THE $\mathcal{F}$-DEPTH OF AN $\mathcal{F}$-PROJECTOR

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Let $\mathfrak{F}$ be a saturated formation and let $G$ be a finite solvable group with $\mathfrak{F}$-projector $F$. In a fundamental work, Carter and Hawkes have shown that for suitably restricted $\mathfrak{F}$ there is a chain of $\mathfrak{F}$-crucial maximal subgroups of $G$ terminating with $F$. It is shown here that the number of links in such a chain is an $\mathfrak{F}$-invariant of $G$, called the $\mathfrak{F}$-depth of $F$ in $G$ and written $d_\mathfrak{F}(F, G)$.

If $\mathfrak{S}(G)$ is the $\mathfrak{S}$-length of $G$ then, provided $\mathfrak{F}$ is normal subgroup-closed, the inequality $\mathfrak{S}(G) \leq 2 \cdot d_\mathfrak{F}(F, G) + 1$ is obtained. If $F$ is also nilpotent of nilpotency class $c(F)$, then it is proved that $\mathfrak{S}(G) \leq d_\mathfrak{F}(F, G) + c(F)$.

If $\mathfrak{F}$ and $\mathfrak{S}$ are two such suitable saturated formations with $\mathfrak{F} \supseteq \mathfrak{S}$, comparisons of the invariants $d_\mathfrak{F}(F, G)$ and $d_\mathfrak{S}(H, G)$ are made, where $F$ and $H$ are respectively the $\mathfrak{F}$- and $\mathfrak{S}$-projectors of the the finite solvable group $G$. In particular, if $H \leq F$ then $d_\mathfrak{S}(F, G) \leq d_\mathfrak{S}(H, G)$, and if in addition $d_\mathfrak{F}(F, G) = d_\mathfrak{S}(H, G)$ then $H = F$.

1. Introduction. In this paper all groups considered are finite and solvable. Throughout we let $\mathfrak{F}$ be a saturated formation which is locally induced by a class of nonempty, integrated formations $\mathfrak{F}(p)$, one for each prime $p$. The concepts, definitions and notation of this article are included in the above-mentioned paper [2] of Carter and Hawkes. However, we make one standard definition not found there.

DEFINITION 1.1. Let $\mathfrak{F}$ be a nonempty formation and let $G$ be a group. Let $G^{(0)} = G^\mathfrak{F}$ and define $G^{(n+1)} = (G^{(n)})^\mathfrak{F}$. If $\mathfrak{F}$ contains all cyclic groups of prime order then $G^{(n)} = \{1\}$ for some integer $n$. The least such integer $n$ is called the $\mathfrak{F}$-length of $G$ and is written $\mathfrak{S}(G)$.

DEFINITION 1.2. Let $S \leq G$. A factor $H/K$ of $G$ ($H, K \leq G$ and $K \leq H$) is called an $S$-composition factor of $G$ if $S$ normalizes both $H$ and $K$ and if $H/K$ has no proper, nontrivial subgroup normalized by $S$. A subnormal series $1 = H_0 \vartriangleleft H_1 \vartriangleleft \cdots \vartriangleleft H_n = G$ is called an $S$-composition series of $G$ if each factor $H_{i+1}/H_i$ is an $S$-composition factor of $G$.

The Jordan-H"{o}lder theorem for operator groups ([9], 2.10.2) implies that any two $S$-composition series for $G$ are equivalent. Further,
because $G$ is solvable we may infer ([9], 4.4.2 and 4.4.5) that each $S$-composition factor of $G$ is an elementary abelian $p$-group for some prime $p$.

2. The $\mathfrak{F}$-depth. In this section we define the notion of the $\mathfrak{F}$-depth of an $\mathfrak{F}$-projector and derive its basic characterization as a “depth”.

**Definition 2.1.** Let $G$ be a group with $\mathfrak{F}$-projector $F$. If an $F$-composition factor $H/K$ of $G$ is a $p$-group, then $H/K$ is called $F$-central whenever it is centralized by $F^{p(p)}$; otherwise it is called $F$-eccentric. If $1 = H_0 < H_1 < \cdots < H_n = G$ is an $F$-composition series for $G$, then the $\mathfrak{F}$-depth of $F$ in $G$ is the number of $F$-eccentric factors in this series and is written as $d_\mathfrak{F}(F, G)$.

Because the action of $F$ (under conjugation) on equivalent factors is the same and because each two $\mathfrak{F}$-projectors of $G$ are conjugate, it follows that $d_\mathfrak{F}(F, G)$ is an invariant of the group $G$ which depends only on the formation $\mathfrak{F}$.

