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(*KE*)-DOMAINS

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A commutative ring R is said to have the (K)-property if for each of its proper ideals A, there exists an ideal A', such that AA' is a nonzero principal ideal of R. A domain D with unity $1 \neq 0$ is said to be a (KE)-domain, if each of its ideals A, considered as a ring, has the (K)-property. The concept of a (KE)-domain had been studied earlier by the author and R. Kumar. In this paper injective modules and flat modules are studied and characterizations of (KE)-domains in terms of these modules are established. Finally the problem of embedding of a (KE)-domain in $\hat{Z}_{(p)}$, the p-adic completion (p a prime number) of the ring Z of integers, is studied.

In [11], the concept of a (KE)-domain was introduced and a structure theorem for the same was established. The study of (KE)-domains was continued in [12], in which, their characterizations in terms of Dedekind domains, Prüfer domains and generalized Krull domains were proved. The present paper is also concerned with the study of (KE)-domains and it contains some further characterizations. Let D be a domain with unity $1 \neq 0$. For any proper ideal A of D, let A^* denote the subring of D generated by $A \cup \{1\}$. In § 1, we study injective modules and prove that, if a proper ideal A of a domain D is such that A^* is Noetherian and every injective D-module is injective as an A^*-module, then $D = A^*$(Theorem 2). This theorem yields a characterization of (KE)-domains given in Theorem 3. In § 3, we study flat modules and prove that a domain D is a (KE)-domain if and only if it is a flat A^*-module for each of its proper ideals A (Theorem 6). Theorem 2 in [12] is deduced as a corollary to Theorem 6. The other important result in § 2 is Theorem 5. Example 1 shows that if a domain D is a fiat A^*-module for some proper ideal A, it need not equal A^*. Let Z be the ring of integers and p any prime number; it was shown in [11, Example 4] that $\hat{Z}_{(p)}$, the p-adic completion of the quotient ring $Z_{(p)}$ is a (KE)-domain. In § 3, we prove that $\hat{Z}_{(p)}$ is a maximal (KE)-domain, in the sense that, if D is any (KE)-domain, different from its quotient field, such that some prime number p is not invertible in it, then D is embeddable in $\hat{Z}_{(p)}$ (Theorem 8). Other results of interest are Proposition 1, Lemma 13, and Theorem 9. The notations and terminology are essentially the same as in [10, 11], except that, all rings considered here are with unity $1 \neq 0$, all modules are unital, and by a proper prime ideal of a ring R is meant a prime ideal different from both (0) and R.

1. Injective modules. A ring R (not necessarily with unity) is said to have the (K)-property if for each of its proper ideals A, there exists an ideal A' of R, such that AA' is a nonzero principal ideal of R [11]. A domain D is said to be a (KE)-domain if each of its ideals A, considered as a ring, has the (K)-property [11, Definition 3]. For any domain D (not necessarily with unity) having F as its quotient field, let D^* denote the subring of F generated by $D \cup \{1\}$, where 1 is the unity of F. The following lemmas, which we state without proof, were proved in [11, Lemma 1 and Theorem 13].

LEMMA 1. *A domain D (not necessarily with unity) has the (K)-property if and only if D^* is a Dedekind domain.*

LEMMA 2. *A proper ideal A of a domain D (with unity) has the (K)-property if and only if $D = A^*$ and D is a Dedekind domain.*

The following lemma is an immediate consequence of the above lemmas.

LEMMA 3. *A domain D is a (KE)-domain if and only if it is a Dedekind domain and for each of its proper ideals A, $A^* = D$.*

For the definitions and fundamental properties of injective modules the reader may refer to Tsai-Chi-Te [13]. A ring R is said to be self-injective ring, if R_R is an injective module. We now establish the following.

PROPOSITION 1. *A domain D is a (KE)-domain if and only if $D = A^*$ for each of its proper ideals A.*

Proof. "Only if" follows from Lemma 3.

Suppose that for every ideal A of D, we have $D = A^*$. Since $D/A = A^*/A \cong Z/(n)$ for some $n \geqslant 0$ and $Z/(n)$ is Noetherian, we get that D is Noetherian. Consider any proper prime ideal P of D. Then $D/P = P^*/P$ is either isomorphic to Z or to $Z/(p)$, for some prime number p. In the former case, for every $k(\neq 0) \in Z$, $k1 \notin P$; consequently $k1 \notin P^2$ and $D/P^2 = (P^2)^*/P^2 \cong Z$. This gives that P^2 is a prime ideal of D: this is not possible in a Noetherian domain. Hence $D/P \cong Z/(p)$, for some prime number p and hence for every proper ideal A of D, $D/A \cong Z/(n)$ for some $n \geqslant 2$. Thus every proper homomorphic image of D is self-injective, since every proper homomorphic image of Z is self-injective. Hence by Levy [6], D is a Dedekind domain. Hence by Lemma 3, D is a (KE)-domain.

