Pacific Journal of Mathematics

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MILTON DON ULMER

Vol. 46, No. 2 December 1973

PACIFIC JOURNAL OF MATHEMATICS
Vol. 46, No. 2, 1973

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Let $X = \prod_{\alpha \in A} X_\alpha$ be a product space and $p \in X$. For each ordinal γ the Σ-space $\Sigma_\gamma(p)$ is given by: $\Sigma_\gamma(p) = \{x \in X\colon \text{card}(\{\alpha \in A\colon x_\alpha \neq p_\alpha\}) < \aleph_\gamma\}$. It is shown that under various hypotheses on X, each continuous real-valued function on $\Sigma_\gamma(p)$ extends continuously over X. A counterexample is constructed to show these hypotheses cannot be weakened in various ways.

Preliminaries. Let $X = \prod_{\alpha \in A} X_\alpha$ be an infinite product space and p be some point in X. Following [8], for each ordinal γ the Σ-space $\Sigma_\gamma(p)$ is the subspace which consists of all points differing from p on fewer than $\aleph_\gamma$ coordinates. Recall that a subspace Y is C-embedded in X provided each continuous real-valued function on Y extends continuously over X. The set of continuous real-valued functions on a space Y shall be denoted by $C(Y)$.

Much work has been done to determine when each continuous real-valued function on a subset of a product space depends on countably many coordinates. (See [3] or [8] for references and a discussion.) In the first section we shall apply these results to the study of C-embedded Σ-spaces. In the second section we show there are many more interesting situations in which each Σ-subspace is C-embedded. In the final section an example is constructed showing that the results of the previous sections cannot be improved in various ways.

1. Functions depending on few coordinates.

The problem of whether a subspace is C-embedded is often solved by showing something stronger. For instance, to show that the space Ω of countable ordinals is C-embedded in the compact space $\Omega^* = \{\gamma\colon \gamma \leqq \aleph_1\}$ one usually shows that each function in $C(\Omega)$ is constant on a tail. Similarly, if for a Σ-space $\Sigma_1(p)$ it is known that each function in $C(\Sigma_1(p))$ factors through a countable subset of A, then $\Sigma_1(p)$ is certainly C-embedded in the product. Mazur used this approach in [7] under the hypothesis that each X_α be second countable.

In [8] it was shown that the best possible results of this kind involve the property pseudo-$\aleph$-compactness. Recall that for any infinite cardinal $\aleph$, a space is pseudo-$\aleph$-compact provided each locally finite collection of open subsets has cardinality less than $\aleph$. From [8; Theorem 3.2] we know that if $\aleph_\gamma$ is not the supremum of a countable set of smaller cardinals and if $\Sigma_\gamma(p)$ is pseudo-$\aleph_\gamma$-compact, then each continuous real-valued function on $\Sigma_\gamma(p)$ depends on fewer than $\aleph_\gamma$ coordinates, and hence $\Sigma_\gamma(p)$ is C-embedded in X.

However, we can say more than this. Suppose $\aleph_\gamma$ is countably accessible and $\Sigma_\gamma(p)$ is pseudo-$\aleph_\gamma$-compact. By [8; Proposition 3.4] we know $C(\Sigma_\gamma(p))$ may contain functions which depend on $\aleph_\gamma$ coordinates. It turns out, that in spite of this $\Sigma_\gamma(p)$ is C-embedded in X.

NOTATION 1.1. *For any subset B of A we let Π_B denote the* canonical projection of $\prod_{\alpha\in A} X_\alpha$ onto $\prod_{\alpha\in B} X_\alpha$. For any point $x \in X$ we set $x_B = \Pi_B(x)$, and we denote by x^B the point in X defined by: $x^B_\alpha = x_\alpha$ provided $\alpha \in B$ and $x^B_\alpha = p_\alpha$ otherwise. For any nonempty basic open set $U \subset X$ we denote by $R(U)$ the finite set of coordinates on which the projection is not the entire factor. Let $A(\gamma)$ denote the family of all subsets of A whose cardinality is less than $\aleph_\gamma$, and let N denote the set of positive integers.

LEMMA 1.2. *let $\aleph$ be any infinite cardinal number, Y any topological space, and E any closed neighborhood of the diagonal in $Y \times Y$. Let Z be any pseudo-$\aleph$-compact subspace of X containing $\Sigma_0(p)$. If f is a continuous function from Z to Y, then there exists a subset C of A such that card $(C) < \aleph$, and such that for any points x and y in Z where $x_c = y_c$, we have $(f(x), f(y)) \in E$.*

Proof. This is a straightforward adaptation of [4; proof of the sufficiency in Theorem 1].

We are now in position to state a theorem which drops the restriction on the cardinal number $\aleph_\gamma$.