We now establish that all $\mathfrak{F}$-crucial chains from $F$ to $G$ have the same number of links, hence the terminology “depth”. The proof given below is due to Trevor Hawkes and is considerably shorter than our original constructive proof.

**Theorem 2.2.** Let $G$ be a group with $\mathfrak{F}$-projector $F$. Let

$$F = M_r < \cdots < M_1 < \cdots < M_i < \cdots < M_1 < \cdots < M_0 = G$$

be an $\mathfrak{F}$-crucial chain from $F$ to $G$. Then $d_\mathfrak{F}(F, G) = r$.

**Proof.** Induct on $|G|$. Because $F$ is an $\mathfrak{F}$-projector of $M_1$ and because the bottom $r - 1$ links of the given chain form an $\mathfrak{F}$-crucial chain from $F$ to $M_1$, we have $d_\mathfrak{F}(F, M_1) = r - 1$.

Suppose that $M_i$ is $p$-maximal and put $K = \text{Core}(M_i)$; let $H/K$ be the unique minimal normal subgroup of $G/K$ (see [6], II. 1.4 and II. 3.2). Because $M_i$ is $\mathfrak{F}$-crucial maximal and because $F$ is an $\mathfrak{F}$-projector of $G$, we have that $G/H \cong M_i/K \in \mathfrak{F}/\mathfrak{F}(p)$, so that $G = FH$. It follows that $H/K$ is an $F$-eccentric $F$-composition factor of $G$ and that an $F$-composition series of $G$ above $H$ is a chief series of $G$ above $H$ with the $G$-action on these factors equivalent to the $F$-action.

Now let $\mathcal{C}$ be an $F$-composition series of $G$ through $H/K$. The $F$-composition factors in $\mathcal{C}$ below $K$ are $F$-composition factors of $M_i$. The observation of the preceding paragraph shows that the factors in $\mathcal{C}$ above $H$ are chief factors of $G$ and because $G/H \in \mathfrak{F}$, each of these factors is $\mathfrak{F}$-central; therefore each is $F$-central since the action of $F$ and $G$ on these factors are equivalent. Hence the isomorphism $G/H = \cdots$
This section concludes with several elementary observations. If \( N \leq G \) then \( FN/N \) is an \( \mathfrak{F} \)-projector of \( G/N \). Thus there is an \( \mathfrak{F} \)-crucial chain in \( G/N \) from \( FN/N \) to \( G/N \). Because \( N \) is contained in the core of any subgroup containing \( FN \), we have the following.

**Proposition 2.3.** If \( N \leq G \) then there is an \( \mathfrak{F} \)-crucial chain in \( G \) from \( FN \) to \( G \). The length of this chain is an \( \mathfrak{F} \)-invariant of \( G \), namely \( d_\mathfrak{F}(FN/N, G/N) \).

If we write \( d_\mathfrak{F}(FN, G) \) for \( d_\mathfrak{F}(FN/N, G/N) \) then, since \( F \) is an \( \mathfrak{F} \)-projector of \( FN \), the following is immediate.

**Proposition 2.4.** Let \( N \leq G \). Then
\[
d_\mathfrak{F}(F, G) = d_\mathfrak{F}(F, FN) + d_\mathfrak{F}(FN, G).
\]

3. The Influence of \( \mathfrak{F} \)-depth. We now study relations among \( d_\mathfrak{F}(F, G) \), the \( \mathfrak{F} \)-length \( \epsilon_\mathfrak{F}(G) \) of \( G \), and, in case \( F \) is nilpotent, the nilpotency class \( c(F) \) of \( F \). For the first result only, we make the additional assumption that \( \mathfrak{F} \) is normal subgroup-closed.

**Theorem 3.1.** Let \( \mathfrak{F} \) be normal subgroup-closed, and let \( G \) be a group with \( \mathfrak{F} \)-projector \( F \). Then
\[
\epsilon_\mathfrak{F}(G) \leq 2 \cdot d_\mathfrak{F}(F, G) + 1.
\]

**Proof.** Let \( r = d_\mathfrak{F}(F, G) \) and induct on \( |G| \). If \( M \) is an \( \mathfrak{F} \)-crucial maximal subgroup containing \( F \), then \( d_\mathfrak{F}(F, M) = r - 1 \). Since \( M < G \) we may write \( \epsilon_\mathfrak{F}(M) \leq 2(r - 1) + 1 = 2r - 1 \).

