

# Pacific Journal of Mathematics

**UNIFORM CROSSNORMS**

BARRY SIMON

## UNIFORM CROSSNORMS

BARRY SIMON

**A crossnorm on a pair of Banach spaces  $(X, Y)$  is a norm,  $\alpha$ , on the algebraic tensor product  $X \odot Y$  obeying  $\alpha(x \otimes y) = \|x\| \|y\|$  for all  $x \in X, y \in Y$ . When Schatten introduced crossnorms, he singled out two general classes of crossnorms: the dualizable crossnorms (called by him "crossnorms whose associates are crossnorms") and the uniform crossnorms. These are crossnorms which induce in a natural way other crossnorms: in the dualizable case, a crossnorm,  $\alpha_d$ , on  $X^* \odot Y^*$ , and in the uniform case, a crossnorm,  $\tilde{\alpha}$ , on  $\mathcal{L}(X) \odot \mathcal{L}(Y)$  where  $\mathcal{L}(X)$  is the algebra of bounded operators on  $X$ . Our main new result is a proof that if  $\alpha$  is a uniform crossnorm, then  $\tilde{\alpha}$ , the induced crossnorm on  $\mathcal{L}(X) \odot \mathcal{L}(Y)$  is dualizable.**

This result will be applied to the theory of tensor products of commutative Banach algebras.

§1. Basic definitions and facts. We recall several definitions from Schatten [5]:

**DEFINITION 1.** A norm  $\alpha$  on  $X \odot Y$ , the algebraic tensor product of two Banach spaces  $X$  and  $Y$ , is called a *crossnorm* if and only if  $\alpha(x \otimes y) = \|x\| \|y\|$  for all  $x \in X, y \in Y$ .

**DEFINITION 2.** A crossnorm,  $\alpha$ , on  $X \odot Y$  is called *dualizable* if and only if for all  $l \in X^*, \mu \in Y^*, z \in X \odot Y$ :

$$|(l \otimes \mu)(z)| \leq \|l\| \|\mu\| \alpha(z).$$

**REMARKS.** 1. We have replaced Schatten's awkward "crossnorm whose associate is a crossnorm" with the term "dualizable crossnorm".

2. It is a simple exercise [5] to show that if  $\alpha$  is dualizable, and  $\lambda \in X^* \odot Y^*$ , then

$$\alpha_d(\lambda) \equiv \sup_{z \in X \odot Y} |\lambda(z)| / \alpha(z)$$

defines a crossnorm,  $\alpha_d$ , on  $X^* \odot Y^*$ .

**DEFINITION 3.**  $\alpha_d$  is called the *dual* crossnorm of  $\alpha$  (Schatten uses the term associated crossnorm).

We will use  $\mathcal{L}(X)$  to denote the Banach algebra of all bounded operators on  $X$ .

DEFINITION 4. A crossnorm,  $\alpha$ , on  $X \odot Y$  is called *uniform* if and only if for all  $A \in \mathcal{L}(X)$ ,  $B \in \mathcal{L}(Y)$  and  $z \in X \odot Y$ :

$$\alpha((A \otimes B)z) \leq \|A\| \|B\| \alpha(z) .$$

Similar to dualizable crossnorms, for  $C \in \mathcal{L}(X) \odot \mathcal{L}(Y)$ , the quantity

$$\tilde{\alpha}(C) = \sup_{z \in X \odot Y} \alpha(Cz) / \alpha(z)$$

defines a crossnorm  $\tilde{\alpha}$  on  $\mathcal{L}(X) \odot \mathcal{L}(Y)$ .

DEFINITION 5.  $\tilde{\alpha}$  is called the induced *crossnorm* of  $\alpha$ .

There is an elementary fact about crossnorms which does not seem to have been noted in the literature:

THEOREM 1. *Every uniform crossnorm is dualizable.*

*Proof.* Let  $l \in X^*$ ,  $\mu \in Y^*$ . Pick  $x \neq 0$  in  $X$ ,  $y \neq 0$  in  $Y$ . Let  $A \in \mathcal{L}(X)$  be given by  $Ax' = l(x')x$  and  $B \in \mathcal{L}(Y)$  by  $By' = \mu(y')y$ . Then  $\|A\| = \|l\| \|x\|$ ,  $\|B\| = \|\mu\| \|y\|$  and

$$\alpha((A \otimes B)z) = |(l \otimes \mu)(z)| (\alpha(x \otimes y) = \|x\| \|y\| |(l \otimes \mu)(z)| .$$