THEOREM 1.3. *If $\Sigma_\gamma(p)$ is pseudo-$\aleph_\gamma$-compact, then it is C-embedded in X.*

Proof. Let $f \in C(\Sigma_\gamma(p))$. By Lemma 1.2 there exists, for each positive integer n, a subset $B_n \in A(\gamma)$ such that whenever $y_{B_n} = z_{B_n}$, then $|f(y) - f(z)| < 1/n$.

To extend f over all of X, let $x \in X - \Sigma_\gamma(p)$. Define $f(x) = \lim_{n\to\infty} f(x^{B_n})$. Since each x^{B_n} is in $\Sigma_\gamma(p)$ and since, by the choice of B_n, the sequence $\{f(x^{B_n}) \colon n \in N\}$ is Cauchy, the limit exists. It suffices to check that f is continuous at $x \in X$.

Fix $\varepsilon > 0$. Let n be any positive integer for which $1/n < \varepsilon/3$. Since x^{B_n} is in $\Sigma_\gamma(p)$, there exists a basic open neighborhood U of x^{B_n} such that for any point y in $U \cap \Sigma_\gamma(p)$ we have $|f(y) - f(x^{B_n})| < \varepsilon/3$. U is of the form: $U = \prod_{\alpha\in R(U)} U_\alpha \times \prod_{\alpha\notin R(U)} X_\alpha$. Let $F = R(U) \cap B_n$, and let $V = \prod_{\alpha\in F} U_\alpha \times \prod_{\alpha\notin F} X_\alpha$.

We will show that V is an ε-neighborhood of x. Let y be any point in V. Since y^{B_n} belongs to U, we have $|f(y^{B_n}) - f(x^{B_n})| < \varepsilon/3$.

Also, $|f(y^{B_n}) - f(y)| \leqq 1/n < \varepsilon/3$, and $|f(x)^{B_n} - f(x)| \leqq 1/n < \varepsilon/3$. Thus $|f(x) - f(y)| \leqq |f(x) - f(x^{B_n})| + |f(x^{B_n}) - f(y^{B_n})| + |f(y^{B_n}) - f(y)| < \varepsilon$. Thus V is the desired neighborhood of x.

The reader is referred to [8; Proposition 3.3] for conditions which insure that $\Sigma_\gamma(p)$ is pseudo-$\aleph_\gamma$-compact.

2. **The main theorem.** In the previous section we found a class of Σ-spaces which were C-embedded because the continuous functions were almost completely determined by sufficiently small subsets of A. In this section we shall show that for any infinite cardinal $\aleph$, there are C-embedded Σ_1-spaces which allow functions which depend on $\aleph$ coordinates. The most immediate example is an appropriate product of discrete spaces. By [8; Theorem 3.4] a product of sufficiently large discrete spaces will admit real-valued continuous functions which depend on any preconceived number of coordinates. However, Theorem 2.2 below yields that every Σ_1-subspace of a product of discrete spaces is C-embedded.

DEFINITION 2.1. A point y in a topological space is said to be a P-point provided the intersection of any countable collection of neighborhoods of y is also a neighborhood of y. A space is said to be a P-space provided each point in the space is a P-point. The reader is referred to [5] for a detailed treatment of this property.

THEOREM 2.2. *If* (i) $\Sigma_\gamma(p)$ *is pseudo-*$\aleph_\gamma$*-compact; or*

(ii) $\aleph_\gamma$ *is a regular uncountable cardinal, and for each index* $\alpha \in A$ *and each point* $x_\alpha \in X_\alpha$, x_α *has a neighborhood base of cardinality less than* $\aleph_\gamma$; *or*

(iii) *for each* $\alpha \in A$, X_α *is a P-space, and* $\gamma > 0$; *then* $\Sigma_\gamma(p)$ *is C-embedded in* X.

Notice that condition (i) places no restrictions on $\aleph_\gamma$. Also, condition (iii) makes no mention of γ other than $\gamma > 0$. Thus $\Sigma_1(p)$ is C-embedded provided X is any product of P-spaces.

Since case (i) has already been proved and since it suffices to prove case (iii) for $\gamma = 1$, we may assume in what follows that $\aleph_\gamma$ is regular and uncountable.

Let $f \in C(\Sigma_\gamma(p))$. Since each Σ-subspace is dense in X, from [2; Theorem 5.3, page 216], to show that f can be extended continuously to all of X it suffices to show that f extends continuously to each space of the form $\Sigma_\gamma(p) \cup \{x\}$, with $x \in X$. In order to do this we first need some lemmas. These lemmas make no use of the hypotheses of Theorem 2.2.