LEMMA 4. *Let D be a domain and A be a proper ideal of D. Then A^* is Noetherian if and only if D is Noetherian and a finite A^*-module.*

Proof. Let A^* be Noetherian. Suppose to the contrary that D is not a finite A^*-module. Then there exists a denumerable subset $S = \{b_i: i = 1, 2, \cdots\}$ of D such that the A^*-submodule of D generated by S cannot be generated by a finite subset of S. Choose a $(\neq 0) \in A$. As A^* is Noetherian and $Sa \subset A^*$, there exists a positive integer n such that the ideal of A^* generated by the elements $b_i a$ $(1 \leqslant i \leqslant n)$ is the same as that generated by Sa. This yields that for each $i \geqslant n+1$, $b_i a = \sum_{j=1}^{n} a_{ij} b_j a$ for some $a_{ij} \in A^*$, and hence $b_i = \sum_{j=1}^{n} a_{ij} b_j$. Consequently the finitely many elements $b_i (1 \leqslant i \leqslant n)$ generate the A^*-submodule of D generated by S; this gives a contradiction. Hence D is a finite A^*-module. It is now immediate that D is Noetherian, since A^* is Noetherian. The converse follows by Eakin [5, Theorem 2]. Finally, the second part is an immediate consequence of [14, Chap. V, p. 255].

If S is a subring of a ring R such that it contains the unity element of R, then every R-module can be regarded as an S-module in a natural way. In the following lemmas, D will be a domain having a proper ideal A, such that A^* is Noetherian and every injective D-module is injective as an A^*-module. For any D-module M $E(M)$ and $E'(M)$ will denote its D-injective hull and A^*-injective hull respectively.

LEMMA 5. *Every indecomposable injective D-module is an indecomposable injective A^*-module.*

Proof. Let M be an indecomposable injective D-module. By the hypothesis M is also an injective A^*-module. Let $M = M_1 \oplus M_2$ for some A^*-submodules $M_i (i = 1, 2)$. As M_1 is an injective A^*-module, it is a divisible A^*-module. Consider $b(\neq 0) \in D$. Choose $a(\neq 0) \in A$. As $ab \in A$ and $ab \neq 0$, $M_1 = M_1 ab$. This implies that $M_1 = M_1 b$ and M_1 is a D-submodule of M. Similarly M_2 is a D-submodule of M. Hence $M_1 = (0)$ or $M_2 = (0)$. This proves the lemma.

LEMMA 6. *Let M and N be any two divisible D-modules. Then:*
(i) *Any A^*-homomorphism of M into N is a D-homomorphism,*

(ii) *M and N are isomorphic as D-modules if and only if they are isomorphic as A^*-modules.*

(iii) $\mathrm{Hom}_D(M, M) = \mathrm{Hom}_{A^*}(M, M)$.

Proof. Let $\sigma: M \to N$ be any A^*-homomorphism. Let $x \in M$ and $b(\neq 0) \in D$. Choose $a(\neq 0) \in A$. Then $ab \in A^*$. As M is a divisible D-module there exists y_ε, M such that $x = ya$. Then $xb = yab$ and $\sigma(xb) = \sigma(yab) = \sigma(y)ab = \sigma(x)b$. Hence σ is a D-homomorphism (ii) and (iii) are immediate consequences of (i).

We need the following two results due to Matlis [7], which we state without proof.

PROPOSITION 2. *Let R be a commutative Noetherian ring. Then there exists a one-to-one correspondence between the prime ideals $P(\neq R)$ of R and the indecomposable injective R-modules, given by $P \leftrightarrow E(R/P)$, where $E(M)$ denotes the injective hull of any R-module M. If Q is an irreducible P-primary ideal, then $E(R/P) = E(R/Q)$.*

THEOREM 1. *With the same notation as in Proposition 2, let $E = E(R/P)$ be an indecomposable injective R-module and*

$$H = \mathrm{Hom}_R(E, E) .$$

Then H is isomorphic to $\hat{R}_P$, the PR_P-adic completion of R_P. More precisely, E is a faithfull $\hat{R}_P$-module and each R-endomorphism of E can be realized by multiplication by an element of $\hat{R}_P$.