It follows that if

$$\alpha((A \otimes B)z) \leq \|A\| \|B\| \alpha(z) ,$$

then

$$|l \otimes \mu(z)| \leq \|l\| \|\mu\| \alpha(z) .$$

Finally we recall the two ‘‘canonical’’ crossnorms of Schatten and some facts about them:

DEFINITION 6.  $\gamma$  is the function on  $X \odot Y$  given by

$$\gamma(z) = \inf \left\{ \sum_{i=1}^n \|x_i\| \|y_i\| \mid z = \sum_{i=1}^n x_i \otimes y_i \right\} .$$

DEFINITION 7. Given  $z \in X \odot Y$ , define  $\Pi_z \in \mathcal{L}(X^*, Y)$ , the bounded operators from  $X^*$  to  $Y$  by

$$\Pi_{\left(\sum_{i=1}^n x_i \otimes y_i\right)}(l) = \sum_{i=1}^n l(x_i) y_i .$$

$\lambda$  is the function on  $X \odot Y$  given by

$$\lambda(z) = \|\Pi_z\|_{\mathcal{L}(X^*, Y)}$$

where  $\|\cdot\|_{\mathcal{L}(X^*, Y)}$  is the operator norm.

**THEOREM 2.** (Schatten [5])

- (a)  $\gamma$  and  $\lambda$  are uniform (dualizable) crossnorms.
- (b) If  $\alpha$  is any crossnorm  $\alpha \leq \gamma$ .
- (c) A norm is a dualizable crossnorm if and only if  $\lambda \leq \alpha \leq \gamma$ .

**REMARKS.** 1. Schatten calls  $\gamma$  the greatest crossnorm and  $\lambda$  the least crossnorm whose associate is a crossnorm.

2. We will use the symbols  $\gamma_{X \odot Y}$  and  $\lambda_{X \odot Y}$  where there might be some confusion as to which algebraic tensor product is intended.

3. The completion of  $X \odot Y$  in the crossnorm,  $\alpha$ , will be denoted  $X \otimes_{\alpha} Y$ .

2. The main result. The main new result of this paper is:

**THEOREM 3.** Let  $\alpha$  be a uniform crossnorm on  $X \odot Y$ . Then the induced norm  $\tilde{\alpha}$  on  $\mathcal{L}(X) \odot \mathcal{L}(Y)$  is dualizable.

This is a rather technical looking result but it is motivated by a fairly simple problem which we discuss in §3. The heart of the proof is the following density lemma.

**LEMMA 1.** Let  $X$  be a Banach space. Endow  $\mathcal{L}(X)^*$  with the weak-\* topology. Given  $l \in \mathcal{L}(X)$  and  $x \in X$ , let  $L_{l,x} \in \mathcal{L}(X)^*$  be defined by  $L_{l,x}(A) = l(Ax)$ . Then the (weak \*-) closed convex hull of  $\{L_{l,x} \mid \|l\| = \|x\| = 1\}$  is the entire unit ball in  $\mathcal{L}(X)^*$ .

*Proof.* Suppose  $L_0 \in \mathcal{L}(X)^*$  and  $L_0$  is not in the closed convex hull of  $\{L_{l,x} \mid \|l\| = \|x\| = 1\}$ . Then by the Hahn-Banach theorem, there exists a weak \*-continuous linear functional,  $A$ , on  $\mathcal{L}(X)^*$  with  $\text{Re } A(L_{l,x}) \leq a$  for all  $l$  and  $x$  with  $\|l\| = 1, \|x\| = 1$ , and with  $\text{Re } A(L_0) > a$ . Since  $L_{l,cx} = cL_{l,x}$  for any scalar  $c$ , by rescaling  $A$ , we can suppose  $|A(L_{l,x})| \leq 1; A(L_0) > 1$ . But every weak \*-continuous functions  $A$  is of the form  $A(L) = L(A)$  for some  $A \in \mathcal{L}(X)$  (see [4], pp. 114-115). Thus  $\sup_{\|l\|=\|x\|=1} |l(Ax)| \leq 1$  and  $L_0(A) > 1$ . The first inequality implies  $\|A\| \leq 1$  so the second implies  $\|L_0\| > 1$ .