LEMMA 2.3. *Let* Γ *be a simply ordered set with no countable* *cofinal subset. A net in the real numbers* $\boldsymbol{R}$, *directed by* Γ, *must have a cluster point in* $\boldsymbol{R}$.

Proof. This is an easy consequence of the fact that $\boldsymbol{R}$ is Lindelöf.

The following lemma tells us how to extend f to the arbitrary point $x \in X$.

LEMMA 2.4. *There exists a countable subset* S *of* A *such that* $f(x^S) = f(x^T)$ *whenever* T *is countable and* $S \subset T \subset A$.

Proof. Fix $\delta > 0$. We will show that there exists a countable subset B_δ of A such that for all countable subsets $B \supset B_\delta$ we have $|f(x^B) - f(x^{B_\delta})| < \delta$.

Suppose no such B_δ exists. Then we can choose a transfinite sequence of countable subsets of A, $\{E_\tau: \tau < \aleph_1\}$, such that for each countable ordinal τ, $E_\tau \supset \bigcup_{\sigma<\tau} E_\sigma$ and $|f(x^{E_\tau}) - f(x^{E_{\tau+1}})| \geqq \delta$.

Now $\{f(x^{E_\tau}): \tau < \aleph_1\}$ is a net in the real numbers, and this net is directed by the countable ordinals. By Lemma 2.3, and the fact that consecutive elements of the net are separated by at least δ, it is clear that this net must have at least two cluster points r_1 and r_2.

Since the real numbers satisfy the first axiom of countability, we can choose a subsequence $\{f(x^{E_{\tau_n}}): n \in N\}$ of the net with the following properties: $E_{\tau_1} \subset E_{\tau_2} \subset \cdots \subset E_{\tau_n} \subset E_{\tau_{n+1}} \subset \cdots$, and $\lim_{n\to\infty} f(x^{E_{\tau_{2n}}}) = r_1$ while $\lim_{n\to\infty} f(x^{E_{\tau_{2n+1}}}) = r_2$.

The set $E = \bigcup_{n \in N} E_{\tau_n}$ is countable, and hence $x^E \in \Sigma_1(p)$. However, since $\{x^{E_{\tau_n}}: n \in N\}$ is a net in $\Sigma_1(p)$ converging to x^E we must have by the continuity of f that $f(x^E) = \lim_{n\to\infty} f(x^{E_{\tau_n}})$, which clearly fails.

The set $S = \bigcup_{m \in N} B_{1/m}$ satisfies the lemma.

Clearly if f has a continuous extension to $\Sigma_\gamma(p) \cup \{x\}$, then the value of the extended function at the point x must be $f(x^S)$. Define $f(x) = f(x^S)$.

DEFINITION 2.5. Let $\varepsilon > 0$. A subset B of A is said to be ε_γ-cofinal for a point $y \in \Sigma_\gamma(p)$ provided card $(B) < \aleph_\gamma$, and provided the following condition is satisfied: Given any set $B' \in A(\gamma)$ which is disjoint from B, there exists a basic open ε-neighborhood of y which is not restricted on B'.

Let S be as in Lemma 2.4.

LEMMA 2.6. *Given any* $\varepsilon > 0$, *there exists a set* T_ε *containing* S *such that* T_ε *is* ε_γ-*cofinal for* x^{T_ε}.

Proof. Suppose that for some $\varepsilon > 0$ no such T_ε exists. We can then choose a sequence $S_1 \subset S_2 \cdots \subset S_n \subset \cdots$ of sets in $A(\gamma)$ containing

S such that for each basic open ε-neighborhood U of x^{S_n} we have $R(U) \cap (S_{n+1} - S_n) \neq \emptyset$.

Take $S_\infty = \bigcup_{n \in N} S_n$. Since $\aleph_\gamma$ is regular and uncountable, we have $S_\infty \in A(\gamma)$, and hence $x^{S_\infty} \in \Sigma_\gamma(p)$. Thus there exists a basic open ε-neighborhood U of x^{S_∞}. However, since $R(U)$ is finite, U is an ε-neighborhood of all but finitely many of the points in the set $\{x^{S_n}: n \in N\}$; and for any such x^{S_k}, we have $R(U) \cap (S_{k+1} - S_k) = \emptyset$. Since this contradicts the construction, there must exist a set T_ε containing S which is ε_γ-cofinal for x^{T_ε}.

LEMMA 2.7. *There exists a set $T \in A(\gamma)$ containing S and such that T is ε_γ-cofinal for x^T for all $\varepsilon > 0$.*

Proof. Let $T_{1/n}$ be $1/n_\gamma$-cofinal for $x^{T_{1/n}}$. Take $T = \bigcup_{n \in N} T_{1/n}$.