Let \( K = \text{Core} (M) \) and, as in the proof of (2.2), let \( H/K \) be the unique minimal normal subgroup of \( G/K \). Since \( M \) is \( \mathfrak{F} \)-crucial we have that \( M/K \cong G/H \in \mathfrak{F} \) and that \( H/K \) is \( \mathfrak{F} \)-eccentric.

Because \( M/K \in \mathfrak{F} \) we obtain \( M^3 \leq K \). But \( K/M^3 \leq M/M^3 \in \mathfrak{F} \) and \( \mathfrak{F} \) is normal subgroup-closed, so that \( K^3 \leq M^3 \). A standard induction argument now shows that \( K^{n3} \leq M^{n3} \) for all \( n \). Therefore \( \epsilon_\mathfrak{F}(K) \leq \epsilon_\mathfrak{F}(M) \leq 2r - 1 \). Finally \( G/H \in \mathfrak{F} \) and \( H/K \) is elementary abelian so that \( \epsilon_\mathfrak{F}(G) \leq \epsilon_\mathfrak{F}(G/K) + \epsilon_\mathfrak{F}(K) \leq 2 + (2r - 1) = 2r + 1 \), the desired inequality.

We remark that if \( \mathcal{N} \) is the class of nilpotent groups and \( G \) is the symmetric group on four letters, then the 2-Sylow subgroups of \( G \) are the \( \mathcal{N} \)-projectors of \( G \) and are maximal. Since \( G \) has nilpotent length 3, the inequality of (3.1) is best possible.

We are grateful to Trevor Hawkes for the following example which shows that the normal subgroup-closed hypothesis of (3.1) is necessary.
EXAMPLE 3.2. Let $H$ be the subgroup of $\text{SL}(2,11)$ of order 120 which corresponds to the alternating group on five letters under the canonical epimorphism from $\text{SL}(2,11)$ to $\text{PSL}(2,11)$. It is known that $H$ acts fixed point freely on an elementary abelian group $V$ of order $11^5$ ([6], p. 500). The normalizer $K$ of a 2-Sylow subgroup of $H$ has order 24. Let $R$ be the split extension $VK$. Let $Q$ be the 2-Sylow subgroup of $K$ and let $P$ be a 3-Sylow subgroup of $K$. It is well-known that $Q$ is a quaternion group and that its central involution $z$ is centralized by the generator $x$ of $P$.

Let $\mathfrak{F}$ be locally induced by the following. If $p \equiv 5, 11$ let $\mathfrak{F}(p) = \{1\}$. Let $\mathfrak{F}(11)$ be the smallest formation containing $Z$. If $\mathfrak{F}$ is the formation of all $[2, 3, 11]$-groups in which the 2-chief factors are central and the nontrivial 11-chief factors are eccentric, then let $\mathfrak{F}(5)$ be the formation of extensions of 5-groups by groups in $\mathfrak{F}$. Then each $\mathfrak{F}(p)$ is integrated. However, $\mathfrak{F}(5)$ is not normal subgroup-closed, for $V \triangleleft V\langle x, z \rangle \in \mathfrak{F}(5)$ and $V$ is an 11-group. A result of Doerk ([3], 2.2) shows that $\mathfrak{F}$ is not normal subgroup-closed.

Finally, $G$ has $\mathfrak{F}$-length 4 and the $\mathfrak{F}$-subgroup $F = WV\langle x, z \rangle$ is maximal. Hence $d_\mathfrak{F}(F, G) = 1$ and the inequality of (3.1) fails for this $G$ and $\mathfrak{F}$.

By way of contrast, we give an example of a group $G$ of $\mathfrak{F}$-length 2 which has a Carter subgroup $A$ of $\mathfrak{F}$-depth $n$, where $n$ is an arbitrary integer.

EXAMPLE 3.3. Let $2 < p_1 < p_2 < \cdots < p_n$ be prime numbers and set $H = Z_{p_1} \times Z_{p_2} \times \cdots \times Z_{p_n}$. For $i = 1, \ldots, n$ let $B_i = \text{Aut}(Z_{p_i}) \cong Z_{p_i-1}$. Set $A = B_1 \times B_2 \times \cdots \times B_n$ and let $A$ act on $H$ coordinate-wise. Put $G = HA$, the split extension. Then $A = C(A) = N(A)$ is nilpotent, so that $A$ is a Carter subgroup of $G$. If $M_i = (Z_{p_1} \times \cdots \times Z_{p_i} \times \{1\} \times \cdots \times \{1\})A$, where the factor $\{1\}$ appears $n-i$ times, then $A < \cdot M_i \cdot < \cdot M_{i+1} \cdot < \cdots < \cdot M_{n-1} \cdot < \cdot M_n \cdot = G$ is an $\mathfrak{F}$-crucial chain of length $n$, so that $d_\mathfrak{F}(A, G) = n$. It is clear that $G$ has nilpotent length 2.

