DUAL SPACES OF CERTAIN VECTOR SEQUENCE SPACES

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This article is an investigation of certain spaces of sequences with values in a locally convex space analogous to the generalized sequence spaces introduced by Pietsch in his monograph Verallgemeinerte Volkommene Folgenräume (Akademie-Verlag, Berlin, 1962). Pietsch combines a perfect sequence space $A$ and a locally convex space $E$ to obtain the space $A(E)$ of all $E$ valued sequences $x = (x_n)$ such that the scalar sequence $\langle \alpha, x_n \rangle$ is in $A$ for every $\alpha \in E'$. Define $A(E)$ to be the space of all $E$ valued sequences $x = (x_n)$ such that the scalar sequence $\langle p(x_n) \rangle$ is in $A$ for every continuous seminorm $p$ on $E$. The spaces $A(E)$ and $A(E)$ are topologized using the topology of $E$ and a certain collection $\mathcal{M}$ of bounded subsets of $A^*$, the $\alpha$ — dual of $A$.

The criteria for bounded sets, compact sets, and completeness are similar for both spaces. The significant difference lies in the duality theory. The dual of $A(E),\mathcal{M}$ is difficult to represent, but the dual of $A(E),\mathcal{M}$ is shown to be easily representable for general $A$ and $E$. For many special cases of $A$ and $E$ the dual of $A(E),\mathcal{M}$ is of the form $A^*(E')$ where $A^*$ is the $\alpha$ — dual of $A$ and $E'$ is the strong dual of $E$.

We begin by recalling basic definitions and elementary facts about sequence spaces and establishing some notation. After defining the space $[A(E),\mathcal{M}]$ and deriving some elementary properties, we proceed to a description of its dual space. We show that the notion of a “fundamentally $A$-bounded” space $E$ provides sufficient conditions for the duality relationship $A(E)' = A^*(E)$. We next show that there are large classes of $A$ and $E$ satisfying these conditions and we conclude by applying our results to the case $A = l^p$ obtain, for example, that the strong dual of $l^p(E)$ is $l^q(E')$ for $E$ a normed, Frechet, or $(DF)$-space, $1 \leq p < \infty$, $p^{-1} + q^{-1} = 1$.

I would like to thank Professor G. M. Köthe for his encouragement during the preparation of this work.

2. Definitions and notations. A sequence space $A$ is a vector space of real or complex sequences with the usual coordinatewise operations. To each sequence space $A$ there corresponds another sequence space $A^*$, called the $\alpha$ — dual of of $A$, consisting of all $\alpha = (\alpha_n)$, such that the scalar products $\langle \alpha, \beta \rangle = \sum \alpha_n \beta_n$ converge absolutely, that is $\sum |\alpha_n \beta_n| < \infty$, for all $\beta$ in $A$. Letting $A^{\alpha^*}$ denote the $\alpha$ — dual
\(A^\ast\), we have \(A \subset A^{\ast\ast}\). If \(A^{\ast\ast} = A\), then \(A\) is called a perfect sequence space.

Every perfect sequence space \(A\) satisfies \(\phi \subset A \subset \omega\), where \(\phi\) is the space of all sequences with only a finite number of nonzero coordinates and \(\omega\) is the space of all scalar sequences. Henceforth we shall consider only perfect spaces \(A\).

A subset \(B\) of \(A\) is called bounded if for every \(\alpha\) in \(A^\ast\) there exists a positive constant \(p\) such that \(\sum |\alpha_n\beta_n| \leq p\) for all \(\beta\) in \(B\). A subset \(M\) of \(A\) is called normal if whenever \(M\) contains \(\alpha\) it also contains all \(\beta\) satisfying \(|\beta_n| \leq |\alpha_n|\) for all \(n\). The normal hull \(N(M)\) of a set \(M\) is the set of all sequences \(\beta\) such that \(|\beta_n| \leq |\alpha_n|\) for all \(n\), for some \(\alpha\) in \(M\). A simple consequence of these definitions is that the normal hull of a bounded set is bounded. Also every perfect sequence space is normal.

The bilinear form \(\langle \alpha, \beta \rangle = \sum \alpha_n\beta_n\) on \(A^\ast \times A\) places \(A^\ast\) and \(A\) in duality with each other. If \(M\) is any bounded subset of \(A^\ast\), then \(M^0 = \{\beta \in A | \langle \alpha, \beta \rangle = |\sum \alpha_n\beta_n| \leq 1\ \text{for all} \ \alpha \in M\}\) is an absorbing absolutely convex subset of \(A\). A family \(\mathcal{M}\), consisting of bounded subsets of \(A^\ast\), is called a normal topologizing system for \(A\) if \(\mathcal{M}\) has the following properties: (i) if \(M_1, M_2 \in \mathcal{M}\), then there exists \(M \in \mathcal{M}\) such that \(M_1 \cup M_2 \subset M\). (ii) if \(M \in \mathcal{M}\) and \(\rho > 0\), then \(\rho M \in \mathcal{M}\). (iii) if \(\alpha \in A^\ast\), then \(\alpha \in M\) for some \(M \in \mathcal{M}\). (iv) the normal hull of every set in \(\mathcal{M}\) is in \(\mathcal{M}\).

(1) If \(\mathcal{M}\) is a normal topologizing system for \(A\), then the collection of all \(M^0, M \in \mathcal{M}\), forms a neighborhood base at 0 for a locally convex topology on \(A\). A base of seminorms for this \(\mathcal{M}\)-topology on \(A\) is given by the seminorms

\[
p_M^0(\beta) = \sup \{|\sum \alpha_n\beta_n| | \alpha \in M\}
\]

where \(M\) ranges over the normal sets in \(\mathcal{M}\).

It is the normality of \(M\) that allows the absolute value to be brought inside the summation above.