LEMMA 2.8. *Given any $\varepsilon > 0$, there exists a finite set $F_\varepsilon \subset T$ which is ε_γ-cofinal for x^T.*

Proof. Since T is ε_γ-cofinal for x^T, we can choose inductively a transfinite sequence $\{U_\tau: \tau < \aleph_\gamma\}$ of basic open ε-neighborhoods of x^T with the property that for any $\sigma < \tau < \aleph_\gamma$, we have $R(U_\sigma) \cap R(U_\tau) \subset T$. Let $\mathscr{F}$ denote the family of finite subsets of T.

Since $\operatorname{card}(\mathscr{F}) = \operatorname{card}(T) < \aleph_\gamma$ and $\aleph_\gamma$ is regular, there must be some finite $F_\varepsilon \in \mathscr{F}$ such that $F_\varepsilon = R(U_\tau) \cap T$ for all τ in some $\aleph_\gamma$-fold subset Γ of $\aleph_\gamma$.

To see that F_ε is ε_γ-cofinal for x^T, let $T' \in A(\gamma)$ be any set for which $F_\varepsilon \cap T' = \emptyset$. Since $\{R(U_\tau) - F_\varepsilon: \tau \in \Gamma\}$ is a collection of $\aleph_\gamma$ pairwise disjoint subsets of A, there is an index $\sigma \in \Gamma$ for which $R(U_\sigma) \cap T' = \emptyset$. This proves the lemma.

We are now in a position to prove Theorem 2.2.

Proof of Theorem. Recall that we defined $f(x) = f(x^S)$, and that $f(x^S) = f(x^T)$, where S and T are as in the preceding lemmas.

Fix $\varepsilon > 0$. Let $\delta = \varepsilon/2$ and let the finite set $F \subset T$ be δ_γ-cofinal for x^T.

We will now show that under the conditions of (ii) or (iii), there is a basic open ε-neighborhood U of x^T which is only restricted on F. Since $f(x) = f(x^T)$, and since $x_F = x^T_F$, we will have that U is also an ε-neighborhood of x.

Case (ii): Let $\{V_\tau: \tau \in \Gamma\}$ be a neighborhood base for x_F in the subproduct $\prod_{\alpha \in F} X_\alpha$. Furthermore, choose this base so that $\operatorname{card}(\Gamma) < \aleph_\gamma$.

Since F is ε_γ-cofinal for x^T, we can choose inductively a family $\{U_\lambda: \lambda < \aleph_\gamma\}$ of basic open ε-neighborhoods of x^T for which the collection

$\{R(U_\lambda) - F: \lambda < \aleph_\gamma\}$ is a family of $\aleph_\gamma$ pairwise disjoint finite subsets of $A - F$.

Since card $(\Gamma) < \aleph_\gamma$, there is an index $\tau_0 \in \Gamma$ such that $\Pi_F(U_\lambda) \supset V_{\tau_0}$ for all λ in Λ, where Λ is some $\aleph_\gamma$-fold subset of $\aleph_\gamma$.

Now, since a point y is in $\Sigma_\gamma(p)$ if and only if the set $\{\alpha: y_\alpha \neq p_\alpha\}$ is of smaller cardinality than $\aleph_\gamma$, we have that $y \in \Sigma_\gamma(p)$ and $y_F \in V_{\tau_0}$ imply that y belongs to infinitely many $U_\lambda, \lambda \in \Lambda$. Thus $V_{\tau_0} \times \prod_{\alpha \notin F} X_\alpha$ is an ε-neighborhood for x^T, and hence also for x.

Case (iii): Suppose that the intersection of any countable collection of neighborhoods of x_F is also a neighborhood of x_F.

Since F is δ_γ-cofinal for x^T, we can choose a countable family $\{U_n: n \in N\}$ of basic open δ-neighborhoods of x^T such that the collection $\{R(U_n) - F: n \in N\}$ is a family of pairwise disjoint subsets of $A - T$.

Since each $\Pi_F(U_n)$ is a neighborhood of x_F in $\prod_{\alpha \in F} X_\alpha$, $V = \bigcap_{n \in N} \Pi_F(U_n)$ is also a neighborhood of x_F. We will show that $U = V \times \prod_{\alpha \notin F} X_\alpha$ is an ε-neighborhood of x.

Suppose there exists a point $y \in \Sigma_\gamma(p) \cap U$ for which $|f(y) - f(x)| \geqq 2\delta = \varepsilon$. Since y belongs to $\Sigma_\gamma(p)$, there exists a basic open neighborhood W of y such that for any point $z \in W \cap \Sigma_\gamma(p)$, we have $|f(z) - f(y)| < \delta$.

