A CHARACTERIZATION OF THE UNITARY AND SYMPLECTIC GROUPS OVER FINITE FIELDS OF CHARACTERISTIC AT LEAST 5

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The following characterization is obtained:

**Theorem.** Let $G$ be a finite group generated by a conjugacy class $D$ of subgroups of prime order $p \geq 5$, such that for any choice of distinct $A$ and $B$ in $D$, the subgroup generated by $A$ and $B$ is isomorphic to $Z_p \times Z_p$, $L_2(p^m)$ or $SL_2(p^m)$, where $m$ depends on $A$ and $B$. Assume $G$ has no nontrivial solvable normal subgroup. Then $G$ is isomorphic to $Sp_n(q)$ or $U_n(q)$ for some power $q$ of $p$.

A much larger class of groups satisfies the analogous property for $p = 2$ or 3, including many of the sporatic simple groups. The classification for $p = 2$ appears in [3]. The classification for $p = 3$ is incomplete, but a partial solution appears in [4].

For the most part the proof here mimics that in the papers mentioned above. The exception comes in handling certain degenerate cases. This is accomplished in § 4 by first showing a minimal counter example possesses a doubly transitive permutation representation, and then utilizing numerous results on doubly transitive groups.

1. **Notation.** In general $G$ is a finite group and $D$ a $G$ invariant collection of subgroups generating $G$. $G$ acts on $D$ by conjugation with this representation denoted by $G^D$. If $\alpha \subseteq D$ is a set of imprimitivity for this action we define

$$D_\alpha = \{\beta \in \alpha^G : [\alpha, \beta] = 1, \alpha \neq \beta\}$$

$$\alpha^\perp = \{\alpha\} \cup D_\alpha$$

$$A_\alpha = \alpha^G - \alpha^\perp$$

$$V_\alpha = \{\beta \in \alpha^G : \alpha^\perp = \beta^\perp\}$$

$$W_\alpha = \{\beta \in \alpha^G : D_\alpha = D_\beta\}$$

$$D_\alpha^* = \{B : B \in \beta \in D_\alpha\}.$$  

For $\Omega \subseteq \alpha^G$, $\mathcal{G} (\Omega)$ is the graph with point set $\Omega$ and edges $(\alpha^\gamma, \alpha^\delta)$ where $\alpha^\gamma \in D_\alpha^*$. $\mathcal{B} (\Omega)$ is the geometry with point set $\Omega$ and block set $\{\beta^\perp \cap \Omega : \beta \in \Omega\}$. For $\alpha, \beta \in \Omega$ the line through $\alpha$ and $\beta$ in $\mathcal{B} (\Omega)$ is

$$\alpha^* \beta = \bigcap_{\gamma \in \alpha^\perp \cap \beta^\perp \cap \Omega} (\gamma^\perp \cap \Omega)$$

$\alpha^* \beta$ is singular if $\beta \in D_\alpha$ and hyperbolic otherwise.
A triangle is a triple \((A, B, C)\) with \(A \in D, C \in D_A\), and \(B \in A_A \cap A_C\).

If \(G\) is a permutation group on a set \(\Omega, \Delta \subseteq \Omega\) and \(X \subseteq G\), then \(X_\Delta, X(\Delta)\) is the pointwise, global stabilizer of \(\Delta\) in \(X\) respectively. \(X' = X(\Delta)/X_\Delta\) with induced permutation representation. \(F(X)\) is the set of fixed points of \(X\).

\(O_\infty(G)\) is the largest normal solvable subgroup of \(G\).

All groups are finite.

2. Locally \(D\)-simple groups. Let \(G\) be a finite group and \(D\) a collection of subgroups of \(G\) such that \(D^\circ = D\). Represent \(G\) as a permutation group on \(G\) by conjugation. \(G\) is said to be \(D\)-simple if \(G\) is generated by any \(G\) invariant subset of \(D\). \(G\) is locally \(D\)-simple if \(D\) generates \(G\) and for any \(A\) and \(B\) in \(D\) either \([A, B] = 1\) or \(\langle A, B \rangle\) is generated by \(A^{x, y}\). \(\alpha\) is a set of imprimitivity for \(G^\circ\) if \(\alpha \cap \alpha^\circ = \emptyset\) for \(g \in G - N_G(\alpha)\), and \(\emptyset \neq \alpha = \langle \alpha \rangle \cap D = D\).

