THE $\mathbb{F}$-DEPTH OF AN $\mathbb{F}$-PROJECTOR

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Let $\mathfrak{F}$ be a saturated formation and let $G$ be a finite solvable group with $\mathfrak{F}$-projector $F$. In a fundamental work, Carter and Hawkes have shown that for suitably restricted $\mathfrak{F}$ there is a chain of $\mathfrak{F}$-crucial maximal subgroups of $G$ terminating with $F$. It is shown here that the number of links in such a chain is an $\mathfrak{F}$-invariant of $G$, called the $\mathfrak{F}$-depth of $F$ in $G$ and written $d_\mathfrak{F}(F, G)$.

If $\zeta_\mathfrak{F}(G)$ is the $\mathfrak{F}$-length of $G$ then, provided $\mathfrak{F}$ is normal subgroup-closed, the inequality $\zeta_\mathfrak{F}(G) \leq 2 \cdot d_\mathfrak{F}(F, G) + 1$ is obtained. If $F$ is also nilpotent of nilpotency class $c(F)$, then it is proved that $\zeta_\mathfrak{F}(G) \leq d_\mathfrak{F}(F, G) + c(F)$.

If $\mathfrak{F}$ and $\mathfrak{G}$ are two such suitable saturated formations with $\mathfrak{F} \supseteq \mathfrak{G}$, comparisons of the invariants $d_\mathfrak{F}(F, G)$ and $d_\mathfrak{G}(H, G)$ are made, where $F$ and $H$ are respectively the $\mathfrak{F}$- and $\mathfrak{G}$-projectors of the finite solvable group $G$. In particular, if $H \leq F$ then $d_\mathfrak{F}(F, G) \leq d_\mathfrak{G}(H, G)$, and if in addition $d_\mathfrak{F}(F, G) = d_\mathfrak{G}(H, G)$ then $H = F$.

1. Introduction. In this paper all groups considered are finite and solvable. Throughout we let $\mathfrak{F}$ be a saturated formation which is locally induced by a class of nonempty, integrated formations $\mathfrak{F}(p)$, one for each prime $p$. The concepts, definitions and notation of this article are included in the above-mentioned paper [2] of Carter and Hawkes. However, we make one standard definition not found there.

**Definition 1.1.** Let $\mathfrak{F}$ be a nonempty formation and let $G$ be a group. Let $G^{13} = G^3$ and define $G^{n3}$ recursively by $G^{(n+1)3} = (G^{n3})^3$. If $\mathfrak{F}$ contains all cyclic groups of prime order then $G^{n3} = \{1\}$ for some integer $n$. The least such integer $n$ is called the $\mathfrak{F}$-length of $G$ and is written $\zeta_\mathfrak{F}(G)$.

**Definition 1.2.** Let $S \leq G$. A factor $H/K$ of $G$ ($H, K \leq G$ and $K \trianglelefteq H$) is called an $S$-composition factor of $G$ if $S$ normalizes both $H$ and $K$ and if $H/K$ has no proper, nontrivial subgroup normalized by $S$. A subnormal series $1 = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_n = G$ is called an $S$-composition series of $G$ if each factor $H_{i+1}/H_i$ is an $S$-composition factor of $G$.

The Jordan-Hölder theorem for operator groups ([9], 2.10.2) implies that any two $S$-composition series for $G$ are equivalent. Further,
because \( G \) is solvable we may infer ([9], 4.4.2 and 4.4.5) that each \( S \)-composition factor of \( G \) is an elementary abelian \( p \)-group for some prime \( p \).

2. The \( \mathcal{F} \)-depth. In this section we define the notion of the \( \mathcal{F} \)-depth of an \( \mathcal{F} \)-projector and derive its basic characterization as a "depth".

**DEFINITION 2.1.** Let \( G \) be a group with \( \mathcal{F} \)-projector \( F \). If an \( F \)-composition factor \( H/K \) of \( G \) is a \( p \)-group, then \( H/K \) is called \( F \)-central whenever it is centralized by \( F^{F(p)} \); otherwise it is called \( F \)-eccentric. If \( 1 = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_n = G \) is an \( F \)-composition series for \( G \), then the \( \mathcal{F} \)-depth of \( F \) in \( G \) is the number of \( F \)-eccentric factors in this series and is written as \( d_{\mathcal{F}}(F, G) \).