The next lemma is the primary inductive tool for the remainder of this section. We make it as general as possible.

LEMMA 3.4. Let $\mathfrak{F}$ be a class of groups which is closed under homomorphic images; and let $f$ be a positive integer-valued set function, defined on the class of groups, satisfying

\begin{equation}
(\#) \quad f(R) \geq f(R\theta), \text{ where } R \text{ is a group and } \theta \text{ is an epimorphism of } R.
\end{equation}

Let $G$ be a group in $\mathfrak{F}$ of minimal order which fails to satisfy

\begin{equation}
(\ast) \quad \phi(G) \leq d_\mathfrak{F}(F, G) + f(F), \text{ where } F \text{ is an } \mathfrak{F}\text{-projector of } G.
\end{equation}

\end{equation}
Then
(a) $G$ has a unique minimal normal subgroup $A$,
(b) $A$ is complemented and $A = C(A)$,
(c) $A \leq F$, and
(d) $\varepsilon_{g}(G/A) = \varepsilon_{g}(G) - 1$.
If, in addition, $F$ is nilpotent, then
(e) $F$ is a $p$-Sylow subgroup of $G$, where $p$ is the exponent of $A$.

Proof. Let $n = \varepsilon_{g}(G)$. It is clear that $G \notin \mathcal{S}$; otherwise $\varepsilon_{g}(G) = 1$, $d_{g}(F, G) = 0$ and (*) obtains since $f(F) \geq 1$. Therefore $n \geq 2$.

Let $\mathcal{R}$ be the class of groups of $\mathcal{S}$-length at most $n - 1$. It is well-known ([4], 4.3) that $\mathcal{R}$ is a saturated formation.

Let $A$ be a minimal normal subgroup of $G$. Since $G/A \in \mathcal{S}$ and $FA/A$ is an $\mathcal{S}$-projector of $G/A$, it follows that
\[ d_{g}(F, G) \geq d_{g}(FA, G) \geq \varepsilon_{g}(G/A) - f(FA/A) \geq \varepsilon_{g}(G/A) - f(F). \]
If $\varepsilon_{g}(G/A) = n$, then (*) holds. Therefore $\varepsilon_{g}(G/A) \leq n - 1$. Since $\mathcal{R}$ is a (saturated) formation and $\varepsilon_{g}(G) = n$, it follows that $A$ is the unique minimal normal subgroup of $G$. Also $A \in \mathcal{S}$, so that $\varepsilon_{g}(G/A) = n - 1$ and $A = G^{(n-1)g}$. This proves (a) and (d).

If any of the inequalities in (i) is strict, then
\[ d_{g}(F, G) \geq (n - 1) - f(F) + 1 = n - f(F), \]
contrary to the choice of $G$. Therefore $d_{g}(F, G) = (n - 1) - f(F)$, whereupon $F = FA$ and $A \leq F$. This proves (c).

If $M$ is an $\mathcal{S}$-projector of $G$, then $G = MA$, $M \cap A = 1$, and $M$ is maximal. Thus $A = C(A)$, proving (b).

Finally, suppose that $F$ is nilpotent and let $A$ be a $p$-group. Because $A \leq F$, a nontrivial $p$-complement of $F$ would centralize $A$, contrary to (b). Hence $F$ is a $p$-group. As $\mathcal{S} \supseteq \mathcal{R}$, it is elementary that $N(F) = F$, consequently $F$ must be a $p$-Sylow subgroup of $G$.

Next we establish one of the main results of this section.

**Theorem 3.5.** If $F$ is nilpotent, then
\[ \varepsilon_{g}(G) \leq d_{g}(F, G) + c(F), \]
where $c(F)$ is the nilpotency class for $F$.

Proof. If a group $R$ is not nilpotent, we extend the function $c$ by defining $c(R) = |R|$. Then the function $c$ satisfies $(\#)$ of (3.4). Let $G$ be a minimal counter-example to the theorem and let $n = \varepsilon_{g}(G)$.