The two extreme cases of \(\mathcal{M}\) are the class \(\mathcal{B} = \mathcal{B}(A^\ast)\) consisting of all normal bounded subsets of \(A^\ast\) and the class \(\mathcal{N} = \mathcal{N}(A^\ast)\) consisting of all normal hulls \(N(\alpha)\) of single elements of \(A^\ast\). The \(\mathcal{B}\)-topology on \(A\) is the so called strong or \(T_\delta(A^\ast)\)-topology on \(A\) and the \(\mathcal{N}\)-topology on \(A\) is the normal topology on \(A\) in the sense of Köthe, [1. §30]. Note that we always have \(\mathcal{N} \subset \mathcal{M} \subset \mathcal{B}\).

We shall need the following result due to Pietsch [2. Satz 1.4].

(2) A subset \(A\) of \(A\) is bounded if, and only if, it is bounded for
Let $\alpha$ be any scalar sequence. We denote by $\alpha(\leq i)$ the $i$th finite section of $\alpha$, that is the sequence with coordinates $\alpha_n$ for $n = 1, 2, \ldots, i$ and 0 for $n > i$. $\alpha(\leq i) = (\alpha_i, \alpha_{i+1}, \ldots, \alpha_n, 0, \ldots)$. Now let $\Lambda_\infty$ denote $\Lambda$ equipped with an $\mathcal{M}$-topology and define $[\Lambda_\infty]$ to be that subspace of $\Lambda_\infty$ consisting of all sequences $\alpha$ which are the $\mathcal{M}$-limits of their finite sections.

(3) For any normal topologizing system $\mathcal{M}$, $\Lambda_\infty$ is complete. $[\Lambda_\infty]$ is a closed subspace of $\Lambda_\infty$ and hence also complete.

(4) (a) $[\Lambda_\infty] = \Lambda_\infty$ for every perfect space $\Lambda$.
(b) If $\Lambda_\infty$ is reflexive, then $[\Lambda_\infty] = \Lambda_\infty$.

The proof of (3) is in Pietsch [2. Satz 1.13, 1.14]. The proofs of (4) are in Köthe [1. § 30.5(8) and § 30.7(1), (5)].

Our terminology for locally convex spaces will be that of Köthe [1]. $E$ will always denote a locally convex Hausdorff space. $E$ has a fundamental system of absolutely convex closed neighborhoods of zero which we denote by $\mathcal{U}(E)$. For every $U \in \mathcal{U}(E)$ there is a continuous seminorm on $E$ denoted by $p_U$ and defined by the formula

$$p_U(x) = \sup \{|\langle u, x \rangle| | u \in U^\circ\}.$$ 

We shall always consider $E'$, the topological dual of $E$, to be equipped with the strong topology, that is, the topology defined by the neighborhoods $B^\circ$ or seminorms

$$p_B^\circ(u) = \sup \{|\langle u, x \rangle| | x \in B\}$$

where $B$ ranges over the bounded subsets of $E$.

Let $U \in \mathcal{U}(E)$ and $p_U$ be the corresponding seminorm. Let $N(U)$ denote the kernel of $p_U$ and let $E_U = E/N(U)$ be the normed quotient space formed by equipping $E/N(U)$ with the quotient norm induced by $p_U$. Dually, let $B$ be a closed absolutely convex bounded subset of $E$ and let $E_B = \bigcup_{n=1}^{\infty} nB$. Then $E_B$ is a linear subspace of $E$ and the Minkowski functional $q_B$ of $B$ is a norm on $E_B$. In particular we may perform these constructions in the dual space $E'$. If $B$ is bounded in $E$ then $B^\circ$ is an absolutely convex closed neighborhood of $0$ in $E'$ and we can form the quotient space $E_B^\circ$ which is a normed space with norm $p_B^\circ(a) = \sup \{|\langle a, x \rangle| | x \in B\}$. Dually if $U \in \mathcal{U}(E)$ then $U^\circ$ is an absolutely convex closed bounded (weakly compact) subset of $E'$ and we can form the subspace $E_U^\circ$ which is a $(B)$-space with norm $q_U^\circ(a) = \sup \{|\langle a, x \rangle| | x \in U\}$. The next proposition is an
easy consequence of these definitions

(5) (a) $E^o_U$ is a (B)-space with norm $q_U^o$ and can be identified with the dual space of $E_U$, $p_U$.

(b) $E_p$ is a norm space with norm $q_p$ and can be identified with a linear subspace of the dual space of $E_p^o$, $p_p^o$.

3. The space $A[E]_\mathcal{A}$. Let $A$ be a perfect sequence space and let $E$ be a locally convex space. $A[E]$ is the vector space of all $E$-valued sequences $x = (x_n)$ such that the sequence of scalars $p_U(x_n)$ is in $A$ for every $U \in \mathcal{U}(E)$. If $\mathcal{A}$ is a normal topologizing system for $A$, $A[E]_\mathcal{A}$ will denote $A[E]$ equipped with the locally convex Hausdorff $\mathcal{A}$-topology defined by the family of seminorms

\[ \pi_{M,U}(x) = \sup \{ \sum |\alpha_n| p_U(x_n) | \alpha \in M \} \text{ where } M \in \mathcal{A}, U \in \mathcal{U}(E). \]

The following two statements are simple consequences of these definitions.

(2) $I_n : A[E]_\mathcal{A} \rightarrow E$ defined by $I_n(x) = x_n$ is a continuous linear map for every $n = 1, 2, \ldots$.

(3) $I_U : A[E]_\mathcal{A} \rightarrow A$ defined by $I_U(x) = (p_U(x_n))$ is uniformly continuous for every $U \in \mathcal{U}(E)$.

A subset $A$ of $A[E]$ is called bounded if for every $\alpha \in A$ and $U \in \mathcal{U}(E)$ there exists a constant $\rho$ such that $\sum |\alpha_n| p_U(x_n) \leq \rho$ for all $x \in A$. For each $x \in A[E]$, define $N(x) = \{ (\lambda_n x_n) | |\lambda_n| \leq 1 \text{ all } n \}$. A subset $A$ of $A[E]$ is called normal if $x \in A$ implies $N(x) \subset A$. The set $N(A) = \bigcup_{x \in A} N(x)$ is called normal hull of $A$. We observe that $A[E]$ is itself normal since $A$ is normal.