Because the action of \( F \) (under conjugation) on equivalent factors is the same and because each two \( \mathcal{F} \)-projectors of \( G \) are conjugate, it follows that \( d_{\mathcal{F}}(F, G) \) is an invariant of the group \( G \) which depends only on the formation \( \mathcal{F} \).

We now establish that all \( \mathcal{F} \)-crucial chains from \( F \) to \( G \) have the same number of links, hence the terminology "depth". The proof given below is due to Trevor Hawkes and is considerably shorter than our original constructive proof.

**THEOREM 2.2.** Let \( G \) be a group with \( \mathcal{F} \)-projector \( F \). Let

\[
F = M_r < \cdots < M_{r-1} < \cdots < M_1 < \cdot G
\]

be an \( \mathcal{F} \)-crucial chain from \( F \) to \( G \). Then \( d_{\mathcal{F}}(F, G) = r \).

**Proof.** Induct on \( |G| \). Because \( F \) is an \( \mathcal{F} \)-projector of \( M_i \) and because the bottom \( r - 1 \) links of the given chain form an \( \mathcal{F} \)-crucial chain from \( F \) to \( M_i \), we have \( d_{\mathcal{F}}(F, M_i) = r - 1 \).

Suppose that \( M_i \) is \( \mathcal{F} \)-maximal and put \( K = \text{Core} (M_i) \); let \( H/K \) be the unique minimal normal subgroup of \( G/K \) (see [6], II. 1.4 and II. 3.2). Because \( M_i \) is \( \mathcal{F} \)-crucial maximal and because \( F \) is an \( \mathcal{F} \)-projector of \( G \), we have that \( G/H \cong M_i/K \in \mathcal{F} \setminus \mathcal{F}(p) \), so that \( G = FH \). It follows that \( H/K \) is an \( F \)-eccentric \( F \)-composition factor of \( G \) and that an \( F \)-composition series of \( G \) above \( H \) is a chief series of \( G \) above \( H \) with the \( G \)-action on these factors equivalent to the \( F \)-action.

Now let \( \mathcal{C} \) be an \( F \)-composition series of \( G \) through \( H/K \). The \( F \)-composition factors in \( \mathcal{C} \) below \( K \) are \( F \)-composition factors of \( M_i \). The observation of the preceding paragraph shows that the factors in \( \mathcal{C} \) above \( H \) are chief factors of \( G \) and because \( G/H \in \mathcal{F} \), each of these factors is \( \mathcal{F} \)-central; therefore each is \( F \)-central since the action of \( F \) and \( G \) on these factors are equivalent. Hence the isomorphism \( G/H \cong \)
This section concludes with several elementary observations. If \( N \leq G \) then \( FN/N \) is an \( \mathcal{S} \)-projector of \( G/N \). Thus there is an \( \mathcal{S} \)-crucial chain in \( G/N \) from \( FN/N \) to \( G/N \). Because \( N \) is contained in the core of any subgroup containing \( FN \), we have the following.

**Proposition 2.3.** If \( N \leq G \) then there is an \( \mathcal{S} \)-crucial chain in \( G \) from \( FN \) to \( G \). The length of this chain is an \( \mathcal{S} \)-invariant of \( G \), namely \( d_\mathcal{S}(FN/N, G/N) \).

If we write \( d_\mathcal{S}(FN, G) \) for \( d_\mathcal{S}(FN/N, G/N) \) then, since \( F \) is an \( \mathcal{S} \)-projector of \( FN \), the following is immediate.

**Proposition 2.4.** Let \( N \leq G \). Then

\[
d_\mathcal{S}(F, G) = d_\mathcal{S}(F, FN) + d_\mathcal{S}(FN, G).
\]

3. The Influence of \( \mathcal{S} \)-depth. We now study relations among \( d_\mathcal{S}(F, G) \), the \( \mathcal{S} \)-length \( \zeta_\mathcal{S}(G) \) of \( G \), and, in case \( F \) is nilpotent, the nilpotency class \( c(F) \) of \( F \). For the first result only, we make the additional assumption that \( \mathcal{S} \) is normal subgroup-closed.

**Theorem 3.1.** Let \( \mathcal{S} \) be normal subgroup-closed, and let \( G \) be a group with \( \mathcal{S} \)-projector \( F \). Then

\[
\zeta_\mathcal{S}(G) \leq 2 \cdot d_\mathcal{S}(F, G) + 1.
\]

**Proof.** Let \( r = d_\mathcal{S}(F, G) \) and induct on \( |G| \). If \( M \) is an \( \mathcal{S} \)-crucial maximal subgroup containing \( F \), then \( d_\mathcal{S}(F, M) = r - 1 \). Since \( M < G \) we may write \( \zeta_\mathcal{S}(M) \leq 2(r - 1) + 1 = 2r - 1 \).