**Lemma 2.1.** Let \(G\) be locally \(D\)-simple and \(\Delta\) a \(G\) invariant subset of \(D\). Then

1. If \(H\) is a \(D\)-subgroup of \(G\) then \(H\) is locally \((H \cap D)\)-simple.
2. If \(\alpha\) is a homomorphism of \(G\) then \(G\alpha\) is locally \(D\alpha\)-simple.
3. Let \(\Gamma = \langle \Delta \rangle \cap D\). Then \([\Gamma, D - \Gamma] = 1\).
4. If \(G^\prime\) is transitive then \(\langle \Delta \rangle\) is transitive.
5. If \(D \cap Z(G)\) is empty and \(G = \langle \Delta \rangle\) for some orbit \(\Delta\) of \(G^\circ\), then \(G\) is \(D\)-simple.

**Proof.** (1) and (2) are straightforward. Let \(H = \langle \Delta \rangle\). Then \(H \subseteq G\). Let \(A \in \Gamma, B \in D - \Gamma\) and assume \([A, B] \neq 1\). Let \(X = \langle A, B \rangle\). Then \(X = \langle A^\circ \rangle \leq H\) so \(B \in \Gamma\), contradicting the choice of \(B\). Therefore, (3) holds.

Assume \(G^\prime\) is transitive. Let \(K = \langle D - \Gamma \rangle\). Then by (3) \(G\) is the central product of \(H\) and \(K\) so for \(A \in \Delta, \Delta = A^\circ = A^{K \cup} = A^\prime\). Thus (4) holds.

Finally assume \(G^\prime\) is transitive, \(G = \langle \Delta \rangle\) and \(Z(G) \cap D\) is empty. Suppose \(\Omega\) is an orbit of \(G^\circ\) with \(K = \langle \Omega \rangle \neq G\). Then as \(G = \langle \Delta \rangle, \Delta \cap K\) is empty, so by (3), \([\Delta, \Omega] = 1\). Thus \(\Omega\) is centralized by \(G\), a contradiction. Thus (5) holds.

**Lemma 2.2.** Let \(G\) be locally \(D\)-simple and \(\alpha\) a set of imprimitivity for \(G^\circ\). Then

1. If \(A \in \alpha, B \in \alpha^\circ \neq \alpha\) and \([A, B] = 1\), then \([\alpha, \alpha^\circ] = 1\).
2. \(\langle \alpha^\circ \rangle\) is locally \(\langle \alpha \rangle^\circ\)-simple.

**Proof.** (1) \(A = A^\circ \in \alpha^\circ\), so \(\alpha^\circ = \alpha\). Thus 2.1.3 applied to \(\langle \alpha, B \rangle\) implies \([\alpha, B] = 1\). But now the same argument shows \([\alpha^\circ, C] = 1\) for each \(C\) in \(\alpha\). (2) Let \(H = \langle \alpha \rangle \neq K = \langle \alpha^\circ \rangle\), and \(X = \langle H, K \rangle\). Assume
\[[H, K] \neq 1 \text{ and let } A \in \alpha, B \in \alpha^g. \text{ Then by (1), } [A, B] \neq 1 \text{ so } B \in \langle A^{(A, B)} \rangle \leq \langle H^X \rangle. \text{ Thus } X = \langle H^X \rangle.\]

**Lemma 2.3.** Let \( G \) be locally D-simple with \( G^D \) transitive, and \( A \) abelian. Then

1. Either \( V_A \) or \( W_A \) equals \( \{A\} \).
2. \( V_A \) and \( W_A \) are sets of imprimitivity for \( G^D \).
3. \( V_{V_A} = \{V_A\} \) and \( W_{W_A} = \{W_A\} \).