*Proof of Theorem 3.* By Theorem 2, we need only show that  $\lambda_{\mathcal{L}(X) \odot \mathcal{L}(Y)} \leq \tilde{\alpha}$ . But, by definition, if  $C = \sum_{i=1}^n A_i \otimes B_i \in \mathcal{L}(X) \odot \mathcal{L}(Y)$ , then  $\lambda(C) = \|\Pi_c\|$  where  $\Pi_c: \mathcal{L}(X)^* \rightarrow \mathcal{L}(Y)$  by  $\Pi_c(L) = \sum_{i=1}^n L(A_i)B_i$ .

Since  $\prod_c$  has this form, it is weak \*-continuous, i.e., if  $L_\alpha \rightarrow L$  in the  $\mathcal{L}(X)^*$  -weak\* topology, then  $\prod_c(L_\alpha) \rightarrow \prod_c(L)$  in  $\mathcal{L}(Y)$ -norm. Thus  $\|\prod_c\| = \sup_{L \in S} \|\prod_c(L)\|_{\mathcal{L}(Y)}$  for any set  $S$  whose closed convex hull is the unit ball of  $\mathcal{L}(X)^*$ . Using the lemma and

$$\|B\| = \sup \{ \|\mu(By)\| \mid \|\mu\| = \|y\| = 1 \},$$

we conclude:

$$\begin{aligned} \lambda_{\mathcal{L}(X) \otimes \mathcal{L}(Y)}(C) &= \\ \sup \{ |(l \otimes \mu)[C(x \otimes y)]| \mid l \in X^*, \mu \in Y^*, x \in X, y \in Y; \|l\| &= \\ = \|\mu\| = \|x\| = \|y\| = 1 \}. \end{aligned}$$

Let  $\alpha$  be a uniform crossnorm, then since  $\alpha$  is dualizable,

$$\begin{aligned} |(l \otimes \mu)[C(x \otimes y)]| &\leq \alpha_a(l \otimes \mu)\alpha(C(x \otimes y)) \\ &\leq \tilde{\alpha}(C)\alpha_a(l \otimes \mu)\alpha(x \otimes y) \\ &= \tilde{\alpha}(C) \|l\| \|\mu\| \|x\| \|y\|. \end{aligned}$$

We conclude  $\lambda \leq \tilde{\alpha}$  and with that, the theorem.

3. Tensor products of commutative Banach algebras. Now let  $\mathfrak{A}_1, \mathfrak{A}_2$  be Banach algebras with identities.

DEFINITION 8. If  $\mathfrak{A}_1$  and  $\mathfrak{A}_2$  are Banach algebras (with identity) a crossnorm,  $\alpha$ , on  $\mathfrak{A}_1 \odot \mathfrak{A}_2$  (which is an algebra) is called a *B-algebra crossnorm* if and only if  $\alpha(xy) \leq \alpha(x)\alpha(y)$  for all  $x, y \in \mathfrak{A}_1 \odot \mathfrak{A}_2$ .

Surprisingly, the following question is open.

Question 1. Let  $\mathfrak{A}_1, \mathfrak{A}_2$  be commutative Banach algebras with identity. Let  $\sigma(\cdot)$  denote the spectrum of the algebra  $\cdot$ . Let  $\alpha$  be a B-algebra crossnorm on  $\mathfrak{A}_1 \odot \mathfrak{A}_2$ . Then  $\mathfrak{A}_1 \otimes_\alpha \mathfrak{A}_2$  is a commutative B-algebra. Is  $\sigma(\mathfrak{A}_1 \otimes_\alpha \mathfrak{A}_2) = \sigma(\mathfrak{A}_1) \times \sigma(\mathfrak{A}_2)$ ?

One can be more explicit. If  $l$  is a multiplicative linear functional on  $\mathfrak{A}_1 \otimes_\alpha \mathfrak{A}_2$ , then  $l(\cdot \otimes 1)$  and  $l(1 \otimes \cdot)$  define elements  $l_1 \in \sigma(\mathfrak{A}_1)$  and  $l_2 \in \sigma(\mathfrak{A}_2)$  with  $l = l_1 \otimes l_2$ . Thus, to conclude that  $\sigma(\mathfrak{A}_1 \otimes_\alpha \mathfrak{A}_2) = \sigma(\mathfrak{A}_1) \times \sigma(\mathfrak{A}_2)$ , it is sufficient to show that for any  $l_1 \in \sigma(\mathfrak{A}_1)$  and  $l_2 \in \sigma(\mathfrak{A}_2)$ ,  $l_1 \otimes l_2$  defines an  $\alpha$ -bounded linear functional on  $\mathfrak{A}_1 \odot \mathfrak{A}_2$  which then extends to  $\mathfrak{A}_1 \otimes_\alpha \mathfrak{A}_2$ . We conclude:

LEMMA 2. [7] *If  $\alpha$  is a Banach algebra crossnorm on commutative algebras which is dualizable, then  $\sigma(\mathfrak{A}_1 \otimes_\alpha \mathfrak{A}_2) = \sigma(\mathfrak{A}_1) \times \sigma(\mathfrak{A}_2)$ .*

Question 2. Is every Banach algebra crossnorm on commutative

Banach algebras dualizable?

An affirmative answer to Question 2 would, of course, imply an affirmative answer to Question 1. Our main remark is that Theorem 3 implies that question 1 has an affirmative answer in a situation which arises quite often in practice.

**THEOREM 4.** *Let  $X$  and  $Y$  be Banach spaces and let  $\alpha$  be a uniform crossnorm. Let  $\mathfrak{A}_1$  be a commutative subalgebra of  $\mathcal{L}(X)$  and let  $\mathfrak{A}_2$  be a commutative subalgebra of  $\mathcal{L}(Y)$ . Let  $\mathfrak{A}$  be the subalgebra of  $\mathcal{L}(X \otimes_\alpha Y)$  generated by*

$$\{A \otimes B \mid A \in \mathfrak{A}_1; B \in \mathfrak{A}_2\} .$$

Then

$$\mathfrak{A} = \mathfrak{A}_1 \otimes_{\tilde{\alpha}} \mathfrak{A}_2, \sigma(\mathfrak{A}) = \sigma(\mathfrak{A}_1) \times \sigma(\mathfrak{A}_2) .$$

*Proof.* That  $\mathfrak{A} = \mathfrak{A}_1 \otimes_{\tilde{\alpha}} \mathfrak{A}_2$  is a trivial fact. That  $\sigma(\mathfrak{A}) = \sigma(\mathfrak{A}_1) \times \sigma(\mathfrak{A}_2)$  follows from Theorem 3 and Lemma 2.

**REMARKS.** 1. This theorem is not new in the case  $X$  and  $Y$  are Hilbert spaces and  $\alpha$  is the Hilbert space inner product. For  $\tilde{\alpha}$  on  $\mathcal{L}(X) \otimes \mathcal{L}(Y)$  is a  $C^*$ -norm, so by a result of Takesaki [6]  $\tilde{\alpha} \geq \lambda_{\mathcal{L}(X) \otimes \mathcal{L}(Y)}$ . By the "local nature" of  $\lambda$ [5], one concludes  $\tilde{\alpha} \geq \lambda_{\mathfrak{A}_1 \otimes \mathfrak{A}_2}$ .

2. The special case of this theorem where  $X$  and  $Y$  are commutative Banach algebras and  $\mathfrak{A}_1 = \{L_x \mid x \in X\}$ ,  $\mathfrak{A}_2 = \{L_y \mid y \in Y\}$  with  $L_a b = ab$ , is due to J. Gil de Lamadrid [2]. He proves in his special case that  $\tilde{\alpha} \geq \lambda$  without requiring a Hahn-Banach argument as in Lemma 1.

3. The special case of this theorem where  $\mathfrak{A}_1$  and  $\mathfrak{A}_2$  are generated by the resolvents of a single operator has been proven by Reed-Simon [4] using the fact that the only compact analytic subvarieties of  $C^2$  are points. We note in passing that Theorem 3 does not allow a simplification of [4] since the machinery needed to prove the special use of Theorem 4 is needed to prove other results.

4. Under the hypotheses of the theorem it is also quite easy to prove  $\partial_{\mathfrak{A}} = \partial_{\mathfrak{A}_1} \times \partial_{\mathfrak{A}_2}$  where  $\partial$  is the Shilov boundary. The proof is the same as in the special case  $\mathfrak{A}_1 \otimes_{\gamma} \mathfrak{A}_2$  [1].

### REFERENCES

1. B. R. Gelbaum, *Tensor products and related questions*, Trans. Amer. Math. Soc., **103** (1962), 525-548.
2. J. Gil de Lamadrid, *Uniform crossnorms and tensor products of Banach algebras*, Duke Math. J., **32** (1965), 359-368.