We now prove the following.

LEMMA 7. *$P \leftrightarrow P \cap A^*$ is a one-to-one correspondence between proper prime ideals P of D and proper prime ideals of A^*.*

Proof. By Lemma 4, D is Noetherian. Thus by Proposition 2, $P \leftrightarrow E(R/P)$ is a one-to-one correspondence between the prime ideals P of D and the indecomposable injective D-modules. By Lemma 5 $E(D/P) = E'(A^*/A^* \cap P)$, the A^*-injective hull of $A^*/A^* \cap P$. From Proposition 2 and Lemma 6 we get that $P \to A^* \cap P$ is a one-to-one mapping of the set of all prime ideals P of D into the set of all prime ideals of A^*. By Lemma 4, D is integral over A^*. Therefore given a prime ideal P' of A^*, there exists a prime ideal P of D such that $P \cap A^* = P'$ [14, p. 223, Theorem 3]. This completes the proof.

LEMMA 8. *Let P be a proper prime ideal of D. There exists a one-to-one inclusion preserving correspondence between the P-primary ideals of D and the $P \cap A^*$-primary ideals of A^*. Further for any irreducible P-primary ideal Q of D, the corresponding primary ideal* *of* A^* *is* $A^* \cap Q$.

Proof. Consider $E = E(D/P) = E'(A^*/A^* \cap P)$. By Lemma 6, $\mathrm{Hom}_A^*(E, E) = \mathrm{Hom}_D(E, E)$. It follows from Theorem 1 that there exists an isomorphism σ of $\hat{D}_P$ onto $\hat{A}^*_{P'}$, where $P' = P \cap A^*$, such that for any $d \in \hat{D}_P$ and $x \in E$, $xd = x\sigma(d)$. By Cohen [3, Theorem 2], for any local ring (R, M), if $\hat{R}$ is the completion of R, then $\hat{M} = M\hat{R}$ is the unique maximal ideal of R and $Q \leftrightarrow Q\hat{R}$ is a one-to-one correspondence between the M-primary ideals Q of R and the $\hat{M}$-primary ideals of $\hat{R}$. Thus $Q \leftrightarrow Q\hat{D}_P$ is a one-to-one correspondence between the P-primary ideals Q of D and $P\hat{D}_P$-primary ideals of $\hat{D}_P$. For any P-primary ideal Q of D, $\sigma(Q\hat{D}_Q) \cap A^*$ is a P'-primary ideal of A^*, and $Q \leftrightarrow \sigma(Q\hat{D}_P) \cap A^*$ is a one-to-one correspondence between the P-primary ideals Q of D, and P'-primary ideals of A^*. Let Q be an irreducible P-primary ideal of D. By Matlis [7, Lemma 32], there exists $x \in E$ for which $\mathrm{ann}_D(x) = Q$. Then $\mathrm{ann}_{\hat{D}_P}(x) = Q\hat{D}_P$ and $\mathrm{ann}_{\hat{A}^*_P}(x) = \sigma(Q\hat{D}_P)$, so that $\mathrm{ann}_{A^*}(x) = \sigma(Q\hat{D}_P) \cap A^*$. At the same time $\mathrm{ann}_{A^*}(x) = \mathrm{ann}_D(x) \cap A^* = Q \cap A^*$. This shows that

$$Q \cap A^* = \sigma(Q\hat{D}_P) \cap A^* .$$

Hence the lemma follows.

THEOREM 2. *If* A *is any proper ideal of a domain* D *such that* A^* *is Noetherian and every injective* D*-module is an injective* A^**-module then* $D = A^*$.

Proof. Let $A = P$ be a prime ideal. Then either $P^*/P \cong Z/(p)$, for some prime number p or $P^*/P \cong Z$. Now $E(D/P) = E(P^*/P)$ implies that $\hat{D}_P = \hat{P}^*_P$. From this we obtain that the quotient field of D/P is isomorphic to the quotient field of P^*/P. If $P^*/P \cong Z/(p)$, then $D/P \cong Z/(p) \cong P^*/P$ and $D = P^*$. If $P^*/P \cong Z$, then the quotient field of D/P is isomorphic to the field R of rational numbers. Since every overring of Z, contained in R, is of the type Z_S, we get that $D/P \cong Z_S$ for some multiplicative subset S of Z. It follows from Lemma 4, that D/P is integral over P^*/P. However Z is integrally closed in R. Consequently $D/P \cong P^*/P \cong Z$. Since Z has no proper subring containing 1, we get that $D = P^* = A^*$.