(4) The following statements are equivalent for a subset $A$ of $A[E]$.

(a) $A$ is bounded.

(b) The normal hull of $A$ is bounded.

(c) $A$ is $\mathcal{A}$-bounded for some (every) $\mathcal{A}$-topology on $A[E]$.

(d) For every $U \in \mathcal{U}(E)$, $I_U(A)$ is bounded in $A$.

(e) For every $U \in \mathcal{U}(E)$, $I_U(A)$ is $\mathcal{A}$-bounded in $A$ for some (every) $\mathcal{A}$-topology on $A$.

Proof. The equivalences (a) $\iff$ (b), (a) $\iff$ (d), and (c) $\iff$ (e) follow directly from the definitions. (d) $\iff$ (e) is a consequence of 2.(2).

(5) If $E$ is complete, then $A[E]_\mathcal{A}$ is complete.

Proof. Let $x^{(\omega)}$ be a Cauchy net in $A[E]_\mathcal{A}$. Continuity of the linear map $I_n$ implies $x_n^{(\omega)}$ is a Cauchy net in $E$ for each fixed $n$ and
hence must converge to some $x_n$ in $E$. Uniform continuity of the map $I_U$ implies $(p_U(x_n^{(v)}))$ is a Cauchy net in $\mathcal{M}$ and hence must converge to some $x^{(v)} = (a_n^{(v)})$ in $\mathcal{M}$. Because of the coordinatewise convergence of $x^{(v)}$ to $x = (x_n)$ we have $p_U(x_n) = a_n^{(v)}$. Thus $(p_U(x_n))$ is in $\mathcal{M}$ and $x$ is therefore in $\mathcal{M}[E]$. Finally $x^{(v)}$ converges to $x$ in the $\mathcal{M}$-topology for if $\varepsilon > 0$ is given and $\nu_v$ is such that

$$\pi_{M,v}(x^{(v)} - x^{(v)}) = \sup \{ \Sigma |\alpha_n| p_U(x^{(v)} - x^{(v)}) | \alpha \in M \} < \varepsilon$$

for all $\nu, \mu \geq \nu_v$, then

$$\pi_{M,u}(x^{(v)} - x) \leq \varepsilon \text{ for all } \nu \geq \nu_v.$$ 

We denote by $x(\leq n) = (x_1, \ldots, x_n, 0 \cdots)$ the $n$th finite section of a sequence $x$ in $\mathcal{M}[E]$. Let $[\mathcal{M}[E],_\mathcal{M}]$ be the subspace of $\mathcal{M}[E],_\mathcal{M}$ consisting of all those $x$ in $\mathcal{M}[E],_\mathcal{M}$ which are the $\mathcal{M}$-limit of their finite sections; that is $[\mathcal{M}[E],_\mathcal{M}]$ consists of those $x$ for which $\pi_{M,v}(x - x(\leq n))$ converges to zero for every $M \in \mathcal{M}$ and $U \in \mathcal{V}(E).

(6) A sequence $x$ in $\mathcal{M}[E]$ is in $[\mathcal{M}[E],_\mathcal{M}]$ if, and only if, for every $U \in \mathcal{V}(E), I_U(x) = (p_U(x_n))$ is in $[\mathcal{M},_\mathcal{M}]$. In general $[\mathcal{M}[E],_\mathcal{M}]$ will be a proper subspace of $\mathcal{M}[E],_\mathcal{M}$, but using (6) and 2.(4) we obtain

(7) (a) $[\mathcal{M}[E],_\mathcal{M}] = \mathcal{M}[E],_\mathcal{M}$.

(b) If $\mathcal{M}$ is reflexive then $[\mathcal{M}[E],_\mathcal{M}] = \mathcal{M}[E],_\mathcal{M}$.

(8) $[\mathcal{M}[E],_\mathcal{M}]$ is a closed subspace of $\mathcal{M}[E],_\mathcal{M}$ and hence complete if $E$ is complete. 

Proof. If $x \in \mathcal{M}[E]$ is the limit of a net $x^{(v)}$ in $[\mathcal{M}[E],_\mathcal{M}]$, then for each $U \in \mathcal{V}(E)$ $I_U(x) = \lim_v I_U(x^{(v)})$ is in $[\mathcal{M},_\mathcal{M}]$ since $[\mathcal{M},_\mathcal{M}]$ is closed in $\mathcal{M}$. But then by (6) $x$ is in $[\mathcal{M}[E],_\mathcal{M}]$.

4. The dual space of $[\mathcal{M}[E],_\mathcal{M}]$. The $\alpha -$ dual of $\mathcal{M}[E]$, denoted $\mathcal{M}[E]^\alpha$, is the vector space of all $E'$-valued sequences $a = (a_n)$ such that $\Sigma |\langle a_n, x_n \rangle | < \infty$ for all $x = (x_n)$ in $\mathcal{M}[E]$.

(1) For every $a$ in $\mathcal{M}[E]^\alpha$ and for every bounded set $B$ in $E$, $(p_B(a_n))$ is in $\mathcal{M}^\ast$. That is $\mathcal{M}[E]^\alpha \subset \mathcal{M}^\ast[E']$. 

Proof. Let $B \in \mathcal{M}$ be arbitrary. For each $n$, there exists $x_n \in B$ such that

$$|\beta_n| p_B(a_n) = p_B(\beta_n a_n) \leq |\langle \beta_n a_n, x_n \rangle | + 2^{-n}.$$ 

Since $(x_n)$ is a bounded sequence in $E$, $(\beta_n x_n)$ is in $\mathcal{M}[E]$ and therefore
\[ \sum |\beta_n| p_{\mathcal{B}}(a_n) \leq \sum |\langle a_n, \beta_n x_n \rangle| + 2^{-n} < \infty. \]

Since \( \beta \in A \) was arbitrary, \( p_{\mathcal{B}}(a_n) \) is in \( A^* \).