Since $\{R(U_n) - F: n \in N\}$ is an infinite collection of pairwise disjoint subsets of A, and since $R(W)$ is finite, there exists an integer m such that $R(U_m) \cap R(W) \subset F$. Define the point $z \in \Sigma_\gamma(p)$ by: $z_\alpha = y_\alpha$ provided $\alpha \in R(W) \cup F$, and $z_\alpha = x^T_\alpha$ otherwise.

Clearly z is in $\Sigma_\gamma(p)$ since x^T is in $\Sigma_\gamma(p)$. Since $z_\alpha = y_\alpha$ for all $\alpha \in R(W)$, we have that $z \in W$. But $y_F \in V$, and $V \subset \Pi_F(U_m)$. Thus $z_F = y_F$ is in $\Pi_F(U_m)$. Also, $z_\alpha = x^T_\alpha$ for all $\alpha \in R(U_m) - F$. Thus z is in U_m as well as W. Since U_m is a δ-neighborhood of x^T, we have: $\varepsilon \leqq |f(y) - f(x)| \leqq |f(y) - f(z)| + |f(z) - f(x^T)| < \delta + \delta = \varepsilon$.

This contradiction proves that U is an ε-neighborhood of x. Thus the theorem is complete.

The following corollary gives a wealth of examples of spaces which satisfy conditions (ii) and (iii), but not necessarily condition (i) of Theorem 2.2.

COROLLARY 2.9. *If X is a product of discrete spaces, then every Σ_1-subspace is C-embedded.*

Clearly a discrete space of cardinality $\aleph$ fails to be pseudo-$\aleph$-compact. Thus, given any cardinal number $\aleph$, by taking a product of suitable discrete spaces, one can construct a product space and a Σ_1-subspace which admit continuous functions depending on more than $\aleph$ coordinates. Since these Σ-spaces are C-embedded, condition (i) of Theorem 2.2 misses many of the most interesting cases.

Theorem 2.2 also yields a proof of the fact that C-embedding is equivalent to C^*-embedding. (Recall that a subspace is C^*-embedded provided each bounded continuous real valued function extends over the entire space.) This could also be shown by using [5; Theorem 1.18].

COROLLARY 2.10. *If $Z \subset X$ contains $\Sigma_1(p)$, then Z is C-embedded in X if and only if Z is C^*-embedded in X.*

Proof. Suppose Z is C^*-embedded in X, and suppose f is an arbitrary function in $C(Z)$. Let $x \in X$. Considering X to be a product of discrete spaces, we know that f extends continuously to $Z \cup \{x\}$, and hence, with this extension $f(x) < \infty$. Now, since Z is C^*-embedded, the extended function is continuous in the original topology.

Since any C-embedded subspace is always C^*-embedded, the corollary is proved.

If X is the product of nonmeasurable discrete spaces, then X is realcompact. Since in this case we know that $\Sigma_1(p)$ is C-embedded, the following corollary is clear.

COROLLARY 2.11. *If X is the product of discrete spaces of nonmeasurable cardinality, then for any point p in X, the Hewitt realcompactification of $\Sigma_1(p)$ coincides with X.*

One also has the following more general, but less elegant, corollary:

COROLLARY 2.12. *If X is a realcompact product space, and X and $\Sigma_\gamma(p)$ satisfy either (i), (ii), or (iii) of Theorem 2.2, then the Hewitt realcompactification of $\Sigma_\gamma(p)$ coincides with X.*

We let $\upsilon(Y)$ denote the Hewitt realcompactification of Y. To see that the Hewitt realcompactification of a Σ-space is often larger than the product space itself, take the coordinate spaces to be noncompact, pseudocompact spaces such as the space Ω of countable ordinals with the usual topology. In this case the coordinate spaces $X_\alpha = \Omega_\alpha$ are not realcompact, but for any point $p \in X = \prod_{\alpha \in A} \Omega_\alpha$, $\Sigma_1(p)$ is C-embedded since these coordinate spaces are first-countable. Thus we have $\Sigma_1(p) \subset \prod_{\alpha \in A} \Omega_\alpha \subset \upsilon(\Sigma_1(p))$, where all the inclusions are proper.

We can say more than this. Using various results from Glicksberg's paper [4], we see first, by [4; Theorem 4], that since Ω is locally compact, X must be pseudocompact, and hence, by [4; Theorem 1], that the Stone-Čech compactification of this product is the product of the Stone-Čech compactifications. But since X is pseudocompact, we have $\upsilon(X) = \beta(X)$. Thus: $\Sigma_1(p) \subset \prod_{\alpha \in A} \Omega_\alpha \subset \upsilon(\Sigma_1(p)) = \upsilon(\prod_{\alpha \in A} \Omega_\alpha) =$

$\beta(\prod_{\alpha\in A}\Omega_\alpha) = \prod_{\alpha\in A}\beta(\Omega_\alpha) = \prod_{\alpha\in A}\Omega_\alpha^* = \prod_{\alpha\in A}\upsilon(\Omega_\alpha)$, where $\Omega^* = \Omega \cup \aleph_1$ with the usual topology.