Let \( K = \text{Core}(M) \) and, as in the proof of (2.2), let \( H/K \) be the unique minimal normal subgroup of \( G/K \). Since \( M \) is \( \mathcal{S} \)-crucial we have that \( M/K \cong G/H \in \mathcal{S} \) and that \( H/K \) is \( \mathcal{S} \)-eccentric.

Because \( M/K \in \mathcal{S} \) we obtain \( M^3 \leq K \). But \( K/M^3 \leq M/M^3 \in \mathcal{S} \) and \( \mathcal{S} \) is normal subgroup-closed, so that \( K^3 \leq M^3 \). A standard induction argument now shows that \( K^{n3} \leq M^{n3} \) for all \( n \). Therefore \( \zeta_\mathcal{S}(K) \leq \zeta_\mathcal{S}(M) \leq 2r - 1 \). Finally \( G/H \in \mathcal{S} \) and \( H/K \) is elementary abelian so that \( \zeta_\mathcal{S}(G) \leq \zeta_\mathcal{S}(G/K) + \zeta_\mathcal{S}(K) \leq 2 + (2r - 1) = 2r + 1 \), the desired inequality.

We remark that if \( \mathcal{N} \) is the class of nilpotent groups and \( G \) is the symmetric group on four letters, then the 2-Sylow subgroups of \( G \) are the \( \mathcal{N} \)-projectors of \( G \) and are maximal. Since \( G \) has nilpotent length 3, the inequality of (3.1) is best possible.

We are grateful to Trevor Hawkes for the following example which shows that the normal subgroup-closed hypothesis of (3.1) is necessary.
Example 3.2. Let $H$ be the subgroup of $\text{SL}(2,11)$ of order 120 which corresponds to the alternating group on five letters under the canonical epimorphism from $\text{SL}(2,11)$ to $\text{PSL}(2,11)$. It is known that $H$ acts fixed point freely on an elementary abelian group $V$ of order $11^2$ ([6], p. 500). The normalizer $K$ of a 2-Sylow subgroup of $H$ has order 24. Let $R$ be the split extension $VK$. Let $W$ be a faithful, irreducible $\mathbb{Z}R$-module and let $G$ be the split extension $WR$. Let $Q$ be the 2-Sylow subgroup of $K$ and let $P$ be a 3-Sylow subgroup of $K$. It is well-known that $Q$ is a quaternion group and that its central involution $z$ is centralized by the generator $x$ of $P$.

Let $\mathcal{F}$ be locally induced by the following. If $p \equiv 5, 11$ let $\mathcal{F}(p) = (1)$. Let $\mathcal{F}(11)$ be the smallest formation containing $\mathbb{Z}_{11}$. If $\mathcal{F}$ is the formation of all $[2, 3, 11]$-groups in which the 2-chief factors are central and the nontrivial 11-chief factors are eccentric, then let $\mathcal{F}(5)$ be the formation of extensions of 5-groups by groups in $\mathcal{F}$. Then each $\mathcal{F}(p)$ is integrated. However, $\mathcal{F}(5)$ is not normal subgroup-closed, for $V \vartriangleleft V\langle x, z \rangle \in \mathcal{F}(5)$ and $V$ is an 11-group. A result of Doerk ([3], 2.2) shows that $\mathcal{F}$ is not normal subgroup-closed.

Finally, $G$ has $\mathcal{F}$-length 4 and the $\mathcal{F}$-subgroup $F = WV\langle x, z \rangle$ is maximal. Hence $d_\mathcal{F}(F, G) = 1$ and the inequality of (3.1) fails for this $G$ and $\mathcal{F}$.

By way of contrast, we give an example of a group $G$ of $\mathcal{F}$-length 2 which has a Carter subgroup $A$ of $\mathcal{F}$-depth $n$, where $n$ is an arbitrary integer.