Using (3.4) we see that $G$ has a unique minimal normal subgroup $A$ with $A = C(A), A \leq F, \varepsilon_{g}(G/A) = n - 1$, and that $F$ is a $p$-Sylow subgroup of $G$, where $A$ is a $p$-group.
Since $Z(F) \leq C(A) = A$ we have that $c(F/A) < c(F)$. By the choice of $G$,
\[ d_\delta(F, G) = d_\delta(F/A, G/A) \geq \epsilon_\delta(G/A) - c(F/A) > (n - 1) - c(F). \]
Therefore $d_\delta(F, G) \geq (n - 1) - c(F) + 1 = n - c(F)$, contrary to the choice $G$. This proves (3.5).

As a special case of (3.5), we have

**COROLLARY 3.6.** $\epsilon_\delta(G) \leq d_\delta(C, G) + c(C)$, where $C$ is a Carter subgroup of the group $G$.

In the remainder of this section we give sufficient conditions for the inequality $\epsilon_\delta(G) \leq d_\delta(F, G) + 1$ to obtain.

A group $G$ is said to belong to $\mathcal{Z}_\delta$ [3] provided the set of $\mathcal{Z}$-projectors of $G$ coincides with the set of $\mathcal{Z}$-normalizers of $G$. Although Doerk [3] has studied $\mathcal{Z}_\delta$ in detail, we need only the elementary fact that if $G \in \mathcal{Z}_\delta$ and $N \trianglelefteq G$ then $G/N \in \mathcal{Z}_\delta$.

**THEOREM 3.7.** The inequality

\[(*) \quad \epsilon_\delta(G) \leq d_\delta(F, G) + 1 \]

obtains provided one of the following holds:

(a) $G$ belongs to $\mathcal{Z}_\delta$, or

(b) $F$ complements $G^\delta$.

**Proof.** The function $f \equiv 1$ satisfies $(\#)$ of (3.4). Let $\mathcal{Z}$ be the class of groups satisfying either (a) or (b). Note that both hypotheses are invariant under homomorphisms.

Let $G$ be a group of minimal order belonging to $\mathcal{Z}$ and failing to satisfy (*). From (3.4) we infer that $G$ has a unique minimal normal subgroup $A$ with $A = C(A)$, $A \leq F$, and $\epsilon_\delta(G/A) = \epsilon_\delta(G) - 1$.

Suppose that (a) holds. Since $F$ is an $\mathcal{Z}$-normalizer of $G$ and $A \leq F$, it follows that $A$ is $\mathcal{Z}$-central. If $A$ is a $p$-group then $G/C(A) = G/A \in \mathcal{Z}(p) \subseteq \mathcal{Z}$; therefore $\epsilon_\delta(G) = 1$ and $G \in \mathcal{Z}$. But then it is easy to see that (*) would hold, contrary to the choice of $G$.

Suppose that (b) holds. Since $1 < A \cap G^\delta \leq G$, it follows that $A \leq G^\delta$. Thus $A \leq F \cap G^\delta = 1$, a contradiction. This completes the proof.

We now restrict our attention to the class $\mathfrak{N}$ of nilpotent groups and obtain further conditions which afford (*) of (3.7) with $\mathcal{Z}$ replaced by $\mathfrak{N}$. For the remainder of this section, $G$ is a group with Carter
A subgroup $H$ of $G$ is called pronormal provided, for every $x \in G$, there is a $y \in \langle H, H^x \rangle$ with $H^y = H^x$. Rose shows ([8], 1.6) that if $H$ is pronormal in $G$ then $N(H)$ is the subnormalizer of $H$ in $G$. Secondly, he shows that the class of groups with pronormal system normalizers is a formation ([8], 3.4) which contains (1) the class of $A$-groups (abelian Sylow subgroups), and (2) the class of groups of nilpotent length at most 2.

**Theorem 3.8.** The inequality

$$\lambda_x(G) \leq d_x(C, G) + 1$$

obtains provided one of the following holds:

(a) $G$ has $p$-length 1 for all primes, $p|G^x|$, or

(b) $G$ has pronormal system normalizers.

**Proof.** Again, let $f = 1$ and let $\mathfrak{G}$ be the class of groups satisfying either (a) or (b). It is immediate that $\mathfrak{G}$ is closed under homomorphic images.

Let $G$ be a group of minimal order which belongs to $\mathfrak{G}$ and fails to satisfy (*). Using (3.4) we find that $G$ has a unique minimal normal subgroup $A$ of exponent $p$ with $A = C(A)$, $A \leq C$, $\lambda_x(G/A) = \lambda_x(G) - 1$, and that $C$ is a $p$-Sylow subgroup of $G$.