Suppose that A is not a prime ideal. Then $A = \bigcap_{i=1}^t Q_i$ for some irreducible ideals Q_i of D such that $\bigcap_{j \neq i} Q_j \not\subset Q_i$ for every i. Now

$$(1) \qquad A = A \cap A^* = \bigcup_{i=1}^{t} (Q_i \cap A^*) .$$

Suppose that A is a prime ideal of A^*. Then (1) yields that

$A = Q_i \cap A^*$ for some i and $Q_i \cap A^* \subset Q_j \cap A^*$ for every j. In view of Lemmas 6(i), 7 and 8, $t = 1$, $A = Q_1 \cap A^*$ and Q_1 is a prime ideal of D, since A is a prime ideal of A^*. Thus $A = Q_1$ is a prime ideal of D. This is a contradiction. Hence A is not a prime ideal of A^*. Consequently $A^*/A \cong Z/(n)$, for some composite integer $n > 2$. Since in $Z/(n)$ every prime ideal different from $Z/(n)$ is a maximal ideal of $Z/(n)$, the prime radical of $Q_i \cap A^*$ in A^* is a maximal ideal of A. Then by Lemma 7, the prime radical of Q_i in D is a maximal ideal of D. Further, since in $Z/(n)$ any family of primary ideals, which have common radical, is totally ordered and by Lemmas 6(i), 7 and 8, $Q_i \cap A^* \not\subset Q_j \cap A^*$ for $i \neq j$, we get that the prime radical of these Q_i are all distinct and maximal. Thus $A = \bigcap_{i=1}^{t} Q_i$ is an irredundant decomposition of A into primary ideals. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_u^{\alpha_u}$ be the factorization of n into distinct prime powers. It is immediate that $t = u$, and we can arrange the $Q_i'^s$ in such a way that $(Q_i \cap A^*)/A \cong (p_i^{\alpha_i})/(n)$. Now by Zariski and Samuel [14, p. 178. Theorem 32], $D/A \cong \bigoplus \sum_{i=1}^{t} D/Q_i$. Further

$$D/Q_i \cong D_{M_i}/Q_i D_{M_i} \cong \hat{D}_{M_i}/Q_i \hat{D}_{M_i} \cong A^*_{M'}/Q'_i A^*_{M_i} = A^*/Q_i \ ,$$

where $M'_i = M_i \cap A^*$ and $Q'_i = Q_i \cap A^*$: as $A^*/Q_i = Z/(p_i^{\alpha_i})$, it follows that $D/A = \bigoplus \sum_{i=1}^{t} Z/(p_i^{\alpha_i}) = Z/(n)$. Thus the additive group of D/A is cyclic and is generated by its unity. Hence $A^* = D$. This proves the theorem.

REMARK. In the above theorem, it can be easily seen from the proof that it is enough to assume that every indecomposable injective D-module is A^*-injective. However in that case a simple application of a theorem due to Matlis [7] yields that every injective D-module is an injective A^*-module. Proposition 1 and the above theorem immediately yield the following characterization of a (KE)-domain.

THEOREM 3. *A domain D is a (KE)-domain if and only if for each of its proper ideals A, A^* is a Noetherian ring and every injective D-module is an injective A^*-module.*

2. **Flat modules.** For definitions and some well known results on flat modules the reader may see Bourbaki [2]. Let D be a domain having K as its quotient field. By an overring of D, we mean any domain D' such that $D \subset D' \subset K$. In [8], Richman studied those overdomains of a domain D which are flat as D-modules. The following theorem which we state without proof was proved by Richman.

THEOREM 4. *Let D' be an over domain of a domain D. Then*

D' is a flat D-module if and only if $D'_M = D_{(M \cap D)}$ for all maximal ideals M of D'.

Let us recall from [11] that a ring R is said to have dimension n, if it contains a chain $P_0 < P_1 < P_2 < \cdots < P_n (\neq R)$ of prime ideals, but it contains no such chain of greater length.