Example 3.3. Let $2 < p_1 < p_2 < \cdots < p_n$ be prime numbers and set $H = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$. For $i = 1, \cdots, n$ let $B_i = \text{Aut}(\mathbb{Z}_{p_i}) \cong \mathbb{Z}_{p_i-1}$. Set $A = B_1 \times B_2 \times \cdots \times B_n$ and let $A$ act on $H$ coordinate-wise. Put $G = HA$, the split extension. Then $A = C(A) = N(A)$ is nilpotent, so that $A$ is a Carter subgroup of $G$. If $M_i = (\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_i} \times \{1\} \times \cdots \times \{1\})A$, where the factor $\{1\}$ appears $n - i$ times, then $A < \cdot M_i < \cdot M_{i+1} < \cdots < \cdot M_{n-1} < \cdot M_n = G$ is an $\mathcal{F}$-crucial chain of length $n$, so that $d_\mathcal{F}(A, G) = n$. It is clear that $G$ has nilpotent length 2.

The next lemma is the primary inductive tool for the remainder of this section. We make it as general as possible.

Lemma 3.4. Let $\mathcal{F}$ be a class of groups which is closed under homomorphic images; and let $f$ be a positive integer-valued set function, defined on the class of groups, satisfying

\[ f(R) \geq f(R \theta), \quad \text{where } R \text{ is a group and } \theta \text{ is an epimorphism of } R. \]

Let $G$ be a group in $\mathcal{F}$ of minimal order which fails to satisfy

\[ \varepsilon_\mathcal{F}(G) \leq d_\mathcal{F}(F, G) + f(F), \quad \text{where } F \text{ is an } \mathcal{F}\text{-projector of } G. \]
Then
(a) \( G \) has a unique minimal normal subgroup \( A \),
(b) \( A \) is complemented and \( A = C(A) \),
(c) \( A \leq F \), and
(d) \( \varepsilon_\Phi(G/A) = \varepsilon_\Phi(G) - 1 \).

If, in addition, \( F \) is nilpotent, then
(e) \( F \) is a \( p \)-Sylow subgroup of \( G \), where \( p \) is the exponent of \( A \).

Proof. Let \( n = \varepsilon_\Phi(G) \). It is clear that \( G \in \mathcal{R} \); otherwise \( \varepsilon_\Phi(G) = 1 \), \( d_\Phi(F, G) = 0 \) and (*) obtains since \( f(F) \geq 1 \). Therefore \( n \geq 2 \).

Let \( \mathcal{R} \) be the class of groups of \( \mathcal{F} \)-length at most \( n - 1 \). It is well-known ([4], 4.3) that \( \mathcal{R} \) is a saturated formation.

Let \( A \) be a minimal normal subgroup of \( G \). Since \( G/A \in \mathcal{F} \) and \( FA/A \) is an \( \mathcal{F} \)-projector of \( G/A \), it follows that

(1) \( d_\Phi(F, G) \geq d_\Phi(FA, G) \geq \varepsilon_\Phi(G/A) - f(FA/A) \geq \varepsilon_\Phi(G/A) - f(F) \).

If \( \varepsilon_\Phi(G/A) = n \), then (*) holds. Therefore \( \varepsilon_\Phi(G/A) \leq n - 1 \). Since \( \mathcal{R} \) is a (saturated) formation and \( \varepsilon_\Phi(G) = n \), it follows that \( A \) is the unique minimal normal subgroup of \( G \). Also \( A \in \mathcal{F} \), so that \( \varepsilon_\Phi(G/A) = n - 1 \) and \( A = G^{(n-1)\Phi} \). This proves (a) and (d).

If any of the inequalities in (1) is strict, then

\[ d_\Phi(F, G) \geq (n - 1) - f(F) + 1 = n - f(F) , \]

contrary to the choice of \( G \). Therefore \( d_\Phi(F, G) = (n - 1) - f(F) \), whereupon \( F = FA \) and \( A \leq F \). This proves (c).

If \( M \) is an \( \mathcal{R} \)-projector of \( G \), then \( G = MA \), \( M \cap A = 1 \), and \( M \) is maximal. Thus \( A = C(A) \), proving (b).

Finally, suppose that \( F \) is nilpotent and let \( A \) be a \( p \)-group. Because \( A \leq F \), a nontrivial \( p \)-complement of \( F \) would centralize \( A \), contrary to (b). Hence \( F \) is a \( p \)-group. As \( \mathcal{F} \supseteq \mathcal{R} \), it is elementary that \( N(F) = F' \), consequently \( F' \) must be a \( p \)-Sylow subgroup of \( G \).

Next we establish one of the main results of this section.