(2) If \( x \in A[E] \) and \( \gamma \in e_c \) (\( e_c = \) scalar sequences convergent to zero), then \( \gamma x = (\gamma_n x_n) \) is in \( [A(E)]^\pi \).

Proof. It follows easily from the definition of the seminorms \( \pi_{M,U} \) that
\[ \pi_{M,U}(\gamma x(\geq i)) \leq \sup_{n > i} |\gamma_n| \pi_{M,U}(x) \]
and the right side converges to zero as \( i \to \infty \), so \( \gamma x \) is the limit of its finite sections.

(3) Every continuous linear form \( F \) on \( [A(E)]^\pi \) has a unique representation of the form
\[ \langle F, x \rangle = \langle a, x \rangle = \sum \langle a_n, x_n \rangle \]
with \( a = (a_n) \) in \( A[E]^z \).

Proof. Define linear forms on \( E \) by \( \langle a_n, x \rangle = \langle F, e_n x \rangle, x \in E, e_n \)
is the \( n \)th unit coordinate vector in \( A \). Continuity of \( F \) implies
\[ |\langle F, x \rangle| \leq \pi_{M,U}(x) \]
for some seminorm \( \pi_{M,U} \) and for every \( x \) in \( [A(E)]^\pi \). Since \( M \) is bounded, we have for each \( n, \rho_n = \sup \{|\alpha_n| | \alpha \in M \} < \infty \).
For every \( x \) in \( E \) we have therefore \( |\langle a_n, x \rangle| = |\langle F, e_n x \rangle| \leq \pi_{M,U}(e_n x) = \sup \{|\alpha_n| p_U(x) | \alpha \in M \} = \rho_n p_U(x) \)
and the continuity of \( a_n \) is established.

Clearly \( a = (a_n) \) represents \( F \) since \( \langle F, x \rangle = \lim_{i \to \infty} \sum_{n=1}^{i} \langle a_n, x_n \rangle = \sum \langle a_n, x_n \rangle. \)

Finally we show \( a \in A[E]^z \). Let \( x \in A[E]^z \) be arbitrary. For every \( \gamma \in e_c \), we can choose \( \lambda = (\lambda_n) \) with \( |\lambda_n| = 1 \) so that \( |\gamma_n \langle a_n, x_n \rangle| = \lambda_n \gamma_n \langle a_n, x_n \rangle \).
By (2), \( \lambda \gamma x = (\lambda_n \gamma_n x_n) \) is in \( [A(E)]^\pi \) and hence
\[ \sum |\gamma_n| |\langle a_n, x_n \rangle| = \sum \lambda_n \gamma_n \langle a_n, x_n \rangle = \langle F, \lambda \gamma x \rangle < \infty. \]
Since \( \gamma \in e_c \) was arbitrary, this shows that \( \sum |\langle a_n, x_n \rangle| < \infty \) and hence that \( a \in A[E]^z \).

REMARKS. Combining (1) and (3) yields \([A(E)]^\pi \subset A[E]^z \subset A^*[E^\ast]\).
Conditions sufficient for the equality of these spaces are given in the next section. We now proceed to an explicit characterization of \([A(E)]^\pi \).

(4) If \( a \in A[E]^z \) defines a continuous linear form on \( [A(E)]^\pi \), then there exists \( U \in \mathcal{Z}(E) \) such that \( a_n \in E^\ast_U \) for all \( n \) and moreover \( \langle q_U(a_n) \rangle \in A^* \).

Proof. Continuity of \( a \) implies \( |\langle a, x \rangle| \leq \pi_{M,U}(x) \) for some seminorm \( \pi_{M,U} \) and for all \( x \in [A(E)]^\pi \). As in the proof of (3), we obtain
that for every $n$, and for every $u \in E$, $|\langle a_n, u \rangle| \leq \rho_n p_U(u)$ from which it follows that $a_n \in E^*_U$ and $q_{U^0}(a_n) \leq \rho_n$. We must show that $(q_{U^0}(a_n)) \in \Lambda^e$.

Let $\beta \in A$ be arbitrary and set $\rho = \sup \{\sum |\alpha_n \beta_n| | \alpha \in M\}$. For each $n$, there exists $y_n \in U$ such that $q_{U^0}(\beta_n a_n) \leq \langle \beta_n a_n, y_n \rangle + 2^{-n}$. For each $i$, the finite section $\beta y(\leq i)$ of the sequence $(\beta_n y_n)$ is in $[A(E)]^*$ and therefore

$$\sum_{n=1}^i \langle \beta_n a_n, y_n \rangle = \langle a, \beta y(\leq i) \rangle \leq \pi_{U, U}(\beta y(\leq i))$$

$$= \sup \left\{ \sum_{n=1}^i |\alpha_n| p_U(\beta_n y_n) | \alpha \in M \right\}$$

$$\leq \sup \left\{ \sum_{n=1}^i |\alpha_n \beta_n| | \alpha \in M \right\} \leq \rho.$$ 

Since $i$ was arbitrary, $\sum \langle \beta_n a_n, y_n \rangle < \infty$. It follows that $\sum |\beta_n| q_{U^0}(a_n) = \sum q_{U^0}(\beta_n a_n) < \infty$ and therefore that $(q_{U^0}(a_n)) \in \Lambda^e$ since $\beta \in A$ was arbitrary.

(5) The dual space of $[A(E)]^*$ is the space of all $E'$-valued sequences $a = (a_n)$ which have a representation of the form $a = \alpha u = (\alpha_n u_n)$ with $\alpha \in \Lambda^e$ and $(u_n)$ an equicontinuous sequence in $E'$.

Proof. If we set $a_n = q_{U^0}(a_n)$ and $u_n = (1/\alpha_n) a_n$, $(u_n = 0$ if $\alpha_n = 0)$, then (4) says that every element in the dual of $[A(E)]^*$ has the given form.

Conversely, if $a = \alpha u = (\alpha_n u_n)$ with $\alpha \in \Lambda^e$ and $(u_n)$ equicontinuous, then, choosing $M$ with $\alpha \in M$ and $U \in \mathcal{U}(E)$ with $(u_n) \subset U^0$, we obtain

$$|\langle a, x \rangle| \leq \sum |\alpha_n| |\langle u_n, x_n \rangle| \leq \pi_{M, U}(x)$$

for all $x$ in $[A(E)]^*$ and hence $a$ is continuous.