**Proof.** Straightforward.

**Lemma 2.4.** Let \( G \) be locally D-simple with \( G^D \) transitive and \( \mathcal{D}(D) \) connected. Let \( A \in D \). Then \( A \) is contained in a unique maximal set of imprimitivity \( \alpha \) of \( G \) and \( \langle D^*_\alpha \rangle \) is \( D^*_\alpha \)-simple.

**Proof.** Let \( H = \langle D_A \rangle, \pi \) an orbit of \( H \) of maximal length on \( D_A \), \( \Delta = (\langle \pi \rangle - Z(\langle \pi \rangle)) \cap D, \Gamma = N_D(\Delta) \) and \( \alpha = \langle \Gamma - \Delta \rangle \cap D \). As \( \mathcal{D}(D) \) is connected, \( |\pi| > 1, \) so \( \Delta \) is nonempty. We will show \( \alpha \) has the properties claimed in the conclusion of the lemma.

By 2.1.3, \( [\alpha, \Delta] = 1. \) By 2.1.4 \( \langle \pi \rangle \) is transitive on \( \pi \). Thus transitivity of \( G^D \) and maximality of \( |\pi| \) imply \( \pi \) is an orbit of \( \langle D_B \rangle \) on \( D_B, \) for \( B \in \alpha \). Therefore \( B^\perp \subseteq \Gamma \).

Suppose \( B \in \alpha \cap \alpha^g \neq \alpha. \) Then \( \Delta \subseteq B^\perp \subseteq \Gamma^g = \alpha^g \cup \Delta^g. \) Now \( \langle \pi \rangle \) is transitive on \( \pi \) so either \( \pi \subseteq \Delta^g \) or \( \pi \subseteq \alpha^g. \) If \( \pi \subseteq \Delta^g \) then \( \Delta \subseteq \langle \pi \rangle \subseteq \langle \alpha^g \rangle, \) so \( \Delta = \Delta^g \) and therefore \( \alpha = \alpha^g, \) a contradiction. Thus \( \pi \subseteq \alpha^g, \) so \( \Delta \subseteq \langle \pi \rangle \subseteq \langle \alpha^g \rangle \) and therefore \( \Delta \subseteq \alpha^g. \)

So \( \Gamma \subseteq \alpha \cup \alpha^g. \) Further \( \Delta^g \subseteq \alpha, \) so \( \alpha^g \subseteq C^\perp \subseteq \Gamma \) for \( C \in D^g. \) Thus \( \Gamma = \alpha \cup \alpha^g. \) From the last remark of the second paragraph it follows that \( \Gamma \) is a component of \( \mathcal{D}(D), \) contradicting the hypothesis that \( \mathcal{D}(D) \) is connected.

It follows that \( \alpha \) is a set of imprimitivity for \( G^D. \) By 2.2.1, \( D^*_\alpha = D_A - \alpha = \Delta - \alpha. \) By construction, \( Z(\langle D \rangle) \cap \Delta \) is empty, so \( D^*_\alpha = \Delta \) and by 2.1.5, \( \langle \Delta \rangle \) is \( \Delta \)-simple.

Finally let \( \beta \) be a set of imprimitivity for \( G \) containing \( D_A \) centralizes \( A, \) so \( \Delta \) normalizes \( \beta. \) If \( B \in \beta \cap \Delta \) then as \( K = \langle \Delta \rangle \) is \( \Delta \)-simple, \( \Delta \subseteq \langle B^\Delta \rangle \subseteq \langle \beta^\Delta \rangle = \langle \beta \rangle. \) Thus \( \Delta \subseteq \beta. \) As \( N_\beta(\beta) \) is transitive on \( \beta, \alpha \subseteq D^*_\alpha \subseteq \beta \) for \( \alpha^g \in D^*_\alpha. \) Thus \( \Delta^\perp \subseteq \beta, \) and transitivity of \( N_\beta(\beta)^g \) implies \( \beta \) is a component of \( \mathcal{D}(D), \) contradicting the hypothesis that \( \mathcal{D}(D) \) is connected.