Suppose that (a) holds. Because $A$ is the unique minimal normal subgroup of $G$, we infer that $O_p(G) = 1$. Since $A \leq G^x$, we have that $G$ has $p$-length 1. Thus $G$ has a normal $p$-Sylow subgroup, namely $C$. But this is contrary to $C = N(C)$, so that (a) cannot hold.

Suppose that (b) holds. Observe that $A$ is $\mathfrak{N}$-eccentric because $\lambda_x(G/A) = \lambda_x(G) - 1$. Therefore a system normalizer $D$ of $G$ avoids $A$. If $D \leq C$ then $D$ is both subnormal and pronormal in $C$, whereupon $D$ is normalized by $C$ ([8], 1.5). But then $[A, D] \leq A \cap D = 1$, contrary to $A = C(A)$. This completes the proof.

4. $\mathfrak{F}$-depth versus $\mathfrak{H}$-depth. In this section we consider two formations $\mathfrak{G}$ and $\mathfrak{H}$ which are locally induced by nonempty, integrated formations $\mathfrak{H}(p)$ and $\mathfrak{H}(p)$ respectively. We call a local formation $\mathfrak{H}(p)$ full if $\mathfrak{H}(p) = \mathfrak{G}_p \mathfrak{F}(p)$, where $\mathfrak{G}_p$ is the class of $p$-groups. The discussion on p. 350-1 of [3] shows that the formations $\mathfrak{F}$ and $\mathfrak{H}$ studied here can always be induced by full local formations. Throughout this section we assume that $\mathfrak{G} \subseteq \mathfrak{F}$ and that all local formations for both $\mathfrak{F}$ and $\mathfrak{H}$ are full; it follows readily from this that $\mathfrak{G}(p) \subseteq \mathfrak{F}(p)$ for each prime $p$. In this setting we compare the invariants...
$d_\delta(F, G)$ and $d_\delta(H, G)$, where $G$ is a group with $\mathcal{F}$-projector $F$ and $\mathcal{S}$-projector $H$.

**Theorem 4.1.** If $H \leq F$ then $d_\delta(F, G) \leq d_\delta(H, G)$.

The referee has observed that this theorem follows immediately from the following more general result which we will need later.

**Theorem 4.2.** If $H \leq F$ and $N \leq G$, then $d_\delta(F, FN) \leq d_\delta(H, HN)$.

**Proof.** Let $\mathcal{C}$ be an $F$-composition series for $N$. Then $\mathcal{C}$ is an $H$-invariant series and can be refined to an $H$-composition series for $N$. From the definition of the depth function it will suffice to show that each $F$-eccentric factor in $\mathcal{C}$ contains at least one $H$-eccentric composition factor. Suppose then that $M/N$ is $F$-eccentric factor in $\mathcal{C}$. If $M/N$ is $H$-hypercentral then $HM/N \in \mathcal{S}$. Since $H$ is an $\mathcal{S}$-projector of $G$, $HN/N$ is an $\mathcal{S}$-projector of $HM/N$, and consequently $H$ covers the factor $M/N$. Thus $F$ covers $M/N$, contradicting the fact that $M/N$ is $F$-eccentric. This completes the proof of (4.2).

Even though $\mathcal{F} \nsubseteq \mathcal{S}$ it is generally the case that an $\mathcal{F}$-projector does not contain an $\mathcal{S}$-projector. In fact, if $H$ is contained in no conjugate of $F$, then the inequality of (4.1) can fail as the following example shows.

**Example 4.3.** Let $R$ be the primitive solvable permutation group of degree 8 and order 168. Let $E$ be the 2-Sylow subgroup of $R$, let $P$ be a 7-Sylow subgroup of $R$ and let $Q$ be a 3-Sylow subgroup of $R$ which normalizes $P$. Let $W$ be a faithful, irreducible $\mathbb{Z}_2R$-module and let $V$ be a faithful, irreducible $\mathbb{Z}_3WR$-module, where $WR$ is the split extension. Let $G$ be the split extension of $V$ by $WR$. Finally let $H = VW EQ$ and $M = VEPQ$. It follows that $H$ and $M$ are maximal subgroups of $G$ of indices 7 and $\mid W \mid$ respectively.