3. The counterexample. In the last section we saw that every Σ_1-space in a product of discrete spaces is C-embedded. In fact, if each coordinate space is first-countable, then every Σ_1-subspace is C-embedded.

Now, if f is a continuous function from any space into the real numbers, then f is also continuous on the topology which is induced by taking the family $\{f^{-1}((a, b)): a \text{ and } b \text{ rational}\}$ as a basis. Clearly this topology is first-countable since it is second-countable. Thus, one might be tempted to conjecture that every Σ_1-space is C-embedded. However, since the sets in this basis may fail to be basic open subsets in the product topology, this weaker topology may be extremely complex.

In fact, we shall see that given any cardinal number $\aleph_\gamma$, there is a product space X_γ and a Σ_γ-subspace which fails to be C-embedded. What is more, the example can be constructed so that each coordinate space is discrete everywhere except at one point, and all but one of the coordinate spaces will be compact.

Thus, in terms of Theorem 2.2 (i), all but one of the coordinate spaces will be well behaved.

Similarly, one can construct a product space in which all but one of the coordinate spaces are first-countable, and yet the product space will contain a Σ_γ-space which is not C-embedded. Also, one might conjecture that condition (ii) of Theorem 2.2 could be weakened. Rather than requiring that each point in each coordinate space admit a neighborhood basis of cardinality less than $\aleph_\gamma$, one might hope that it would be sufficient to require only that each point in each coordinate space is the intersection of fewer than $\aleph_\gamma$ neighborhoods. It turns out, however, that the example given below can be altered so that every point in each coordinate space is a G_δ-point, and yet there exists a Σ_γ-subspace which fails to be C-embedded. Since $\aleph_\gamma$ was an arbitrary cardinal number, and G_δ-points are those points which are the intersection of a countable set of neighborhoods, this conjecture fails completely.

Finally, by altering the counterexample in another way, it is possible to have every coordinate space except one be a P-space, and to have the one special space be discrete everywhere except one point.

The construction of the product space X_γ employs the concept of a regular ultrafilter on a set of cardinality $\aleph_\gamma$. The following definition is a special case of a definition given by Keisler in [6].

DEFINITION 3.1. An ultrafilter on a set of cardinality $\aleph$ is said to be regular provided there exists an $\aleph$-fold subfamily $\mathscr{F}$ such that any infinite subfamily of $\mathscr{F}$ has empty intersection.

We shall be most interested in the subfamily $\mathscr{F}$.

The existence of regular ultrafilters is well known. The following construction is particularly well suited for our later use.

CONSTRUCTION 3.2. Let $\aleph_\gamma$ be any infinite cardinal, and let B be an index set of cardinality $\aleph_\gamma$. For each $\beta \in B$, let N_β be a copy of the positive integers. Let A^* be another index set of cardinality $\aleph_\gamma$. For each positive integer k, let $\mathscr{P}(k)$ denote the family of all k-element subsets of A^*. Let I_k be any bijection from $\{k_\beta : \beta \in B\}$ onto $\mathscr{P}(k)$. For each $\alpha \in A^*$, set $D_\alpha = \{k_\beta : \alpha \in I_k(k_\beta)\}$, and finally set $\mathscr{F} = \{D_\alpha : \alpha \in A^*\}$.

To see that $\mathscr{F}$ has the finite intersection property, let F be any finite subset of A^*. Suppose F contains k elements; then F belongs to $\mathscr{P}(k)$, and there is an element $\beta \in B$ for which $I_k(k_\beta) = F$. But then $k_\beta \in \bigcap_{\alpha \in F} D_\alpha$.

To see that the intersection of any infinite subfamily of $\mathscr{F}$ is empty, simply notice that for each index $\beta \in B$, k_β belongs to exactly k of the sets in $\mathscr{F}$.

It is also clear from this property of $\mathscr{F}$ that each member of $\mathscr{F}$ must have cardinality $\aleph_\gamma$.

We will now construct the product space $X_\gamma = \prod_{\alpha \in A} X_\alpha$ and the Σ_γ-subspace which is not C-embedded.

EXAMPLE 3.3. Let A be the index set which contains A^* and one other point, say 0. Let $\mathscr{F}$ be the family of subsets of $D_0 = \bigcup_{\beta \in B} N_\beta$ which we constructed in 3.2.