LEMMA 9. *Let P be a proper prime ideal of a domain D such that for every nonzero primary ideal Q of D contained in P (not necessarily a P-primary ideal), D is a flat Q^*-module. Then:*

(i) *Height $P \leqslant 2$.*

(ii) *If P is not a minimal proper prime ideal, then P is a maximal ideal.*

(iii) *There exists a P-primary ideal $Q \neq P$.*

Proof. Suppose that P is not a minimal prime ideal. Then there exists a proper prime ideal $P' < P$. Let M be a maximal ideal of D containing P. Since by the hypothesis, D is a flat P'^*-module, Theorem 4 yields that $D_M = (P')^*_{(P' \cap M)}$. Since $(P')^*/P' \cong Z/(n)$ for some n and $\dim Z/(n) \leqslant 1$, we have $\dim (P')^*/P' \leqslant 1$: thus

$$\dim D/P' \leqslant 1 .$$

It follows that there exists no prime ideal of D properly between P' and M. Consequently $M = P$. By considering P' instead of P, we also get that P' is a minimal prime. Hence height $P \leqslant 2$. This proves (i) and (ii).

Let P be a minimal prime ideal of D. The contraction in D of any proper ideal of D_P, not equal to PD_P is a P-primary ideal of D different from P. Now let P be not a minimal prime ideal. Then there exists a proper prime ideal $P' < P$. By (i) $D_P/P'D_P$ is a one dimensional domain. Choose any proper ideal $T/P'D_P$ of D_P/PD_P, not equal to its maximal ideal, then the contraction of T in D is a P-primary ideal of D, not equal to P. This proves (iii).

LEMMA 10. *Let P be a proper prime ideal of D, satisfying the hypothesis of Lemma 9. Then $P^* = D$, $P^*/P \cong Z/(p)$, for some prime number p, and P is a maximal ideal of D.*

Proof. By Lemma 9, there exists a P-primary ideal $Q \neq P$. Let M be a maximal ideal of D containing P. Theorem 4 yields that,

$$D_M = P^*_{(P^* \cap M)} = Q^*_{(Q^* \cap M)} . \tag{2}$$

Now $P^*/P \cong Z$ or $P^*/P = Z/(p)$, for some prime number p. Let

$P^*/P = Z$. Then for every $n(\neq 0) \in Z$, $n1 \notin P$: consequently $n1 \notin Q$. This yields that $Q^*/Q \cong Z$ and that Q is a prime ideal of Q^*. Then from (2) it follows that Q is a prime ideal of D. This is a contradiction. Hence $P^*/P \cong Z/(p)$ and that P is a maximal ideal of P^*. Consequently (2) yields that $M \cap P^* = P$ and $D_M = P^*_P$. So that $P = M$ and $D/P \cong P^*/P \cong Z/(p)$. Thus P^*/P is a subring of D/P such that both of them have p elements. Hence $P^* = D$ and the lemma follows.

COROLLARY 1. *If P is a proper prime ideal of a domain D, satisfying the hypothesis of Lemma* 9, *then height* $P = 1$.

Proof. If P' is any proper prime ideal of D contained in P, then P' also satisfies the hypothesis of Lemma 9. By Lemma 10, P' is a maximal ideal of D. Hence $P' = P$ and height $P = 1$.

THEOREM 5. *Let P be a proper prime ideal of domain D such that for every nonzero primary ideal Q of D contained in P, D is a flat Q^*-module. Then every nonzero primary ideal Q of D contained in P is P-primary, $D/Q \cong Z/(p^\alpha)$ for some power p^α of a prime number p and $Q^* = D$.*

Proof. By Corollary 1, height $P = 1$. So that $\sqrt{Q} = P$. In case $P = Q$, the result follows from Lemma 10. Let $Q \neq P$. Since D is a flat Q^*-module, by Theorem 4,

$$D_P = Q^*_{(Q^* \cap P)} \; . \tag{3}$$

This equation along with Lemma 10, yields that there exists a prime number p such that $Z/(p) \cong D/P \cong Q^*/Q^* \cap P$. However $Q^*/Q \cong Z/(n)$, for some n, and Q is a $(Q^* \cap P)$-primary ideal of Q^*. Therefore $n = p^\alpha$, for some $\alpha > 2$. Then from (3) $D/Q \cong Q^*/Q \cong Z/(p^\alpha)$: as a consequence we get that $D = Q^*$. This proves the theorem.

Henceforth the domain D will always be assumed to be different from its quotient field. The following corollary is an immediate consequence of the above theorem.

COROLLARY 2. *If D is a flat A^*-module for each of its proper ideals A, then* $\dim D = 1$.