3. M. Reed and B. Simon, *Tensor products of closed operators on Banach spaces*, J. Func. Anal. to appear and *A spectral mapping theorem for tensor products of unbounded operators*, Bull. Amer. Math. Soc., **78** (1972), 730-733.
4. ———, *Methods of Modern Mathematical Physics, I: Functional Analysis*, Academic Press, 1972.
5. R. Schatten, *A Theory of Cross Spaces*, Ann. Math. Study No. 26, Princeton University Press, 1950.
6. M. Takesaki, *On the crossnorm of the direct product of  $C^*$ -algebras*, Tôhoku Math. J., **16** (1964), 111-122.
7. J. Tomiyama, *Tensor products of commutative Banach algebras*, Tôhoku Math. J., (2), **12** (1960), 147-154.

Received April 18, 1972. A Sloan Foundation Fellow.

PRINCETON UNIVERSITY

# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

D. GILBARG AND J. MILGRAM

Stanford University  
Stanford, California 94305

J. DUGUNDJI

Department of Mathematics  
University of Southern California  
Los Angeles, California 90007

R. A. BEAUMONT

University of Washington  
Seattle, Washington 98105

RICHARD ARENS

University of California  
Los Angeles, California 90024

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA  
STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY  
NAVAL WEAPONS CENTER

---

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. 39. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$48.00 a year (6 Vols., 12 issues). Special rate: \$24.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.



Christopher Allday, <i>Rational Whitehead products and a spectral sequence of Quillen</i> .....	313
James Edward Arnold, Jr., <i>Attaching Hurewicz fibrations with fiber preserving maps</i> .....	325
Catherine Bandle and Moshe Marcus, <i>Radial averaging transformations with various metrics</i> .....	337
David Wilmot Barnette, <i>A proof of the lower bound conjecture for convex polytopes</i> .....	349
Louis Harvey Blake, <i>Simple extensions of measures and the preservation of regularity of conditional probabilities</i> .....	355
James W. Cannon, <i>New proofs of Bing's approximation theorems for surfaces</i> .....	361
C. D. Feustel and Robert John Gregorac, <i>On realizing HNN groups in 3-manifolds</i> .....	381
Theodore William Gamelin, <i>Iversen's theorem and fiber algebras</i> .....	389
Daniel H. Gottlieb, <i>The total space of universal fibrations</i> .....	415
Yoshimitsu Hasegawa, <i>Integrability theorems for power series expansions of two variables</i> .....	419
Dean Robert Hickerson, <i>Length of period simple continued fraction expansion of <math>\sqrt{d}</math></i> .....	429
Herbert Meyer Kamowitz, <i>The spectra of endomorphisms of the disc algebra</i> .....	433
Dong S. Kim, <i>Boundedly holomorphic convex domains</i> .....	441
Daniel Ralph Lewis, <i>Integral operators on <math>\mathcal{L}_p</math>-spaces</i> .....	451
John Eldon Mack, <i>Fields of topological spaces</i> .....	457
V. B. Moscatelli, <i>On a problem of completion in bornology</i> .....	467
Ellen Elizabeth Reed, <i>Proximity convergence structures</i> .....	471
Ronald C. Rosier, <i>Dual spaces of certain vector sequence spaces</i> .....	487
Robert A. Rubin, <i>Absolutely torsion-free rings</i> .....	503
Leo Sario and Cecilia Wang, <i>Radial quasiharmonic functions</i> .....	515
James Henry Schmerl, <i>Peano models with many generic classes</i> .....	523
H. J. Schmidt, <i>The <math>\mathcal{F}</math>-depth of an <math>\mathcal{F}</math>-projector</i> .....	537
Edward Silverman, <i>Strong quasi-convexity</i> .....	549
Barry Simon, <i>Uniform crossnorms</i> .....	555
Surjeet Singh, <i>(KE)-domains</i> .....	561
Ted Joe Suffridge, <i>Starlike and convex maps in Banach spaces</i> .....	575
Milton Don Ulmer, <i>C-embedded <math>\Sigma</math>-spaces</i> .....	591
Wolmer Vasconcelos, <i>Conductor, projectivity and injectivity</i> .....	603
Hidenobu Yoshida, <i>On some generalizations of Meier's theorems</i> .....	609