

Pacific Journal of Mathematics

RADIAL QUASIHARMONIC FUNCTIONS

LEO SARIO AND CECILIA WANG

RADIAL QUASIHARMONIC FUNCTIONS

LEO SARIO AND CECILIA WANG

A function s on a Riemannian manifold is called **quasi-harmonic** if it satisfies $\Delta s = 1$, where Δ is the Laplace-Beltrami operator $d\delta + \delta d$. Existence of quasiharmonic functions with various boundedness properties has thus far been investigated by means of useful implicit tests. We now ask: Can such functions be formed by direct construction, in a manner accessible to computation if need be?

1. We shall present our approach to the problem in the setup of a Riemannian N -ball

$$(1) \quad B_\alpha = \{r < 1 \mid ds\}$$

endowed with the generalized Poincaré metric

$$(2) \quad ds = \lambda(r) |dx|, \lambda(r) = (1 - r^2)^\alpha, \alpha \in \mathbf{R},$$

where $r = |x|$, $x = (x^1, \dots, x^N)$. In [16] we proved that there exist bounded quasiharmonic functions on B_α if and only if $\alpha \in (-1, 1/(N-2))$. We shall now show that this in turn is necessary and sufficient for the boundedness of an explicitly constructed function $s(r)$, given in No. 3 below. *Thus the boundedness of this single function characterizes the existence of bounded quasiharmonic functions on B_α .*

We shall call, for brevity, a function radial if it depends on r only. A simple consequence of our result is that there exist bounded radial quasiharmonic functions if and only if there exist bounded quasiharmonic functions.

We expect that our approach is extendable to other classes of quasiharmonic and biharmonic functions as well, and to other Riemannian manifolds which are invariant under rotation. In particular, *there exist negative radial quasiharmonic functions on every B_α .*

2. The proof of our main result will be divided into Lemmas 1-6. We start by formulating the equation:

LEMMA 1. *A function $s(r)$ satisfies*

$$(3) \quad \Delta s = 1$$

on B_α if and only if

$$(4) \quad s'' + \left(\frac{N-1}{r} - \frac{2(N-2)\alpha r}{1-r^2} \right) s' + (1-r^2)^{2\alpha} = 0.$$

Proof. The metric tensor (g_{ij}) is diagonal, with elements

$$\lambda^2, \lambda^2 r^2, \lambda^2 r^2 \varphi_1, \dots, \lambda^2 r^2 \varphi_{N-2},$$

where $\varphi_1, \dots, \varphi_{N-2}$ are functions of the coordinate angles $\theta^1, \dots, \theta^{N-1}$. We set $\varphi = (\varphi_1 \cdots \varphi_{N-2})^{1/2}$, and have $\sqrt{g} = \lambda^N r^{N-1} \varphi$, $g^{rr} = \lambda^{-2}$, and

$$\begin{aligned} \Delta s &= -\frac{1}{\sqrt{g}} \frac{\partial}{\partial r} (\sqrt{g} g^{rr} s') \\ &= -\lambda^{-2} \left[s'' + \left(\frac{N-1}{r} + \frac{(N-2)\lambda'}{\lambda} \right) s' \right], \end{aligned}$$

hence the lemma.

For convenience in later calculation, we rewrite (4) in the form:

$$(4') \quad \begin{aligned} r^2(1-r^2)s'' + r[(N-1)(1-r^2) \\ - 2(N-2)\alpha r^2]s' + r^2(1-r^2)^{2\alpha+1} = 0. \end{aligned}$$

3. We are ready to give the function $s(r)$ referred to in the introduction. Here and later \sum_m^n with $n < m$ will mean 0.

LEMMA 2. Equation (3) is satisfied by the function

$$(5) \quad s(r) = -\sum_{i=0}^{\infty} b_i r^{2i+2},$$

where

$$(6) \quad b_0 = \frac{1}{2N}$$

and for $i > 0$,

$$(7) \quad b_i = \frac{1}{2N} \prod_{j=1}^i p_j + \sum_{j=1}^{i-1} q_j \prod_{k=j+1}^i p_k + q_i,$$

with

$$(8) \quad p_i = \frac{2i[2i + (N-2)(2\alpha+1)]}{(2i+2)(2i+N)},$$

$$(9) \quad q_i = \frac{\prod_{j=1}^i (j-2\alpha-2)j^{-1}}{(2i+2)(2i+N)}.$$

Proof. Substitution of (5) into (4') gives

$$\begin{aligned} & -\sum_{i=0}^{\infty} (1-r^2)(2i+2)(2i+1)b_i r^{2i+2} - \sum_{i=0}^{\infty} (N-1)(2i+2)b_i r^{2i+2} \\ & + \sum_{i=0}^{\infty} [(N-1) + 2(N-2)\alpha](2i+2)b_i r^{2i+4} + r^2 \\ & + r^2 \sum_{i=1}^{\infty} \left(\prod_{j=1}^i \frac{j-2\alpha-2}{j} \right) r^{2i} = 0. \end{aligned}$$