LEMMA 11. *Let D be a domain such that D is a flat A^*-module for each of its proper ideals A. If P_1 and P_2 are two distinct proper prime ideals of D, such that $D/P_1 \cong Z/(p_1)$ and $D/P_2 \cong Z/(p_2)$, then $p_1 \neq p_2$.*

Proof. Suppose that $p_1 = p_2 = p$. Then $p1 \in P_1 \cap P_2 = P_1P_2$. Hence $(P_1P_2)^*/P_1P_2 \cong Z/(p)$ and $N = P_1P_2$ is a maximal ideal of $(P_1P_2)^*$. Consequently $P_1 \cap (P_1P_2)^* = N = P_2 \cap (P_1P_2)^*$. By Theorem 4,

$$D_{P_1} = (P_1P_2)^*_N = D_{P_2}\,.$$

This yields that $P_1 = P_2$. Hence the lemma follows.

THEOREM 6. *A domain D is a (KE)-domain if and only if it is a flat A^*-module for each of its proper ideals A.*

Proof. Let D be a (KE)-domain. By Proposition 1, given any proper ideal A of, $D = A^*$. Then obviously D is a flat A^*-module for each of its proper ideals A.

Conversely let D be a flat A^*-module for each of its proper ideals A. Consider any proper prime ideal P of D. By Theorem 5, P is a maximal ideal and there exists a prime number p such that for any nonzero primary ideal Q of D contained in P, $D/Q \cong Z/(p^\alpha)$ for some $\alpha \geqslant 1$. Consequently $D_P/QD_P \cong Z/(p^\alpha)$, a *PIR* with d.c.c. So that D_P is a discrete valuation ring of rank one. As an immediate consequence we get that every nonzero primary ideal of D contained in P is a power of P and $D/P^\alpha \cong Z/(p^\alpha)$ for every α. Thus $p1 \in P\backslash P^2$. Now for any given proper prime ideal $P' \neq P$, $D/P' \cong Z/(p')$, for some prime number p', which, because of Lemma 11, is not equal to p. So that $p1 \notin P'$. Then using the fact that for any ideal A of D, $A = \bigcap AD_T$, where T runs over all the maximal ideals of D, we get that $P = (p1)$, a principal ideal of D. By Cohen [4, Theorem 2], D is Noetherian. Let A be a proper ideal of D and $A = \bigcap_{i=1}^t Q_i$ be an irredundant decomposition of A into primary ideals. For each i, since $D/Q_i \cong Z/(p_i^{\alpha_i})$, for some prime power $p_i^{\alpha_i}$ and the prime number p_i are all distinct, we get that, $D/A \cong \bigoplus \sum_{i=1}^t D/Q_i \cong \bigoplus \sum_{i=1}^t Z/(p_i^{\alpha_i}) \cong Z/(n)$, where $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_t^{\alpha_t}$. Since the ring $Z/(n)$ is generated by its unity element, it follows that $D = A^*$. Hence by Proposition 1, D is a (KE)-domain.

We now obtain Theorem 2 of [12] as a corollary to the above theorem.

COROLLARY 3. *A domain D is a (KE)-domain if and only if for each proper ideal A of D, one of the following holds:*

(i) *A^* is a Dedekind domain.*

(ii) *A^* is a Prüfer domain.*

(iii) *A^* is a generalized Krull domain.*

(iv) *A^* is an almost Krull domain.*

Proof. If D is a (KE)-domain, then by Lemma 3, D satisfies the given conditions.

Let D satisfy the given conditions. Let A be a proper ideal of D. If A^* satisfies any of the conditions: (i), (iii), and (iv) then for each of its minimal prime ideals P', $A^*_{P'}$ is a rank one valuation ring and A^* is an intersection of these rings. Now AP' is a nonzero ideal of D contained in P'. For $S = A^*\backslash P$, $A^*_{P'} \subset D_S$. Since

$$S \cap AP' = \varnothing ,$$

D is not a field. However $A^*_{P'}$ is a maximal subring of its quotient field. Consequently $D_S = A^*_{P'}$ and $D \subset A^*_{P'}$. Hence $D = A^*$. In this case D is trivially an A^*-flat module. If A^* is a Prüfer domain, then again by Richman [8], D is a flat A^*-module. Hence, by Theorem 6, D is a (KE)-domain.

The following theorem is also an immediate consequence of Theorem 6. It also follows from Lemma 13 given below, and which is analogous to Theorem 2.