Using the methods of the proofs of (4) and (5), one can show

(6) The equicontinuous subsets of $[A(E)]^*$ are the sets of the form

$$\{\alpha u | \alpha = (\alpha_n) \in M, u = (u_n) \subset U^0\}$$

where $M \in \mathcal{M}$ and $U \in \mathcal{U}(E)$.

5. Fundamentally $\Lambda$-bounded spaces. In the previous section, we saw that $[A(E)]^* \subset \Lambda(E)^e \subset \Lambda^e(E')$. In this section we establish conditions sufficient for the equality $\Lambda(E)^e = \Lambda^e(E')$ and for the more interesting equality $[A(E)]^* = \Lambda^e(E')$. We also give conditions which insure the strong dual of $[A(E)]^*$ is $\Lambda^e(E')^*$. Finally we give suffi-
cient conditions for $A(E)_\infty$ to be reflexive.

The important concept in all these conditions is that of a "fundamentally $A$-bounded" space $E$. A locally convex space $E$ is fundamentally $A$-bounded if all the bounded subsets of $A(E)$ can be obtained in a natural way from the bounded subsets of $A$ and $E$.

Let $R$ be a normal bounded subset of $A$ and let $B$ be a closed absolutely convex bounded subset of $E$. Define $[R, B] = \{x \in A(E) | x_n \in E_B$ and $(q_B(x_n)) \in R\}$. The following are simple consequences of this definition.

1. $[R, B]$ is a bounded subset of $A(E)$.
2. If $R \subseteq R'$ and $B \subseteq B'$, then $[R, B] \subseteq [R', B']$.

Let $V$ be a vector space in which the notion of a bounded set has been defined. A collection $\mathcal{B}$ of subsets of $V$ is called a fundamental system of bounded sets for $V$ if every bounded set in $V$ is contained in some set in $\mathcal{B}$.

We shall say that a locally convex space $E$ is fundamentally $A$-bounded if the collection of all sets of the form $[R, B]$ form a fundamental system of bounded sets for $A(E)$, where $R$ and $B$ run through a fundamental system of bounded sets for $A$ and $E$ respectively.

3. If $E$ is fundamentally $A$-bounded, then $A(E)^* = A^*\{E^*\}$.

Proof. We need only show the inclusion $A^*\{E^*\} \subseteq A(E)^*$. Let $a \in A^*\{E^*\}$ and let $x \in A(E)$. Then there exist $R$ and $B$ with $x \in [R, B]$ and hence $(q_B(x_n)) \in A$. But $(p_B(a_n)) \in A^*$, and therefore

$$\sum |\langle a_n, x_n \rangle| \leq \sum p_B(a_n) q_B(x_n) < \infty.$$ 

Since $x$ was arbitrary, this shows $a \in A(E)^*$.

Recall that a locally convex space $E$ is called $\sigma$-infrabarreled if every countable strongly bounded subset of $E'$ is equicontinuous. Clearly every infrabarreled space is $\sigma$-infrabarreled.

The next theorem is the main result of this section.

4. Let $E$ be a $\sigma$-infrabarreled space and let $A$ be a perfect sequence space.
   a. If $E'$ is fundamentally $A^*$-bounded, then the dual of $[A(E)_\infty]$ is $A^*\{E^*\}$.
   b. If moreover $E$ is fundamentally $A$-bounded, then the strong dual of $[A(E)_\infty]$ is $A^*\{E^*\}_{\sigma}$.

Proof. (a) We need only show the inclusion $A^*\{E^*\} \subseteq [A(E)_\infty]$. Let $a \in A^*\{E^*\}$. By hypothesis there exists a bounded set $D$ in $E'$ such that $(q_D(a_n)) \in A^*$. For each $n$, set $u_n = q_D(a_n)^{-1}a_n$ ($u_n = 0$ if $q_D(a_n) = 0$).
Then \( u_n \) is in \( D \) for each \( n \). Since \( E \) is \( \sigma \)-infrabarreled, \( \{u_n \mid n = 1, 2, \cdots \} \) is equicontinuous and hence \( a = (a_n) = (q_p(a_n)u_n) \) is in \( [A(E)_{\sigma}]' \) by 4.(5).

(b) If \( E \) is fundamentally \( \Lambda \)-bounded, then the strong topology on \( [A(E)_{\sigma}]' = \Lambda^*(E') \) is defined by the seminorms

\[
\sigma_{(R, B)}(a) = \sup |\sum \langle a_n, x_n \rangle | = \sup |\sum \langle a_n, x_n \rangle |
\]

where the sup is taken over \( x \) in \([R, B] \cap [A(E)_{\sigma}]\). The topology on \( \Lambda^*(E')_{\sigma} \) is defined by the seminorms

\[
\pi_{R, B_0}(a) = \sup \{ \sum |\alpha_n| p_{B_0}(a_n) \mid \alpha \in R \} .
\]

In both cases, \( R \) ranges over all normal bounded subsets of \( \Lambda \) and \( B \) over all absolutely convex bounded subsets of \( E \). We show these seminorms coincide.