**Theorem 3.5.** If \( F \) is nilpotent, then

\[ \varepsilon_\Phi(G) \leq d_\Phi(F, G) + c(F) , \]

where \( c(F) \) is the nilpotency class for \( F \).

Proof. If a group \( R \) is not nilpotent, we extend the function \( c \) by defining \( c(R) = |R| \). Then the function \( c \) satisfies (2) of (3.4). Let \( G \) be a minimal counter-example to the theorem and let \( n = \varepsilon_\Phi(G) \).

Using (3.4) we see that \( G \) has a unique minimal normal subgroup \( A \) with \( A = C(A) \), \( A \leq F \), \( \varepsilon_\Phi(G/A) = n - 1 \), and that \( F \) is a \( p \)-Sylow subgroup of \( G \), where \( A \) is a \( p \)-group.
Since $Z(F) \leq C(A) = A$ we have that $c(F/A) < c(F)$. By the choice of $G,$

$$d_\circ(F, G) = d_\circ(F/A, G/A) \geq \zeta(G/A) - c(F/A) > (n - 1) - c(F).$$

Therefore $d_\circ(F, G) \geq (n - 1) - c(F) + 1 = n - c(F)$, contrary to the choice $G$. This proves (3.5).

As a special case of (3.5), we have

**Corollary 3.6.** $\zeta(G) \leq d_\circ(C, G) + c(C)$, where $C$ is a Carter subgroup of the group $G$.

In the remainder of this section we give sufficient conditions for the inequality $\zeta_\circ(G) \leq d_\circ(F, G) + 1$ to obtain.

A group $G$ is said to belong to $\mathcal{U}_\circ$ [3] provided the set of $\circ$-projectors of $G$ coincides with the set of $\circ$-normalizers of $G$. Although Doerk [3] has studied $\mathcal{U}_\circ$ in detail, we need only the elementary fact that if $G \in \mathcal{U}_\circ$ and $N \leq G$ then $G/N \in \mathcal{U}_\circ$.

**Theorem 3.7.** The inequality

$$\zeta_\circ(G) \leq d_\circ(F, G) + 1$$

obtains provided one of the following holds:

(a) $G$ belongs to $\mathcal{U}_\circ$, or

(b) $F$ complements $G^\circ$.

**Proof.** The function $f = 1$ satisfies ($\#$) of (3.4). Let $\mathcal{G}$ be the class of groups satisfying either (a) or (b). Note that both hypotheses are invariant under homomorphisms.

Let $G$ be a group of minimal order belonging to $\mathcal{G}$ and failing to satisfy (*). From (3.4) we infer that $G$ has a unique minimal normal subgroup $A$ with $A = C(A), A \leq F$, and $\zeta(G/A) = \zeta(G) - 1$.

Suppose that (a) holds. Since $F$ is an $\circ$-normalizer of $G$ and $A \leq F$, it follows that $A$ is $\circ$-central. If $A$ is a $p$-group then $G/C(A) = G/A \in \mathcal{G}(p) \leq \mathcal{G}$; therefore $\zeta(G) = 1$ and $G \in \mathcal{G}$. But then it is easy to see that (*) would hold, contrary to the choice of $G$.

Suppose that (b) holds. Since $1 < A \cap G^\circ \leq G$, it follows that $A \leq G^\circ$. Thus $A \leq F \cap G^\circ = 1$, a contradiction. This completes the proof.

We now restrict our attention to the class $\mathcal{N}$ of nilpotent groups and obtain further conditions which afford (*) of (3.7) with $\circ$ replaced by $\mathcal{N}$. For the remainder of this section, $G$ is a group with Carter
subgroup $C$.

A subgroup $H$ of $G$ is called pronormal provided, for every $x \in G$, there is a $y \in \langle H, H^x \rangle$ with $H^y = H^x$. Rose shows ([8], 1.6) that if $H$ is pronormal in $G$ then $N(H)$ is the subnormalizer of $H$ in $G$. Secondly, he shows that the class of groups with pronormal system normalizers is a formation ([8], 3.4) which contains (1) the class of $A$-groups (abelian Sylow subgroups), and (2) the class of groups of nilpotent length at most 2.

**Theorem 3.8.** The inequality

\[ \zeta_n(G) \leq d_n(C, G) + 1 \]

obtains provided one of the following holds:

(a) $G$ has $p$-length 1 for all primes, $p | |G^s|$, or
(b) $G$ has pronormal system normalizers.