If $\pi$ is a set of primes, let $\mathcal{G}_\pi$ be the set of $\pi$-groups. Define $\mathcal{S}$ and $\mathcal{F}$ locally as follows:

- $\mathcal{S}(2) = \mathcal{F}(2) = \mathcal{G}_{\{2,3,7\}}$ and $\mathcal{S}(5) = \mathcal{F}(5) = \mathcal{G}_{\{3,5\}}$
- $\mathcal{S}(7) = \mathcal{G}_{\{7\}}$
- $\mathcal{F}(7) = \mathcal{G}_{\{7\}} \mathcal{A}_6$, where $\mathcal{A}_6$ is the formation of abelian groups of exponent dividing 6
- $\mathcal{S}(11) = \mathcal{F}(11) = \mathcal{G}_{\{11\}} \mathcal{L}$, where $\mathcal{L} = \{G: G/F(G) has order prime to 7\}$

For $p \neq 2, 5, 7, 11$ let $\mathcal{S}(p) = \mathcal{F}(p) = \mathcal{G}_p$. Then $\mathcal{S}$ and $\mathcal{F}$ satisfy the general conditions imposed in this section, and $\mathcal{F} \nRightarrow \mathcal{S}$ since $PQ \in \mathcal{F} \setminus \mathcal{S}$.
It follows that $H$ is an $\mathfrak{S}$-projector of $G$; thus $d_\mathfrak{S}(H, G) = 1$. Further, $M$ is an $\mathfrak{S}$-crucial maximal subgroup of $G$ because $M/\mathrm{Core}(M) = M/V \cong EPQ \in \mathfrak{S}(5)$. Hence $M$ contains an $\mathfrak{S}$-projector of $G$ ([2], 5.4). If $L$ is the largest normal 11-nilpotent subgroup of $M$, then $M/L = M/V \cong EPQ \in \mathfrak{S}(11)$, so that $M \not\approx \mathfrak{S}$ ([2], 2.5). It now follows that $M$ must contain an $\mathfrak{S}$-projector $F$ of $G$ as a proper subgroup. Therefore $d_\mathfrak{S}(F, G) \geq 2 > 1 = d_\mathfrak{S}(H, G)$. It is obvious that $F$ does not contain a conjugate of $H$.

We give still another example to demonstrate the complex relationship between $\mathfrak{S}$-depth and $\mathfrak{S}$-depth. In particular, we display two chains of $\mathfrak{S}$-abnormal maximal subgroups of different lengths which terminate with a given Carter subgroup.

**Example 4.4.** Let $\mathfrak{S}$ be the class of 3-nilpotent groups and let $\mathfrak{S} = \mathfrak{R}$. Then $\mathfrak{R}$ can be induced by the local formations $\mathfrak{R}(p) = \mathfrak{S}_p$, for each prime $p$, and $\mathfrak{S}$ by the local formations $\mathfrak{S}(3) = \mathfrak{S}_3$ and $\mathfrak{S}(p) = \mathfrak{S}_p$ for $p = 3$.

Let $G$ be the group of all semi-linear transformations over $GF(3^3)$ of the form $x \mapsto ax^t + b$, where $a, b \in GF(3^3)$ with $a \neq 0$ and where $t$ belongs to the Galois group of $GF(3^3)$ over its prime field. Then $|G| = 27 \cdot 26 \cdot 3$. Let $F$ be those transformations in which $b = 0$, $C$ those in which $b = 0$ and $a = \pm 1$, $P$ those in which $a = 1$, and $A$ those in which $a = 1$ and $t = 1$.

Certainly $F$ is 3-nilpotent and maximal in $G$, hence $F$ is an $\mathfrak{S}$-projector of $G$ and $d_\mathfrak{S}(F, G) = 1$. Also $C < \cdot F$ because $|F: C| = 13$. Since $P$ is the 3-Sylow subgroup of $G$ it has a central series $1 \triangleleft B_1 \triangleleft \cdots \triangleleft B_i \triangleleft A \triangleleft P$ with factors of order 3; consequently the series $1 \triangleleft B_1 \triangleleft B_2 \triangleleft A$ is $C$-invariant. Because the Galois group fixes the prime field elementwise (here identified with $a = \pm 1$), $C$ is nilpotent. It now follows that $C < \cdot CB_1 < \cdot CB_2 < \cdot CA < \cdot G$ is an $\mathfrak{R}$-crucial chain in $G$. Therefore $C$ contains a Carter subgroup of $G$ ([2], 5.4). However, since $C$ is nilpotent it must be a Carter subgroup of $G$.

Finally, note the numerical relations: $d_\mathfrak{S}(C, G) = 4$, $d_\mathfrak{S}(C, F) = 1$ and $d_\mathfrak{S}(F, G) = 1$. In particular, $d_\mathfrak{S}(C, F) + d_\mathfrak{S}(F, G) \equiv d_\mathfrak{S}(C, G)$.