One inequality is easy:

\[
\sigma_{(R, B)}(a) = \sup \{ \sum |\alpha_n| p_{B_0}(a_n) \mid \alpha \in R \} \leq \pi_{R, B_0}(a) .
\]

Now the reverse inequality. Let \( a \in \Lambda^*(E') \) and let \( \varepsilon > 0 \). By definition of \( \pi_{R, B_0} \) there exists \( \alpha \in R \) with \( \pi_{R, B_0}(a) \leq \varepsilon + \sum |\alpha_n| p_{B_0}(a_n) \). For each \( n \) there exists \( y_n \in B \) such that \( p_{B_0}(a_n) \leq |\langle a_n, y_n \rangle| + \varepsilon 2^{-n} |\alpha_n|^{-1} \). (If \( \alpha_n \) or \( a_n \) is zero, let \( y_n \) be any element in \( B \).) Let \( z_n = \alpha_n y_n \). Then \( z \in [R, B] \) and

\[
\pi_{R, B_0}(a) \leq \varepsilon + \sum |\alpha_n| p_{B_0}(a_n) \\
\leq \varepsilon + \sum |\alpha_n| |\langle a_n, y_n \rangle| + \varepsilon 2^{-n} \\
= 2\varepsilon + \sum |\langle a_n, z_n \rangle| \\
= 2\varepsilon + \sup \{ \sum |\gamma_n| \mid |\langle a_n, \gamma_n \rangle| \mid \gamma \in c_0, ||\gamma||_\infty \leq 1 \} \\
= 2\varepsilon + \sup \{ \sum |\langle a_n, \gamma_n z_n \rangle| \mid \gamma \in c_0, ||\gamma||_\infty \leq 1 \} \\
\leq 2\varepsilon + \sigma_{(R, B)}(a) .
\]

The last inequality follows from the fact that \( \gamma z \in [R, B] \cap [A(E)_{\sigma}] \). Since \( \varepsilon \) was arbitrary the theorem is proved.

(5) Let \( E \) be locally convex and let \( \Lambda \) be a perfect sequence space such that

(i) \( \Lambda_{\sigma} \) and \( E \) are both reflexive, and

(ii) \( E \) is fundamentally \( \Lambda \)-bounded and \( E' \) is fundamentally \( \Lambda^* \)-bounded. Then both \( \Lambda(E)_{\sigma} \) and its strong dual \( \Lambda^*(E')_{\sigma} \) are reflexive.

**Proof.** Since \( E \) is reflexive, both \( E \) and \( E' \) are \( \sigma \)-infrabarreled.
Also $E''$ is fundamentally $\Lambda^\ast$-bounded since $E = E''$ and $\Lambda = \Lambda^\ast$. Since $\Lambda_\varphi$ is reflexive, so also is its strong dual $\Lambda^\ast_\varphi$. It follows from 2.(7)(b) that $[\Lambda(E)_\varphi] = \Lambda(E)_\varphi$ and $[\Lambda^\ast(E')_\varphi] = \Lambda^\ast(E')_\varphi$. This theorem now follows by applying (4) twice, first to $[\Lambda(E)_\varphi]$ and then to $[\Lambda^\ast(E')_\varphi]$.

6. Examples of fundamentally $\Lambda$-bounded spaces. In this section, we show that there exist nontrivial classes of spaces $E$ and $\Lambda$ for which $E$ is fundamentally $\Lambda$-bounded.

(1) Every normed space $E$ is fundamentally $\Lambda$-bounded for every perfect sequence space $\Lambda$.

Proof. Let $\Lambda$ be any bounded subset of $\Lambda(E)$, and let $B$ denote the unit ball of $E$. Then $I_\Lambda(A) = \{|\|x_n\|\mid x \in A\}$ is a bounded subset of $\Lambda$ and hence contained in some normal bounded set $R$. Thus $\Lambda \subset [R, B]$.

(2) (a) If $E$ is normed and if $\Lambda$ is any perfect sequence space, then the strong dual of $[\Lambda(E)_\varphi]$ is $\Lambda^\ast(E')_\varphi$.

(b) If $E$ is reflexive $(B)$-space and if $\Lambda_\varphi$ is reflexive, then $\Lambda(E)_\varphi$ and its strong dual $\Lambda^\ast(E')_\varphi$ are reflexive.

This follows from (1) above and 5.(4), (5).

The next lemma is due to Pietsch [3. Satz 1.5.8].

(3) Every metrizable locally convex space $E$ is fundamentally $\ell\ast$-bounded.

We shall also use the following well-known fact. (See e.g. [1. §29.1.(5)].)

(4) If $E$ is a metrizable locally convex space, and if $B_k$ is a sequence of bounded subsets of $E$, then there always exist positive scalars $\lambda_k$ such that $B = \bigcup_{k=1}^\infty \lambda_k B_k$ is also bounded.

(5) Let $\Lambda$ and $\Lambda^\ast$ be perfect sequences spaces which are $\alpha$ — dual to one another. Suppose $\Lambda^\ast$ has a countable fundamental system of bounded sets $N_1 \subset N_2 \subset N_3 \subset \cdots$. Then:

(a) Every metrizable locally convex space is fundamentally $\Lambda$-bounded.

(b) Every $(DF)$-space is fundamentally $\Lambda^\ast$-bounded.

See [1. §29], for example, for the definition and basic properties of $(DF)$-spaces.

Proof. (a) Let $E$ be metrizable and let $\Lambda$ be a bounded subset of $\Lambda(E)$. Then by $\Lambda$ is $\mathcal{B}$-bounded in $\Lambda(E)$. Thus for each $k$ and
each $U \in \mathcal{V}(E)$, there exists a constant $\rho_{k,U}$ such that for all $x \in A$,

$$\pi_{N_k,U}(x) = \sup \{ \sum |\alpha_n| p_U(x_n) | \alpha \in N_k \} \leq \rho_{k,U}.$$ 

This implies that the set $A_k = \{ \alpha x = (\alpha_n x_n) | \alpha \in N_k, x \in A \}$ is a bounded subset of $l^1(E)$. By Lemma (3), there exists a bounded set $B_k$ in $E$ such that $A_k \subseteq [R_i, B_k]$ where $R_i$ denotes the unit ball of $U_i$, or equivalently

$$(*) \quad \sum |\alpha_n| q_{B_k}(x_n) = \sum q_{B_k}(\alpha_n x_n) = \| (q_{B_k}(\alpha_n x_n)) \|_{l^1} \leq 1$$

for all $\alpha \in N_k, x \in A$. By (4) there exist positive scalars $\lambda_k$ such that $B = \bigcup_{k=1}^{r} \lambda_k B_k$ is bounded. Since $B_k \subseteq \lambda_k^{-1} B$ we have for all $x \in E_{N_k}$ that $q_{B}(x) \leq \lambda_k^{-1} q_{B_k}(x)$. Thus for every $k$ and for all $x \in A$, we have

$$p_{N_k}(q_{B}(x_n)) = \sup \{ \sum |\alpha_n| q_{B}(x_n) | \alpha \in N_k \}$$

$$\leq \sup \{ \sum |\alpha_n| \lambda_k q_{B_k}(x_n) | \alpha \in N_k \}$$

$$\leq \lambda_k$$

by (*). This implies that the set $\{(q_{B}(x_n)) | x \in A \}$ is $\mathcal{B}$-bounded and hence bounded in $A$, and is therefore contained in some normal bounded subset $R$ of $A$. Thus $A \subseteq [R, B]$ and (a) is proved.