Let $X_0 = D_0 \cup \{\infty_0\}$, where X_0 has the discrete topology everywhere except at the special point ∞_0. Let the family $\{D_\alpha \cup \{\infty_0\} : \alpha \in A^*\}$ be a subbasis for the neighborhood system of the point ∞_0 in X_0.

For each index $\alpha \in A^*$, let $X_\alpha = D_\alpha \cup \{\infty_\alpha\}$ where X_α is the one-point compactification of the discrete space D_α. Finally, set $X_\gamma = \prod_{\alpha \in A} X_\alpha$, and denote the induced product topology by T.

Now in this product space every coordinate space except one is compact. Before showing the existence of a Σ_γ-subspace which is not C-embedded, we shall construct a similar product in which every coordinate space except one is first-countable.

EXAMPLE 3.4. Let the sets A, X_0, and X_α for α in A^* be the same as in the previous example. Let X_0 have the same topology as above, and for each coordinate space X_α, $\alpha \neq 0$, let X_α have the discrete topology everywhere except the point ∞_α. For each integer n, define $V(\alpha, n) = \{k_\beta : \alpha \in I_k(k_\beta) \text{ and } k > n\} \cup \{\infty_\alpha\}$. Let the family $\{V(\alpha, n) : n \in N\}$ be a neighborhood system for the point ∞_α in X_α, and let $X_\gamma =$

$\prod_{\alpha \in A} X_\alpha$ have the product topology T' induced by these new topologies. Notice that each coordinate space other than X_0 is first countable, and hence each point in these spaces is a G_δ-point, and notice that each point in X_0 is a G_δ-point (by the construction on $\mathscr{F}$)!

Finally, we shall give the product a topology in which every coordinate space except X_0 is a P-space.

EXAMPLE 3.5. Let X_0 retain its previously defined topology. For each element $\alpha \in A^*$, let X_α have the discrete topology everywhere except at the point ∞_α. A basic open neighborhood of the point ∞_α shall be any set of the form $X_\alpha - C$, where C is a countable subset of D_α. Now each X_α, $\alpha \neq 0$, is a P-space.

To prevent each X_α, $\alpha \neq 0$, from being a discrete space under this topology, we will require that the set D_0, and hence each subset D_α, be uncountable.

Let this topology on the product space $X_\gamma = \prod_{\alpha \in A} X_\alpha$ be denoted by T''.

Now we have three different topologies on the same product X_γ. We will define a single Σ_γ-subspace which fails to be C-embedded in each of these topologies.

Since each coordinate space was constructed primarily from a subset of D_0, there are many identifications we can make. In particular, if x is a point in the product X_γ, and α is an element of A, then not only is x_α an element of X_α, but it can also be considered to be a point in many of the other coordinate spaces. This is of course an abuse of notation, but hopefully the context will make the situation more comprehensible than any additional notation.

EXAMPLE 3.6. Let $p \in X_\gamma$ be any point such that $p_\alpha \neq \infty_\alpha$ for each $\alpha \in A$. Define the function f from $\Sigma_\gamma(p)$ into $\boldsymbol{R}$ by:

$$f(x) = \begin{cases} 1 & \text{provided } x_0 \neq \infty_0, \text{ and } x_0 = x_\alpha \text{ whenever } x_0 \in D_\alpha\,; \\ 0 & \text{otherwise.} \end{cases}$$

We must check that f is continuous on $\Sigma_\gamma(p)$ in each of the three topologies. However, since T is a coarser topology than either T' or T'', it suffices to check the continuity of f on $\Sigma_\gamma(p)$ with the relative topology induced by (X_γ, T). To this end, let x be an arbitrary point in $\Sigma_\gamma(p)$.

Suppose first that $f(x) = 1$. Let F be the finite set $\{\alpha: x_0 \in D_\alpha\}$. Define $V = \prod_{\alpha \in F} \{x_\alpha\} \times \prod_{\alpha \notin F} X_\alpha$. Clearly V is a basic open neighborhood of x and for any $y \in V \cap \Sigma_\gamma(p)$, $f(y) = 1$.

Suppose now $f(x) = 0$. If $x_0 \neq \infty_0$, there must be some $\alpha' \in A^*$ such that $x_0 \in D_{\alpha'}$ but $x_0 \neq x_{\alpha'}$. Define $V = \{x_0\} \times \{x_{\alpha'}\} \times \prod_{\alpha \neq 0, \alpha'} X_\alpha$. If

$x_0 = \infty_0$, let $\alpha' \in A^*$ be any index for which $x_{\alpha'} \neq \infty_{\alpha'}$ and let $D_{\alpha''} \in \mathscr{F}$ be any set not containing $x_{\alpha'}$. Set $U = (D_{\alpha'} \cap D_{\alpha''}) \cup \{\infty_0\}$. Define $V = U \times \{x_{\alpha'}\} \times \prod_{\alpha \neq 0, \alpha'} X_\alpha$. By either definition, V is a basic open neighborhood of x and for any $y \in V \cap \Sigma_\gamma(p)$, $f(y) = 0$. Hence f is a continuous function on $\Sigma_\gamma(p)$ in the relative topology induced by (X_γ, T).