On changing by unity the summation index in the coefficients of r^{2i+4} we obtain

$$\sum_{i=0}^{\infty} (2i + 2)(2i + N)b_i r^{2i+2} = \sum_{i=1}^{\infty} 2i[2i + (N - 2)(2\alpha + 1)]b_{i-1} r^{2i+2} + r^2 + \sum_{i=1}^{\infty} \left(\sum_{j=1}^i \frac{j - 2\alpha - 2}{j} \right) r^{2i+2} = 0 .$$

We equate the coefficient of r^2 to 0 and have (6). The coefficient of r^{2+2i} for $i > 0$ gives in notation (8), (9)

$$(10) \quad b_i = p_i b_{i-1} + q_i ,$$

which by induction yields (7).

4. We recall that we are only interested in $\alpha \in (-1, 1/(N - 2))$, and we shall at this point introduce the condition $\alpha < 1/(N - 2)$. To estimate b_i given by (7), we start with Πp_j . Let i_0 be any integer such that

$$(11) \quad i_0 \geq 1 - \alpha(N - 2) - \frac{N}{2} .$$

Further conditions on i_0 will be imposed in the course of our reasoning.

LEMMA 3. For $\alpha < 1/(N - 2)$ and $i > i_0$,

$$(12) \quad \prod_{j=i_0+1}^i p_j < \frac{i_0 + 1}{i + 1} \left(\frac{2i_0 + N + 2}{2i + N + 2} \right)^{1-\alpha(N-2)} .$$

Proof. In p_i , consider first the factor

$$\delta_i = \frac{2i + (N - 2)(2\alpha + 1)}{2i + N} = 1 - \frac{2[1 - \alpha(N - 2)]}{2i + N} .$$

For $\alpha < 1/(N - 2)$ and $i > i_0$, we have $0 < \delta_i < 1$ and

$$\log \delta_i < - \frac{2[1 - \alpha(N - 2)]}{2i + N} < 0 .$$

Therefore

$$\log \prod_{j=i_0+1}^i \delta_j < - 2[1 - \alpha(N - 2)] \int_{i_0+1}^{i+1} \frac{dx}{2x + N}$$

and

$$\prod_{j=i_0+1}^i \delta_j < \left(\frac{2i_0 + N + 2}{2i + N + 2} \right)^{1-\alpha(N-2)} .$$

In view of

$$p_i = \frac{i}{i+1} \delta_i,$$

the lemma follows.

5. To proceed with the estimation of b_i we now utilize also the condition $\alpha > -1$ and impose on i_0 the additional requirement

$$(13) \quad i_0 \geq 2(\alpha + 1).$$

In the sequel c will stand for a positive constant, not always the same.

LEMMA 4. For $\alpha \in (-1, 1/(N-2))$ and $i > i_0$,

$$(14) \quad |q_i| < \frac{c}{(2i+2)(2i+N)} \left(\frac{i_0+1}{i+1} \right)^{2(\alpha+1)}.$$

Proof. For $j > i_0$

$$0 < 1 - \frac{2(\alpha+1)}{j} < 1,$$

and therefore

$$\log \prod_{j=i_0+1}^i \left(1 - \frac{2(\alpha+1)}{j} \right) < -2(\alpha+1) \sum_{j=i_0+1}^i \frac{1}{j} < -2(\alpha+1) \int_{i_0+1}^{i+1} \frac{dx}{x}.$$

This gives

$$\prod_{j=i_0+1}^i \frac{j-2\alpha-2}{j} < \left(\frac{i_0+1}{i+1} \right)^{2(\alpha+1)},$$

hence (14).

6. We now come to the main step in estimating b_i . It will be necessary to consider separately the cases $\alpha \in (-1/N, 1/(N-2))$, $\alpha = -1/2$, and $\alpha \in (-1, 0) - \{-1/2\}$.