THEOREM 7. *A domain D is a (KE)-domain if and only if it is a projective A^*-module for each of its proper ideals A.*

LEMMA 13. *If for a proper ideal A of a domain D, D is a projective A^*-module, then $D = A^*$.*

Proof. As D is a projective A^*-module, by the dual basis theorem for projective modules, there exists a family $\{\sigma_\alpha\}_{\alpha \in \Lambda}$ of elements of $\mathrm{Hom}_{A^*}(D, A^*)$ and a corresponding family $\{d_\alpha\}_{\alpha \in \Lambda}$ of elements of D such that for each $d \in D$, $\sigma_\alpha(d) = 0$, for all but a finite number of values of α, and $d = \sum_\alpha \sigma_\alpha(d) d_\alpha$.

Let $\sigma \in \mathrm{Hom}_{A^*}(D, A^*)$. Consider b, $c \in D$. Choose $a \ (\neq 0) \in A$. Then $\sigma(bc)a = \sigma(bca) = \sigma(b)ca$, since $ca \in A^*$: consequently $\sigma(bc) = \sigma(b)c$. Thus σ is a D-homomorphism. Hence for any

$$d \in D, d = \sum_\alpha \sigma_\alpha(d) d_\alpha = \sum_\alpha \sigma_\alpha(dd_\alpha) \in A^* .$$

This proves that $D = A^*$.

The above lemma does not hold for flat modules, as is evident from the following example.

EXAMPLE 1. Consider the formal power series ring $D = R[[X]]$, over the field R of rational numbers. Its maximal ideal is $M = (X)$. Now $M^* = Z + M \neq D$ and $D = M^*_S$, where S is the set of all nonzero integers. Hence D is a flat M^*-module, but $D \neq M$.

3. The ring $\hat{Z}_{(p)}$. In [11, Example 4], it was shown that for any prime number p, $\hat{Z}_{(p)}$, the p-adic completion of $Z_{(p)}$, is a (KE)-domain. In this section we prove that $\hat{Z}_{(p)}$ is a maximal (KE)-domain, in the sense that if in a (KE)-domain D, which is not a field, some prime number p is not invertible, then D is embeddable in $\hat{Z}_{(p)}$. Some other results on (KE)-domains are also established. The following structure theorem on (KE)-domains was proved in [11, Theorem 14].

THEOREM 8. *Any domain D, which is not a field, is a (KE)-domain if and only if it satisfies the following*:

(i) *There exists a multiplicative subset S of the ring of integers Z, such that Z_S is embeddable in D.*

(ii) *The correspondence $A \leftrightarrow A \cap Z_S$ is one-to-one between the ideals A of D and those of Z_S.*

(iii) *For every proper prime ideal P of D, $D/P \cong Z_S/P \cap Z_S$.*

If a (KE)-domain D satisfies conditions (i) to (iii) of Theorem 8 we say that D is a (KE)-domain associated with Z_S: in that case it is immediate that a prime number p is invertible in D if and only if it is invertible in Z_S.

DEFINITION 1. A (KE)-domain D associated with Z_S is said to be a maximal (KE)-domain associated with Z_S, if there exists no (KE)-domain D' associated with Z_S such that it contains D properly.

THEOREM 9. *Let D be a (KE)-domain, which is not a field and in which some prime number p is not invertible, then D is embeddable in $\hat{Z}_{(p)}$.*

Proof. Let D be associated with Z_S. Since Z_S is a PID of characteristic zero, Theorem 8 yields that D is a PID of characteristic zero. Further as pZ_S is a maximal ideal of Z_S, Theorem 8 also yields that $P = pD$ is a maximal ideal of D such that $D/P \cong Z/(p)$. By Theorem 5, for each $n \geqslant 1$, $D/P^n = Z/(p^n)$ and hence every element of D is of the form $k1 + p^n a$; $k \in Z$, $a \in D$. Consequently there exists a natural homomorphism $\sigma_n \colon D \to Z/(p^n)$ such that

$$\sigma_n(k1 + p^n a) = k + (p^n) .$$

For $m \leqslant n$, we have the natural homomorphism $\pi_n^m \colon Z/(p^n) \to Z/(p^m)$. Then $\{Z/(p^n), \pi_n^m\}$ form a projective system and $\varprojlim Z/(p^n) = \hat{Z}_{(p)}$ [9, Chap. 1, p. 55]. For each n, let $\pi_n \colon \hat{Z}_{(p)} \to Z/(p^n)$ be the canonical mapping. It can be easily seen that $\sigma_m = \pi_n^m \sigma_n$ whenever $m \leqslant n$.