**Proof.** Again, let $f = 1$ and let $\mathfrak{S}$ be the class of groups satisfying either (a) or (b). It is immediate that $\mathfrak{S}$ is closed under homomorphic images.

Let $G$ be a group of minimal order which belongs to $\mathfrak{S}$ and fails to satisfy (*). Using (3.4) we find that $G$ has a unique minimal normal subgroup $A$ of exponent $p$ with $A = C(A)$, $A \leq C$, $\zeta_n(G/A) = \zeta_n(G) - 1$, and that $C$ is a $p$-Sylow subgroup of $G$. Suppose that (a) holds. Because $A$ is the unique minimal normal subgroup of $G$, we infer that $O_{p'}(G) = 1$. Since $A \leq G^s$, we have that $G$ has $p$-length 1. Thus $G$ has a normal $p$-Sylow subgroup, namely $C$. But this is contrary to $C = N(C)$, so that (a) cannot hold.

Suppose that (b) holds. Observe that $A$ is $R$-eccentric because $\zeta_n(G/A) = \zeta_n(G) - 1$. Therefore a system normalizer $D$ of $G$ avoids $A$. If $D \leq C$ then $D$ is both subnormal and pronormal in $C$, whereupon $D$ is normalized by $C$ ([8], 1.5). But then $[A, D] \leq A \cap D = 1$, contrary to $A = C(A)$. This completes the proof.

4. ***$\mathfrak{S}$-depth versus $\mathfrak{T}$-depth.*** In this section we consider two formations $\mathfrak{S}$ and $\mathfrak{T}$ which are locally induced by nonempty, integrated formations $\{\mathfrak{S}(p)\}$ and $\{\mathfrak{T}(p)\}$ respectively. We call a local formation $\mathfrak{S}(p)$ full if $\mathfrak{S}(p) = \mathfrak{G}_p \mathfrak{S}(p)$, where $\mathfrak{G}_p$ is the class of $p$-groups. The discussion on p. 350-1 of [3] shows that the formations $\mathfrak{S}$ and $\mathfrak{T}$ studied here can always be induced by full local formations. Throughout this section we assume that $\mathfrak{S} \subseteq \mathfrak{F}$ and that all local formations for both $\mathfrak{S}$ and $\mathfrak{T}$ are full; it follows readily from this that $\mathfrak{T}(p) \subseteq \mathfrak{F}(p)$ for each prime $p$. In this setting we compare the invariants
$d_\delta(F, G)$ and $d_\phi(H, G)$, where $G$ is a group with $\mathfrak{F}$-projector $F$ and $\mathfrak{S}$-projector $H$.

**Theorem 4.1.** If $H \leq F$ then $d_\delta(F, G) \leq d_\phi(H, G)$.

The referee has observed that this theorem follows immediately from the following more general result which we will need later.

**Theorem 4.2.** If $H \leq F$ and $N \leq G$, then $d_\delta(F, FN) \leq d_\phi(H, HN)$.

**Proof.** Let $\mathfrak{C}$ be an $F$-composition series for $N$. Then $\mathfrak{C}$ is an $H$-invariant series and can be refined to an $H$-composition series for $N$. From the definition of the depth function it will suffice to show that each $F$-eccentric factor in $\mathfrak{C}$ contains at least one $H$-eccentric composition factor. Suppose then that $M/N$ is $F$-eccentric factor in $\mathfrak{C}$. If $M/N$ is $H$-hypercentral then $HM/N \in \mathfrak{S}$. Since $H$ is an $\mathfrak{S}$-projector of $G$, $HN/N$ is an $\mathfrak{S}$-projector of $HM/N$, and consequently $H$ covers the factor $M/N$. Thus $F$ covers $M/N$, contradicting the fact that $M/N$ is $F$-eccentric. This completes the proof of (4.2).

Even though $\mathfrak{F} \supseteq \mathfrak{S}$ it is generally the case that an $\mathfrak{F}$-projector does not contain an $\mathfrak{S}$-projector. In fact, if $H$ is contained in no conjugate of $F$, then the inequality of (4.1) can fail as the following example shows.

**Example 4.3.** Let $R$ be the primitive solvable permutation group of degree 8 and order 168. Let $E$ be the 2-Sylow subgroup of $R$, let $P$ be a 7-Sylow subgroup of $R$ and let $Q$ be a 3-Sylow subgroup of $R$ which normalizes $P$. Let $W$ be a faithful, irreducible $Z_2 R$-module and let $V$ be a faithful, irreducible $Z_7 WR$-module, where $WR$ is the split extension. Let $G$ be the split extension of $V$ by $WR$. Finally let $H = VWEQ$ and $M = VEPQ$. It follows that $H$ and $M$ are maximal subgroups of $G$ of indices 7 and $|W|$ respectively.