LEMMA 5. For $\alpha \in (-1/N, 1/(N-2))$, and $i > i_0$,

$$(15) \quad |b_i| < c \left(\frac{1}{i} \right)^{2-\alpha(N-2)} + d \left(\frac{1}{i} \right)^{(3/2)-(1/2)\alpha(N-2)} + e \left(\frac{1}{i} \right)^{2(\alpha+2)},$$

where c, d, e are positive constants.

Proof. By (7)

$$(16) \quad b_i = b_{i_0} \prod_{j=i_0+1}^i p_j + \sum_{j=i_0+1}^{i-1} q_j \prod_{k=j+1}^i p_k + q_i,$$

and by (12)

$$(17) \quad \left| b_{i_0} \prod_{j=i_0+1}^i p_j \right| < \frac{c}{i+1} \left(\frac{1}{2i+N+2} \right)^{1-\alpha(N-2)} < c \left(\frac{1}{i} \right)^{2-\alpha(N-2)}.$$

In view of (12) and (14) we have

$$(18) \quad \left| q_j \prod_{k=j+1}^i p_k \right| < \frac{c}{(2j+2)(2j+N)} \left(\frac{1}{j+1} \right)^{2(\alpha+1)} \cdot \frac{j+1}{i+1} \left(\frac{2j+N+2}{2i+N+2} \right)^{1-\alpha(N-2)}.$$

For $\alpha \in (-1/N, 1/(N-2))$,

$$1 - \alpha(N-2) < 2(1+\alpha).$$

We therefore may and do require of i_0 further that for $j > i_0$

$$\frac{(2j+N+2)^{1-\alpha(N-2)}}{(j+1)^{2(1+\alpha)}} < 2^{1-\alpha(N-2)}.$$

We obtain

$$\left| \sum_{j=i_0+1}^{i-1} q_j \prod_{k=j+1}^i p_k \right| < \frac{c}{2i+2} \left(\frac{1}{2i+N+2} \right)^{1-\alpha(N-2)} \sum_{j=i_0+1}^{i-1} \frac{1}{2j+N}$$

where

$$\sum_{j=i_0+1}^{i-1} \frac{1}{2j+N} < \int_{i_0}^{i-1} \frac{dx}{2x+N} = \frac{1}{2} \log \frac{2i+N-2}{2i_0+N}.$$

Accordingly, we set on i_0 the additional condition that for $i > i_0$

$$\left(\frac{1}{i} \right)^{(1/2)[1-\alpha(N-2)]} \log \frac{2i+N+2}{2i_0+N} < 1.$$

Then

$$(19) \quad \left| \sum_{j=i_0+1}^{i-1} q_j \prod_{k=j+1}^i p_k \right| < c \left(\frac{1}{i} \right)^{(3/2)-(1/2)\alpha(N-2)}.$$

A bound for the last term in (16) is immediate by (14):

$$(20) \quad |q_i| < c \left(\frac{1}{i} \right)^{2(\alpha+2)}.$$

We combine (16), (17), (19), and (20), and obtain (15).

7. We are ready to state:

LEMMA 6. For $\alpha \in (-1, 1/(N - 2))$, the function $s(r)$ of Lemma 2 is bounded quasiharmonic.

In fact, for $\alpha \in (-1/N, 1/(N - 2))$, all three exponents in (15) are > 1 , and therefore

$$(21) \quad |s(r)| = \left| \sum_{i=0}^{\infty} b_i r^{2i+2} \right| < \sum_{i=0}^{\infty} |b_i| < \infty .$$

The case $\alpha = -1/2$ is simple, as all $q_i = 0$, and by (8)

$$|b_i| = |b_{i_0}| \prod_{j=i_0+1}^i p_j < |b_{i_0}| \frac{(2i_0 + 2)^2}{(2i + 2)(2i + N)} < c \left(\frac{1}{i}\right)^2 ,$$

whence $\sum_{i=0}^{\infty} |b_i| < \infty$.

