuous function from \((X, \mathcal{P})\) to a \( c \)-regular compact \( T_2 \) space \((Y, \mathcal{G})\). It is easy to check that \( f \) is Cauchy-continuous. Since \( Y \) is compact and \( T_2 \), the image of a filter in \( \mathcal{G} \) has a unique limit in \( Y \). Moreover, the images of equivalent filters have the same limit. This defines a map \( h: X^* \to Y \); namely, \( h(p) \) is the limit of the \( f \)-image of any filter in \( p \). Notice \( hj = f \). We need to establish that \( h \) is \( p \)-continuous. This is where \( c \)-regularity is used.

Let \( < \in \mathcal{G}_2 \) and select \( \sigma \in \Sigma \). Choose \( <, \in \mathcal{P} \) so \( <, \subseteq < \) and set \( <, = f(<,) \cap <, \). We claim \( <, \subseteq h(<) \). This is based on the following observations.

(i) If \( A \subseteq B \) then \( h(A') \subseteq f(A)^\sigma \).
(ii) If \( C^- \subseteq D \) then \( f^-1(C)^c \subseteq h^-1(D)^c \).
(iii) If \( B \) \( f(<,) C \) then \( f^-1(B)^c < f^-1(C)^c \).

Note \( h \) is unique, since every continuous extension of \( f \) must agree with \( h \) on the dense subset \( j(X) \).

**References**


Received February 24, 1972.

UNIVERSITY OF MASSACHUSETTS
The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the Pacific Journal of Mathematics should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscritps, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. 39. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics is issued monthly as of January 1966. Regular subscription rate: $48.00 a year (6 Vols., 12 issues). Special rate: $24.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.
<table>
<thead>
<tr>
<th>Title</th>
<th>Authors</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rational Whitehead products and a spectral sequence of Quillen</td>
<td>Christopher Allday</td>
<td>313</td>
</tr>
<tr>
<td>Attaching Hurewicz fibrations with fiber preserving maps</td>
<td>James Edward Arnold, Jr.</td>
<td>325</td>
</tr>
<tr>
<td>Radial averaging transformations with various metrics</td>
<td>Catherine Bandle and Moshe Marcus</td>
<td>337</td>
</tr>
<tr>
<td>A proof of the lower bound conjecture for convex polytopes</td>
<td>David Wilmot Barnette</td>
<td>349</td>
</tr>
<tr>
<td>Simple extensions of measures and the preservation of regularity</td>
<td>Louis Harvey Blake</td>
<td>355</td>
</tr>
<tr>
<td>New proofs of Bing’s approximation theorems for surfaces</td>
<td>James W. Cannon</td>
<td>361</td>
</tr>
<tr>
<td>On realizing HNN groups in 3-manifolds</td>
<td>C. D. Feustel and Robert John Gregorac</td>
<td>381</td>
</tr>
<tr>
<td>Iversen’s theorem and fiber algebras</td>
<td>Theodore William Gamelin</td>
<td>389</td>
</tr>
<tr>
<td>The total space of universal fibrations</td>
<td>Daniel H. Gottlieb</td>
<td>415</td>
</tr>
<tr>
<td>Integrability theorems for power series expansions of two variables</td>
<td>Yoshimitsu Hasegawa</td>
<td>419</td>
</tr>
<tr>
<td>Length of period simple continued fraction expansion of \sqrt{d}</td>
<td>Dean Robert Hickerson</td>
<td>429</td>
</tr>
<tr>
<td>The spectra of endomorphisms of the disc algebra</td>
<td>Herbert Meyer Kamowitz</td>
<td>433</td>
</tr>
<tr>
<td>Boundedly holomorphic convex domains</td>
<td>Dong S. Kim</td>
<td>441</td>
</tr>
<tr>
<td>Integral operators on L_p-spaces</td>
<td>Daniel Ralph Lewis</td>
<td>451</td>
</tr>
<tr>
<td>Fields of topological spaces</td>
<td>John Eldon Mack</td>
<td>457</td>
</tr>
<tr>
<td>On a problem of completion in bornology</td>
<td>V. B. Moscatelli</td>
<td>467</td>
</tr>
<tr>
<td>Proximity convergence structures</td>
<td>Ellen Elizabeth Reed</td>
<td>471</td>
</tr>
<tr>
<td>Dual spaces of certain vector sequence spaces</td>
<td>Ronald C. Rosier</td>
<td>487</td>
</tr>
<tr>
<td>Absolutely torsion-free rings</td>
<td>Robert A. Rubin</td>
<td>503</td>
</tr>
<tr>
<td>Radial quasiharmonic functions</td>
<td>Leo Sario and Cecilia Wang</td>
<td>515</td>
</tr>
<tr>
<td>Peano models with many generic classes</td>
<td>James Henry Schmerl</td>
<td>523</td>
</tr>
<tr>
<td>The {}F-depth of an {}F-projector</td>
<td>H. J. Schmidt</td>
<td>537</td>
</tr>
<tr>
<td>Strong quasi-convexity</td>
<td>Edward Silverman</td>
<td>549</td>
</tr>
<tr>
<td>Uniform crossnorms</td>
<td>Barry Simon</td>
<td>555</td>
</tr>
<tr>
<td>(KE)-domains</td>
<td>Surjeet Singh</td>
<td>561</td>
</tr>
<tr>
<td>Starlike and convex maps in Banach spaces</td>
<td>Ted Joe Suffridge</td>
<td>575</td>
</tr>
<tr>
<td>C-embedded {}Sigma-spaces</td>
<td>Milton Don Ulmer</td>
<td>591</td>
</tr>
<tr>
<td>Conductor, projectivity and injectivity</td>
<td>Wolmer Vasconcelos</td>
<td>603</td>
</tr>
<tr>
<td>On some generalizations of Meier’s theorems</td>
<td>Hidenobu Yoshida</td>
<td>609</td>
</tr>
</tbody>
</table>