(b) Let $E$ be a $(DF)$-space. Then $E$ has a countable fundamental system of bounded sets $B_1 \subseteq B_2 \subseteq B_3 \subseteq \cdots$.

Suppose $E$ is not fundamentally $A^s$-bounded, then there exists a bounded subset $A$ in $A^s(E)$ such that $A$ is not contained in any of the sets $[N_k, B_k], k = 1, 2, \cdots$. We show this leads to a contradiction.

For every index $k$, $A$ not a subset of $[N_k, B_k]$ implies that there exists $x^{(k)} \in A$ such that $(q_{B_k}(x^{(k)})) \in N_k$. Thus there exists $\beta^{(k)} \in N_k^{*}$ such that $\sum \beta^{(k)} q_{B_k}(x^{(k)}) > 1$. In fact for each $k$, there exists a finite set $\{u^{(k)}_n \} \subseteq B_k^{*}$, $n = 1, 2, \cdots, f_k$, such that

$$\sum_{n=1}^{f_k} \beta^{(k)}_n |\langle u^{(k)}_n, x^{(k)} \rangle | > 1.$$ 

Let $G = \{u^{(k)}_n | k = 1, 2, \cdots, \text{ and } n = 1, 2, \cdots, f_k \}$. Then $G$ is a countable subset of $E'$. If $G$ is strongly bounded in $E'$, then $G$ is equicontinuous since $E$ is a $(DF)$-space. We show $G$ is strongly bounded. Fix $m$. Since $\{u^{(k)}_n | k = 1, 2, \cdots, m, n = 1, 2, \cdots, f_k \}$ is finite, there exists a positive constant $\rho_m \geq 1$ with $u^{(k)}_n \in \rho_m B_m$ for $k = 1, \cdots, m$ and $n = 1, \cdots, f_k$, since $B_m$ is an absorbing subset of $E'$. For $k > m$, $B_k \supseteq B_m$ and hence $B_k \subseteq B_m$ so $u^{(k)}_n \in B_k \subseteq B_m$ for all $k > m$ and $n = 1, 2, \cdots, f_k$. Thus for every $m$, there exists a positive constant $\rho_m$ with $G \subseteq \rho_m B_m$. The sets $B_1 \supset B_2 \supset B_3 \cdots$ form a neighborhood base for the strong topology on $E'$, so $G$ is strongly bounded and hence equi-continuous.
Let $U \in \mathcal{Z}(E)$ be such that $G \subset U^\circ$. Since $A$ is bounded in $A^\ast(E)$, the set $\{p_u(x_n) | x \in A\}$ is bounded in $A^\ast$ and hence contained in some $N_k$. Since $\beta^{(k)} \in N_k$, this implies $\sum \beta_n^{(k)} p_u(x_n) \leq 1$ for all $x \in A$. But taking $x = x^{(k)}$, we obtain $\sum \beta_n^{(k)} p_u(x_n^{(k)}) > \sum_{n=1}^k \beta_n^{(k)} |\langle u^{(k)}_n, x^{(k)}_n \rangle| > 1$ which is a contradiction.

As in theorem (5), let $A$ and $A^\ast$ be $\alpha$-dual perfect sequence spaces such that $A^\ast$ has a countable fundamental system of bounded sets. The results of (5) cannot be improved to include either of the following assertions.

(a) Every $(DF)$-space is fundamentally $A$-bounded.
(b) Every metrizable locally convex space is fundamentally $A^\ast$-bounded.

Counterexamples are provided by (9) and (8) below.

Recall that $\omega$ is the space of all scalar sequences and $\phi$ is the space of all scalar sequences with only finitely many nonzero coordinates. $\phi$ and $\omega$ are perfect and $\alpha$-dual to each other. Moreover $\phi$ has a countable fundamental system of bounded sets $N_k \subset N_k \subset \ldots$, where $N_k = \{\alpha \in \phi | |\alpha_n| \leq k$ if $n \leq k$ and $\alpha_n = 0$ if $n > k\}$. The following lemma is due to Pietsch [2, Satz 3.19].

(6) Let $E$ be a metrizable locally convex space which has no continuous norm. Then there exists $x \in \phi(E)$ such that for every index $n$, $x_n \neq 0$.

Proof. Let $p_1 \leq p_2 \leq \ldots$ be a fundamental system of seminorms for $E$. No $p_k$ is a norm. Thus for each integer $k$ there exists $x_k \in E$ with $x_k \neq 0$ but $p_k(x_k) = 0$. Set $x = (x_n)$. Fix $k$. For all $n \geq k$ we have $p_n(x_n) = 0$ but $p_k \leq p_n$, so $p_n(x_n) = 0$ for all $n \geq k$. Thus $(p_k(x_n)) \in \phi$ for each seminorm $p_k$.

(7) For any locally convex space $E$, $\omega(E)$ is the space of all $E$-valued sequences.

(8) There exist metrizable locally convex spaces $E$ such that $E$ is not fundamentally $\phi$-bounded.

Proof. Let $E$ be a metrizable space with no continuous norm. By (6) there exists $x \in \phi(E)$ with $x_n \neq 0$ for all $n$. Therefore there exist $a_n \in E'$ with $\langle a_n, x_n \rangle = 1$. But by (7), $a = (a_n) \in \omega(E')$. Since $\langle a, x \rangle = \sum \langle a_n, x_n \rangle = \infty$, we conclude $\phi(E') = \omega(E') = \phi^\ast(E')$. By 5.(3) this implies $E$ is not fundamentally $\phi$-bounded.