It remains to consider the case $\alpha \in (-1, 0) - \{-1/2\}$. We obtain at once

$$p_k < \frac{2k(2k + N - 2)}{(2k + 2)(2k + N)} ,$$

$$\prod_{j+1}^i p_k < \frac{(2j + 2)(2j + N)}{(2i + 2)(2i + N)} ,$$

and by (14)

$$|q_j| < \frac{c}{(2j + 2)(2j + N)} \cdot \left(\frac{1}{j + 1}\right)^{2(\alpha+1)}$$

for $j > i_0$. Therefore

$$\sum_{j=i_0+1}^{i-1} \left| q_j \prod_{k=j+1}^i p_k \right| < \frac{c}{(2i + 2)(2i + N)} \int_{i_0}^{i-1} \frac{dx}{(x + 1)^{2(\alpha+1)}} ,$$

where the integral has the value

$$\frac{1}{-2\alpha - 1} [i^{-2\alpha-1} - (i_0 + 1)^{-2\alpha-1}]$$

since $\alpha \neq -1/2$. As a consequence

$$(22) \quad \sum_{j=i_0+1}^{i-1} \left| q_j \prod_{k=j+1}^i p_k \right| < c \left(\frac{1}{i}\right)^{3+2\alpha} + d \left(\frac{1}{i}\right)^2 .$$

Similarly

$$(23) \quad \left| b_{i_0} \prod_{k=i_0+1}^i p_k \right| < c \left(\frac{1}{i}\right)^2$$

and

$$(24) \quad |q_i| < e\left(\frac{1}{i}\right)^{2\alpha+4}.$$

Since all exponents in (22)–(24) are > 1 , it follows again by (16) and (21) that the function $s(r)$ is bounded.

8. We have established our result:

THEOREM. *There exist bounded quasiharmonic functions on the Riemannian ball B_α if and only if the function $s(r)$ of Lemma 2 is bounded.*

In fact, we know that there exist bounded quasiharmonic functions on B_α if and only if $\alpha \in (-1, 1/(N - 2))$ (Sario-Wang [16]). This together with Lemma 6 gives the theorem.

A simple consequence is perhaps worth stating. Let R be the family of *radial* functions, characterized by the dependence on r only. Denote by O_{QBR} and O_{QB} the classes of Riemannian manifolds which do not carry bounded radial quasiharmonic functions, or bounded quasiharmonic functions, respectively, and set $B = \{\cup B_\alpha | \alpha \in \mathbf{R}\}$.

COROLLARY 1. $B \cap O_{QBR} = B \cap O_{QB}$.

That is, there exist bounded radial quasiharmonic functions on B_α if and only if there exist bounded quasiharmonic functions.

COROLLARY 2. $B \cap O_{QNR} \neq \emptyset$.

For $\alpha \in (-1, 1/(N - 2))$, we have $s - \sup_{B_\alpha} |s| \in QNR$. For all α , it is readily seen that the function

$$-1 - \int_0^r \int_0^\sigma \left(\frac{\rho}{\sigma}\right)^{N-1} \frac{(1 - \rho^2)^{N\alpha}}{(1 - \sigma^2)^{(N-2)\alpha}} d\rho d\sigma$$

is radial, negative, and quasiharmonic.

REFERENCES

1. Y. K. Kwon, L. Sario and B. Walsh, *Behavior of biharmonic functions on Wiener's and Royden's compactifications*, Ann. Inst. Fourier (Grenoble), **21** (1971), 217-226.
2. M. Nakai, *Dirichlet finite biharmonic functions on the plane with distorted metric*, (to appear).
3. M. Nakai and L. Sario, *Biharmonic classification of Riemannian manifolds*, Bull. Amer. Math. Soc., **77** (1971), 432-436.
4. ———, *Quasiharmonic classification of Riemannian manifolds*, Proc. Amer. Math. Soc., **31** (1972), 165-169.