If $\pi$ is a set of primes, let $\mathfrak{S}_\pi$ be the set of $\pi$-groups. Define $\mathfrak{S}$ and $\mathfrak{F}$ locally as follows:

- $\mathfrak{S}(2) = \mathfrak{F}(2) = \mathfrak{S}_{\{2,3,7\}}$ and $\mathfrak{S}(5) = \mathfrak{F}(5) = \mathfrak{S}_{\{2,3,5\}}$
- $\mathfrak{S}(7) = \mathfrak{S}_7$
- $\mathfrak{F}(7) = \mathfrak{S}_{\{7\}} \mathfrak{A}_6$, where $\mathfrak{A}_6$ is the formation of abelian groups of exponent dividing 6
- $\mathfrak{S}(11) = \mathfrak{F}(11) = \mathfrak{S}_{\{11\}} \mathfrak{L}$, where $\mathfrak{L} = \{G: G/F(G) \text{ has order prime to 7}\}$.

For $p \neq 2, 5, 7, 11$ let $\mathfrak{S}(p) = \mathfrak{F}(p) = \mathfrak{S}_p$. Then $\mathfrak{S}$ and $\mathfrak{F}$ satisfy the general conditions imposed in this section, and $\mathfrak{F} \supseteq \mathfrak{S}$ since $PQ \in \mathfrak{F} \setminus \mathfrak{S}$.
It follows that \( H \) is an \( S \)-projector of \( G \); thus \( d_S(H, G) = 1 \). Further, \( M \) is an \( S \)-crucial maximal subgroup of \( G \) because \( M/\text{Core}(M) = M/V \cong \text{EPQ} \in S(3) \). Hence \( M \) contains an \( S \)-projector of \( G \) ([2], 5.4). If \( L \) is the largest normal 11-nilpotent subgroup of \( M \), then \( M/L = M/V \cong \text{EPQ} \in S(11) \), so that \( M \in S \) ([2], 2.5). It now follows that \( M \) must contain an \( S \)-projector \( F \) of \( G \) as a proper subgroup. Therefore \( d_S(F, G) \geq 2 > 1 = d_S(H, G) \). It is obvious that \( F \) does not contain a conjugate of \( H \).

We give still another example to demonstrate the complex relationship between \( S \)-depth and \( S \)-depth. In particular, we display two chains of \( S \)-abnormal maximal subgroups of different lengths which terminate with a given Carter subgroup.

**Example 4.4.** Let \( S \) be the class of 3-nilpotent groups and let \( \mathfrak{N} = \mathfrak{S} \). Then \( \mathfrak{N} \) can be induced by the local formations \( \mathfrak{N}(p) = S_p \), for each prime \( p \), and \( S \) by the local formations \( S(3) = S \) and \( S(p) = S \) for \( p \equiv 3 \).

Let \( G \) be the group of all semi-linear transformations over \( GF(3^3) \) of the form \( x \mapsto ax^t + b \), where \( a, b \in GF(3^3) \) with \( a \neq 0 \) and where \( t \) belongs to the Galois group of \( GF(3^3) \) over its prime field. Then \( |G| = 27 \cdot 26 \cdot 3 \). Let \( F \) be those transformations in which \( b = 0 \), \( C \) those in which \( b = 0 \) and \( a = \pm 1 \), \( P \) those in which \( a = 1 \), and \( A \) those in which \( a = 1 \) and \( t = 1 \).

Certainly \( F \) is 3-nilpotent and maximal in \( G \), hence \( F \) is an \( S \)-projector of \( G \) and \( d_S(F, G) = 1 \). Also \( C < F \) because \( |F: C| = 13 \). Since \( P \) is the 3-Sylow subgroup of \( G \) it has a central series \( 1 < B_1 < B_2 < A < P \) with factors of order 3; consequently the series \( 1 < B, \triangleleft B_2 < A \triangleleft P \) is \( C \)-invariant. Because the Galois group fixes the prime field elementwise (here identified with \( a = \pm 1 \)), \( C \) is nilpotent. It now follows that \( C < \cdot CB_2 < \cdot CB_2 \cdot C A < \cdot G \) is an \( \mathfrak{N} \)-crucial chain in \( G \). Therefore \( C \) contains a Carter subgroup of \( G \) ([2], 5.4). However, since \( C \) is nilpotent it must be a Carter subgroup of \( G \).