Next we will show that f cannot be extended to a continuous function on the entire product space with either the topology T, T', or T''. In particular, we will show that f cannot be extended continuously to the point $q = \prod_{\alpha \in A} \{\infty_\alpha\}$.

To see this, let V be a basic open neighborhood of q in (X_γ, T), (X_γ, T'), or (X_γ, T''). Since V is restricted on only finitely many coordinates, there must be a point $y \in V \cap \Sigma_\gamma(p)$ such that $y_0 = \infty_0$. Hence $f(y) = 0$.

Thus it suffices to find a point $z \in V \cap \Sigma_\gamma(p)$ such that $f(z) = 1$.

Case (i). V is open in T. Since $T \subset T'$, and $T \subset T''$, both Case (ii) and Case (iii) serve to demonstrate Case (i). Since Case (ii) is more difficult, we will do Case (iii) first.

Case (iii). V is open in T''. In this case, as we remarked in Example 3.5, $\aleph_\gamma$ is taken to be uncountable. V is of the form: $V = V_0 \times \prod_{\alpha \in F} V_\alpha \times \prod_{\alpha \notin F \cup \{0\}} X_\alpha$, where each V_α, $\alpha \in F$, has countable complement C_α in X_α. Since V_0 must be uncountable, $V_0 - (\bigcup_{\alpha \in F} C_\alpha)$ is infinite. Let d be any point in $V_0 - (\bigcup_{\alpha \in F} C_\alpha)$ other than ∞_0, and let F' be the set of all indices $\alpha \in A$ for which $d \in D_\alpha$. Define the point $z \in \Sigma_\gamma(p) \cap V$ by: $z_\alpha = d$ provided $\alpha \in F'$, and $z_\alpha = p_\alpha$ otherwise.

By the definition of f we have $f(z) = 1$. Thus f admits no continuous extension over all of (X_γ, T'').

Case (ii). V is open in T'. In this case V is of the form: $V = V_0 \times \prod_{\alpha \in F} V(\alpha, n_\alpha) \times \prod_{\alpha \notin F \cup \{0\}} X_\alpha$, where $V_0 = (\bigcap_{\alpha \in G} D_\alpha) \cup \{\infty_0\}$, and G is a finite subset of A^*.

We claim now that $(\bigcap_{\alpha \in F} V(\alpha, n_\alpha)) \cap (\bigcap_{\alpha \in G} D_\alpha)$ is nonempty. To see this, we may assume without loss of generality that the cardinality of G is larger than $\max \{n_\alpha : \alpha \in F\}$. Thus, since each k_β, $\beta \in B$, belongs to at most k of the sets in $\mathscr{F}$, we have the following inclusion: $(\bigcap_{\alpha \in F} V(\alpha, n_\alpha)) \cap (\bigcap_{\alpha \in G} D_\alpha) \supset \bigcap_{\alpha \in F \cup G} D_\alpha$. But $\mathscr{F}$ has the finite intersection property, so there is a point $d \in (\bigcap_{\alpha \in F} V(\alpha, n_\alpha)) \cap (\bigcap_{\alpha \in G} D_\alpha)$.

Now, let z be any point in $\Sigma_\gamma(p)$ such that $z_\alpha = d$ whenever $d \in D_\alpha$. By the choice of d, we must have $z \in V$, and $f(z) = 1$.

Thus f cannot be extended continuously over X_γ with either the topology T, T', or T''. This completes the counterexample.

REMARK 3.7. In the previous section it was shown that there are non-realcompact product spaces in which every Σ_1-subspace is C-embedded. We are now in a position to destroy a related conjecture by proving that there are realcompact product spaces containing Σ_γ-subspaces which fail to be C-embedded.

In Example 3.3 each coordinate space except X_0 is compact and hence realcompact. It is easy to see that if $\aleph_\gamma$ is nonmeasurable, then X_0 is also realcompact. Thus there is no clear relationship between a product space being realcompact and each Σ_γ-subspace being C-embedded.

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Received August 14, 1971. This paper contains parts of the author's doctoral dissertation [**10**] which was written under the helpful guidance of Professor W. W. Comfort. The author also wishes to express gratitude to Professor A. W. Hager for his suggestions. Work on this paper was partially supported by the National Sciences Foundation under grant NSF-GP-8357.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics
Vol. 46, No. 2 December, 1973