In the previous example it was seen that the equality $d_\mathfrak{S}(H, G) = d_\mathfrak{S}(H, F) + d_\mathfrak{S}(F, G)$ failed even though $H \trianglelefteq F$. We now give a rather strong sufficient condition for this equality to obtain.

**Theorem 4.5.** If $G \in \mathfrak{R}\mathfrak{S}$ then

(a) $H$ is contained in a conjugate of $F$, and
(b) $d_\mathfrak{S}(H, G) = d_\mathfrak{S}(H, F) + d_\mathfrak{S}(F, G)$.

**Proof.** Note that $G$ also belongs to $\mathfrak{R}\mathfrak{S}$ since $\mathfrak{S} \subset \mathfrak{R}$.

(a) Every $\mathfrak{S}$-crucial link above $F$ is $\mathfrak{S}$-abnormal since $\mathfrak{S}(p) \subset \mathfrak{S}(3)$.
for each prime $p$; thus ([2], 4.2) implies that $F$ contains an $\mathfrak{S}$-normalizer of $G$. As $G \in \mathfrak{R}_S$ it follows that $H$ is an $\mathfrak{S}$-normalizer of $G$, whereupon $H$ is contained in a conjugate of $F$.

(b) Consider an $\mathfrak{S}$-crucial chain from $F$ to $G$. Again, because $\mathfrak{S}(p) \subseteq \mathfrak{S}(p)$ for each prime $p$, each $\mathfrak{S}$-crucial link is $\mathfrak{S}$-critical. Because $G \in \mathfrak{S}_S$ every $\mathfrak{S}$-critical link above $H$ is also $\mathfrak{S}$-crucial. Thus by adjoining an $\mathfrak{S}$-crucial chain from $F$ to $G$ to an $\mathfrak{S}$-crucial chain from $H$ to $F$ we obtain an $\mathfrak{S}$-crucial chain from $H$ to $G$. This is (b).

Since the derived group of a supersolvable group is nilpotent, we have

**Corollary 4.6.** If $G$ is supersolvable, then

(a) $H$ is contained in a conjugate of $F$, and

(b) $d_\mathfrak{S}(H, G) = d_\mathfrak{S}(H, F) + d_\mathfrak{S}(F, G)$.

**Theorem 4.7.** If $H \leq F$ and $d_\mathfrak{S}(H, G) = d_\mathfrak{S}(F, G)$, then $H = F$.

We remark that if (b) of (4.5) held for all groups $G$, then a proof of (4.7) would proceed as follows. Combine the equalities $d_\mathfrak{S}(H, G) = d_\mathfrak{S}(F, G)$ and $d_\mathfrak{S}(H, G) = d_\mathfrak{S}(H, F) + d_\mathfrak{S}(F, G)$ to obtain that $d_\mathfrak{S}(H, F) = 0$, and hence infer that $H = F$. Because the restrictions on $G$ in (4.5) are quite strong, it is surprising that (4.7) holds in general.

**Proof of (4.7).** Let $G$ be a minimal counter-example and let $A$ be a minimal normal subgroup of $G$. Then $HA \leq FA$ and (4.2) implies that $d_\mathfrak{S}(F, FA) \leq d_\mathfrak{S}(H, HA)$. Since $d_\mathfrak{S}(F, G) = d_\mathfrak{S}(H, G)$, the equality $d_\mathfrak{S}(F, FA) + d_\mathfrak{S}(FA, G) = d_\mathfrak{S}(H, HA) + d_\mathfrak{S}(HA, G)$ and (4.1) imply that $d_\mathfrak{S}(F, FA) = d_\mathfrak{S}(H, HA)$ and $d_\mathfrak{S}(HA, G) = d_\mathfrak{S}(FA, G)$. By the choice of $G$ we can conclude that $FA = HA$.

If $A \leq F$, then $d_\mathfrak{S}(F, FA) = 0$, so that $d_\mathfrak{S}(H, HA) = 0$; hence $A \leq H$. But then, $F = FA = HA = H$, a contradiction. Therefore $A \not\leq F$.

If $FA < G$, then because of the choice of $G$, we have that $F = H$ since $d_\mathfrak{S}(H, HA) = d_\mathfrak{S}(F, FA)$. Thus $G = FA$ and $F \cap A = 1$. But then $F = F \cap FA = F \cap HA = H(F \cap A) = H$, the final contradiction.

**References**


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