Thus there exists a homomorphism σ of D into $\hat{Z}_{(p)}$ such that $\sigma_n = \pi_n\sigma$ for every n. Since $\bigcap_n \ker \sigma_n = (0)$, σ is a monomorphism. Hence the theorem follows.

THEOREM 10. *Let* $\{D_\alpha, \pi_\alpha^\beta\}_{\alpha,\beta\in\Lambda}$ *be an injective system of* (KE)-*domains associated with the same* Z_S ($\neq$ *the field of rational numbers*). *Then the injective limit* $D = \varinjlim D_\alpha$ *is a* (KE)-*domain associated with* Z_S. (*It is assumed that each of* π_α^β *is a nonzero mapping.*)

Proof. For each $\alpha \in \Lambda$, there exists a homomorphism $\pi_\alpha\colon D_\alpha \to D$ satisfying the following:

(i) $\pi_\alpha = \pi_\beta\pi_\alpha^\beta$ for α, $\beta \in \Lambda$ such that $\alpha \leqslant \beta$.

(ii) $D = \bigcup \pi_\alpha(D_\alpha)$

(iii) If for some α, there exists $x_\alpha \in D_\alpha$ such that $\pi_\alpha(x_\alpha) = 0$, then there exists $\beta \geqslant \alpha$ such that $\pi_\alpha^\beta(x_\alpha) = 0$.

Using the above properties, it follows that D is an integral domain. As $\pi_\alpha^\beta \neq 0$, $\pi_\alpha^\beta(1) = 1$. We get that π_α^β is an identity map on Z_S. Consequently each π_α is also identity map on Z_S. Consider any $x_\alpha(\neq 0) \in D_\alpha$. As seen in the proof of Corollary 3 in [11], $x_\alpha = n_\alpha u_\alpha$ for some $n_\alpha \in Z$ and a unit u_α in D_α: thus $\pi_\alpha(x_\alpha) = n_\alpha\pi_\alpha(u_\alpha)$. Clearly $\pi_\alpha(u_\alpha)$ is a unit in D. It follows that every element of D is of the type nu; $n \in Z$ and u a unit in D. Consider any proper ideal A of D. Now for every α, $A_\alpha = \pi_\alpha^{-1}(A)$ is a proper ideal of D and $A = \bigcup \pi_\alpha(A_\alpha)$. Thus $A^* = \bigcup \pi_\alpha(A_\alpha^*) = \bigcup \pi_\alpha(D_\alpha) = D$. Hence by Proposition 1, D is a (KE)-domain. Since every prime number invertible in Z_S is invertible in every D_α, we get it is also invertible in D. Conversely if any prime number p is invertible in D, then the above properties of D imply that p is invertible in some D_α and hence p is invertible in Z_S. This shows that D is associated with Z_S.

We end this paper with a few remarks.

1. Some of the lemmas, for example Lemmas 4 to 8, and 12 can be proved by replacing A^* by any Noetherian subring of D, containing a nonzero ideal of D and keeping the other hypotheses unchanged. It is not clear whether in that case, we obtain $B = D$, as in Theorem 2.

2. Theorems 9 and 10 can be proved in more general settings. To explain the point, let T be a fixed Noetherian domain, which is not a field. Let us call a domain D containing T lattice equivalent to T if it has the following properties:

(i) $A \leftrightarrow A \cap T$, is a one-to-one correspondence between the ideals A of D and those of T.

(ii) For any proper ideal A of D, $D = A + T$.

Take any proper prime ideal P of T. Then as in Theorem 8, it can be shown that D is embeddable in $\hat{T}_P$, the PT_P-adic completion of T_P. In Theorem 9, we had $T = Z_S$. In Theorem 10, if we replace each D_α by a domain lattice equivalent to a fixed Noetherian domain T and let each π_α^β be identity on T, then their injective limit is also lattice equivalent to T. The only reason for not proving Theorems 9 and 10 in this more general setting is that the paper is essentially concerned with (KE)-domains.

3. By Theorem 9, given a Z_S (not equal to the field of rational numbers), all (KE)-domains associated with Z_S can be regarded as subrings of a fixed $\hat{Z}_{(p)}$. It can be easily seen that the family of all (KE)-domains associated with the same Z_S is inductive. Hence by Zorn's lemma it has maximal members. It remains open whether any two maximal (KE)-domains associated with a Z_S are isomorphic or not.

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