Finally, note the numerical relations: \( d_S(C, G) = 4, d_S(C, F) = 1 \) and \( d_S(F, G) = 1 \). In particular, \( d_S(C, F) + d_S(F, G) = d_S(C, G) \).

In the previous example it was seen that the equality \( d_S(H, G) = d_S(H, F) + d_S(F, G) \) failed even though \( H \leq F \). We now give a rather strong sufficient condition for this equality to obtain.

**Theorem 4.5.** If \( G \in \mathfrak{N}_S \) then
(a) \( H \) is contained in a conjugate of \( F \), and
(b) \( d_S(H, G) = d_S(H, F) + d_S(F, G) \).

**Proof.** Note that \( G \) also belongs to \( \mathfrak{N}_S \) since \( S \subseteq S \).

(a) Every \( S \)-crucial link above \( F \) is \( S \)-abnormal since \( S(p) \subseteq
for each prime \( p \); thus ([2], 4.2) implies that \( F \) contains an \( \mathfrak{G} \)-normalizer of \( G \). As \( G \in \mathfrak{N} \) it follows that \( H \) is an \( \mathfrak{G} \)-normalizer of \( G \), whereupon \( H \) is contained in a conjugate of \( F \).

(b) Consider an \( \mathfrak{G} \)-crucial chain from \( F \) to \( G \). Again, because \( \mathfrak{G}(p) \subseteq \mathfrak{G}(p) \) for each prime \( p \), each \( \mathfrak{G} \)-crucial link is \( \mathfrak{G} \)-critical. Because \( G \in \mathfrak{N} \) every \( \mathfrak{G} \)-critical link above \( H \) is also \( \mathfrak{G} \)-crucial. Thus by adjoining an \( \mathfrak{G} \)-crucial chain from \( F \) to \( G \) to an \( \mathfrak{G} \)-crucial chain from \( H \) to \( F \) we obtain an \( \mathfrak{G} \)-crucial chain from \( H \) to \( G \). This is (b).

Since the derived group of a supersolvable group is nilpotent, we have

**Corollary 4.6.** If \( G \) is supersolvable, then

(a) \( H \) is contained in a conjugate of \( F \), and

(b) \( d_\varphi(H, G) = d_\varphi(H, F) + d_\varphi(F, G) \).

**Theorem 4.7.** If \( H \leq F \) and \( d_\varphi(H, G) = d_\varphi(F, G) \), then \( H = F \).

We remark that if (b) of (4.5) held for all groups \( G \), then a proof of (4.7) would proceed as follows. Combine the equalities \( d_\varphi(H, G) = d_\varphi(F, G) \) and \( d_\varphi(H, G) = d_\varphi(H, F) + d_\varphi(F, G) \) to obtain that \( d_\varphi(H, F) = 0 \), and hence infer that \( H = F \). Because the restrictions on \( G \) in (4.5) are quite strong, it is surprising that (4.7) holds in general.

**Proof of (4.7).** Let \( G \) be a minimal counter-example and let \( A \) be a minimal normal subgroup of \( G \). Then \( HA \leq FA \) and (4.2) implies that \( d_\varphi(F, FA) \leq d_\varphi(H, HA) \). Since \( d_\varphi(F, G) = d_\varphi(H, G) \), the equality \( d_\varphi(F, FA) + d_\varphi(FA, G) = d_\varphi(H, HA) + d_\varphi(HA, G) \) and (4.1) imply that \( d_\varphi(F, FA) = d_\varphi(H, HA) \) and \( d_\varphi(HA, G) = d_\varphi(FA, G) \). By the choice of \( G \) we can conclude that \( FA = HA \).

If \( A \leq F \), then \( d_\varphi(F, FA) = 0 \), so that \( d_\varphi(H, HA) = 0 \); hence \( A \leq H \). But then, \( F = FA = HA = H \), a contradiction. Therefore \( A \not\leq F \).

If \( FA < G \), then because of the choice of \( G \), we have that \( F = H \) since \( d_\varphi(H, HA) = d_\varphi(F, FA) \). Thus \( G = FA \) and \( F \cap A = 1 \). But then \( F = F \cap FA = F \cap HA = H(F \cap A) = H \), the final contradiction.

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