(9) There exist $(DF')$-spaces $E$ such that $E$ is not fundamentally $\omega$-bounded.
Proof. Let $E$ be a $(DF)$-space whose strong dual $E'$ is an $(F)$-space with no continuous norm. By (6) there exists $a \in \phi(E')$ such that $a_n \neq 0$ for all $n$. Let $x_n \in E$ be such that $\langle a_n, x_n \rangle = 1$. Then $x = (x_n) \in \omega(E)$ but $\langle a, x \rangle = \sum \langle a_n, x_n \rangle = \infty$ so we conclude $\omega(E) \neq \phi(E') = \omega^*(E')$. By 5.(3) this implies $E$ is not fundamentally $\omega$-bounded.

The space $\omega$ may be viewed as a topological product of countably many copies of the scalar field. With the product topology it is a $(F)$-space with no continuous norm. It is the strong dual of the $(DF)$-space $\phi$ viewed as a locally convex direct sum of countably many copies of the scalar field. Thus the examples in (8) and (9) can be made more explicit by taking $E = \omega$ in (8) and $E = \phi$ in (9).

7. The spaces $l^p(E)$ $1 \leq p \leq \infty$. It is well known that for $1 \leq p \leq \infty$ the $\alpha$—dual of $l^p$ is $l^q$ where $p^{-1} + q^{-1} = 1$. The bounded subsets of $l^p$ are easily seen to be the sets which are bounded in $l^p$-norm $||\alpha||_p = (\sum |\alpha_n|^p)^{1/p}$. Thus every $l^p$ space has a countable fundamental system of bounded sets consisting of positive integer multiples of the unit ball.

A sequence $x = (x_n)$ in a locally convex space $E$ is called absolutely $p$-summable, $1 \leq p < \infty$, if for every continuous seminorm $p_U$ on $E$, $\sum p_U(x_n)^p < \infty$.

1. $l^p(E)$, $1 \leq p < \infty$, is the vector space of all absolutely $p$-summable sequences in $E$. $l^\infty(E)$ is the vector space of all bounded sequences in $E$.

The seminorms defining the $\mathscr{B} = \mathscr{B}(l^p)$ topology on $l^p(E)$, $1 \leq p < \infty$, are given by

$$
\pi_{k,B,U}(x) = \sup \{ \sum |\alpha_n| p_U(x_n) \mid \alpha \in kB \}
= \sup \{ \sum |\alpha_n| p_{kB}^U(x_n) \mid \alpha \in B \}
= (\sum p_{kB}^U(x_n)^p)^{1/p}
$$

where $k$ is a positive integer, $B$ is the unit ball in $l^p$, and $U$ is an absolutely convex neighborhood of 0. Since $k^{-1}U$ is also such a neighborhood, we have

2. $1 \leq p < \infty$. A base of seminorms for $l^p(E)_\mathscr{B}$ is given by the family of seminorms

$$
\pi_{U}^f(x) = (\sum p_U(x_n)^p)^{1/p} \quad U \in \mathscr{B}(E) .
$$

A similar argument for the case $p = \infty$ yields

3. A base of seminorms for $l^\infty(E)_\mathscr{B}$ is given by the family of
semimnorms

\[ \pi^u(x) = \sup \{ p_U(x_n) | n = 1, 2, \cdots \} . \]

It follows that an element \( x \) in \( l^\infty(E) \) will be the limit of its finite sections if and only if \( p_U(x_n) \) converges to 0 for every \( U \in \mathcal{U}(E) \).

Clearly every element of \( l^p(E) \) is the limit of its finite sections.

(4) \[ [l^p(E)] = l^p(E), \text{ for } 1 \leq p < \infty \]

\[ [l^\infty(E)] = c_0(E) = \text{vector space of all sequences in } E \text{ converging to 0}. \]

We now show how the results of the previous sections can be applied to the duality theory of the \( l^p(E) \) spaces.

(5) Every metrizable locally convex space and every (DF)-space is fundamentally \( l^p \)-bounded for every \( p, 1 \leq p \leq \infty \).

Proof. Since every \( l^r, 1 \leq q \leq \infty \), has a countable fundamental system of bounded sets, and since \( (l^p)^* = l^q \) with \( p^{-1} + q^{-1} = 1 \), this result follows immediately from 6.(5).

(6) Let \( E \) be a metrizable locally convex space or a (DF)-space.

For \( 1 \leq p < \infty \), the strong dual of \( l^p(E) \) is \( l^q(E') \), and the strong dual of \( [l^\infty(E)] = c_0(E) \) is \( l^q(E') \).

Proof. This is a direct application of (5) above and 5.(4). (We are also using the facts that the dual of a metrizable space is a (DF)-space and the dual of a (DF')-space is metrizable.)

(7) If \( E \) is a reflexive (B)-, (F)-, or (DF)-space, then for \( 1 < p < \infty \), \( l^p(E) \) is a reflexive (B)-, (F)-, or (DF)-space respectively.

Proof. By (6) above and 5.(5), \( l^p(E) \) is reflexive. If \( E \) is a \( (B) \)- or \( (F) \)-space, then it is clear from the fact that the seminorms \( \pi^u, U \in \mathcal{U}(E) \), define the \( \mathcal{B} \)-topology on \( l^p(E) \), that \( l^p(E) \) is a \( (B) \)- or \( (F) \)-space respectively. If \( E \) is a reflexive (DF')-space, then \( E' \) is an (F')-space and \( l^p(E) \) as the strong dual of the (F')-space \( l^p(E') \) must be a (DF')-space.

References


Received April 18, 1972.

*Georgetown University*
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The Pacific Journal of Mathematics is issued monthly as of January 1966. Regular subscription rate: $48.00 a year (6 Vols., 12 issues). Special rate: $24.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

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