5. ———, *Dirichlet finite biharmonic functions with Dirichlet finite Laplacians*, Math. Z., **122** (1971), 203-216.
6. ———, *A property of biharmonic functions with Dirichlet finite Laplacians*, Math. Scand., **29** (1971), 307-316.
7. ———, *Existence of Dirichlet finite biharmonic functions*, Ann. Acad. Sci. Fenn., (to appear).
8. ———, *Existence of bounded biharmonic functions*, J. Reine Angew. Math., (to appear).
9. ———, *Existence of bounded Dirichlet finite biharmonic functions*, Ann. Acad. Sci. Fenn. A. I., **505** (1972), 1-12.
10. ———, *Biharmonic functions on Riemannian manifolds*, Continuum Mechanics and Related Problems of Analysis, Nauka, Moscow, (1972), 329-335.
11. H. O'Malla, *Dirichlet finite biharmonic functions on the unit disk with distorted metrics*, Proc. Amer. Math. Soc., **32** 1972, 521-524.
12. L. Sario, *Lectures on biharmonic and quasiharmonic functions on Riemannian manifolds*, Duplicated lecture notes, 1969-70, University of California, Los Angeles.
13. L. Sario and C. Wang, *The class of (p, q) -biharmonic functions*, Pacific J. Math., **41** (1972), 799-808.
14. ———, *Counterexamples in the biharmonic classification of Riemannian 2-manifolds*, (to appear).
15. ———, *Generators of the space of bounded biharmonic functions*, Math. Z., **127** (1972), 273-280.
16. ———, *Quasiharmonic functions on the Poincaré N -ball*, Rend. Mat. (to appear).
17. ———, *Existence of Dirichlet finite biharmonic functions on the Poincaré 3-ball*, Pacific J. Math., (to appear).
18. L. Sario, C. Wang and M. Range, *Biharmonic projection and decomposition*, Ann. Acad. Sci. Fenn. A. I., **494** (1971), 1-14.
19. C. Wang and L. Sario, *Polyharmonic classification of Riemannian manifolds*, Kyoto Math. J., **12** (1972), 129-140.

Received April 18, 1972 and in revised form November 15, 1972. The work was sponsored by the U.S. Army Research Office-Durham, Grant DA-ARO-D-31-124-71-G181, University of California, Los Angeles.

UNIVERSITY OF CALIFORNIA, LOS ANGELES

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

D. GILBARG AND J. MILGRAM
Stanford University
Stanford, California 94305

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT
University of Washington
Seattle, Washington 98105

RICHARD ARENS
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. 39. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$48.00 a year (6 Vols., 12 issues). Special rate: \$24.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 46, No. 2

December, 1973

Christopher Allday, <i>Rational Whitehead products and a spectral sequence of Quillen</i>	313
James Edward Arnold, Jr., <i>Attaching Hurewicz fibrations with fiber preserving maps</i>	325
Catherine Bandle and Moshe Marcus, <i>Radial averaging transformations with various metrics</i>	337
David Wilmot Barnette, <i>A proof of the lower bound conjecture for convex polytopes</i>	349
Louis Harvey Blake, <i>Simple extensions of measures and the preservation of regularity of conditional probabilities</i>	355
James W. Cannon, <i>New proofs of Bing's approximation theorems for surfaces</i>	361
C. D. Feustel and Robert John Gregorac, <i>On realizing HNN groups in 3-manifolds</i>	381
Theodore William Gamelin, <i>Iversen's theorem and fiber algebras</i>	389
Daniel H. Gottlieb, <i>The total space of universal fibrations</i>	415
Yoshimitsu Hasegawa, <i>Integrability theorems for power series expansions of two variables</i>	419
Dean Robert Hickerson, <i>Length of period simple continued fraction expansion of \sqrt{d}</i>	429
Herbert Meyer Kamowitz, <i>The spectra of endomorphisms of the disc algebra</i>	433
Dong S. Kim, <i>Boundedly holomorphic convex domains</i>	441
Daniel Ralph Lewis, <i>Integral operators on \mathcal{L}_p-spaces</i>	451
John Eldon Mack, <i>Fields of topological spaces</i>	457
V. B. Moscatelli, <i>On a problem of completion in bornology</i>	467
Ellen Elizabeth Reed, <i>Proximity convergence structures</i>	471
Ronald C. Rosier, <i>Dual spaces of certain vector sequence spaces</i>	487
Robert A. Rubin, <i>Absolutely torsion-free rings</i>	503
Leo Sario and Cecilia Wang, <i>Radial quasiharmonic functions</i>	515
James Henry Schmerl, <i>Peano models with many generic classes</i>	523
H. J. Schmidt, <i>The \mathcal{F}-depth of an \mathcal{F}-projector</i>	537
Edward Silverman, <i>Strong quasi-convexity</i>	549
Barry Simon, <i>Uniform crossnorms</i>	555
Surjeet Singh, <i>(KE)-domains</i>	561
Ted Joe Suffridge, <i>Starlike and convex maps in Banach spaces</i>	575
Milton Don Ulmer, <i>C-embedded Σ-spaces</i>	591
Wolmer Vasconcelos, <i>Conductor, projectivity and injectivity</i>	603
Hideobu Yoshida, <i>On some generalizations of Meier's theorems</i>	609