So \( \beta \cap \Delta \) is empty and by 2.1.3, \( [\beta, \Delta] = 1. \) Thus \( \beta \subseteq N_\beta(\Delta) - \Delta = \alpha. \) Thus \( \alpha \) is maximal as claimed.

Lemmas 2.6 and 2.7 are from §2 of [4]. 2.6 is a slight generalization of its counterpart, but the same proof goes through.
**Lemma 2.6.** Let $G$ be locally $\Omega$-simple, let $\Lambda \subseteq \Omega$, and let $H$ be a $\Omega$-subgroup of $G$. Assume

(i) $H$ takes the edge set of $\mathcal{D}(\Lambda)$ onto the edge set of $\mathcal{D}(\Omega)$ under conjugation.

(ii) There exists a partition $\Lambda = \sum A_i$ of $\Lambda$ such that if $\alpha^h \in \Lambda$ for some $\alpha \in A_i$, $h \in H$, then there exists $r \in N_H(A_i)$ with $\alpha^h = \alpha^r$.

Let $\bar{G}$ be a second group satisfying the hypothesis of $G$ for which there exists a permutation isomorphism $T$ of $H^0 \bar{H}^0$ and an isomorphism $S$ of $\mathcal{D}(\Lambda)$ and $\mathcal{D}(\bar{A})$ such that

(iii) $T$ restricted to $N_H(A_i)$ commutes with $S$ and $N_H(\alpha)T = N_H(\alpha S)$ for each $\alpha \in A$.

Then $S$ extends to an isomorphism of $\mathcal{D}(D)$ and $\mathcal{D}(\bar{D})$.

A triangle in $D$ is a triple $(A, B, C)$ with $A \in D, C \in D_A$, and $B \in A_A \cap A_C$. $D$ is locally conjugate in $G$ if for $A, B \in D, A$ is conjugate to $B$ in $\langle A, B \rangle$, or $[A, B] = 1$.

**Lemma 2.7.** Let $\Omega$ be locally conjugate in $G$ with $G^\Omega$ primitive and $\mathcal{D}(\Omega)$ connected. Assume

(*) If $(\alpha, \beta, \gamma)$ is a triangle and $X = \langle \alpha, \beta, \gamma \rangle$, then $\beta^1 \cap X \subseteq \beta^{(\alpha^1 \cap X)}$ and $\beta^r \subseteq (\beta^1 \cap X)^{\alpha}$.

Then $\langle \alpha^1 \rangle$ is transitive on $A_\alpha$ and $G^\alpha$ is rank 3.

3. $p$-transvections. Let $G$ be a finite group, $p$ a prime. A set of $p$-transvections of $G$ is a $G$ invariant collection $D$ of subgroups generating $G$ such that for any $A, B \in D, |A| = p$ and $\langle A, B \rangle$ is the homomorphic image of a subgroup of $SL_2(p^n)$, with $n$ and the image depending on $A$ and $B$.

If $p = 2$ then $D$ is a set of odd transpositions. Groups generated by odd transpositions have been classified [3]; they include the sporadic simple groups discovered by Fischer plus many infinite classes of simple groups. Conway’s sporadic simple group $1$ is generated by 3-transvections, as is the Hall-Janko group and Suzuki’s sporadic simple group.

**Lemma 3.1.** Let $D$ be a set of $p$-transvections of $G, p > 2$, and let $M = O_\alpha(G)$. Then

(1) $G$ is locally $D$-simple

(2) If $G$ is a $p$-group then $G$ is abelian

(3) If $G = M$ is not a $p$-group then $p = 3$ and $G$ is a $\{2, 3\}$ group

(4) If $p > 3$ then $M/O_p(G) = Z(G/O_p(G))$.

(5) Let $M = 1$. Then $G$ is a simple unless $p = 3$ and $G \cong PGL_3(2)$.

**Proof.** Let $A, B \in D, [A, B] \neq 1$. Set $X = \langle A, B \rangle$. Then $X$ is isomorphic to $SL_2(p^n)$ or $L_2(p^n)$ unless $p = 3$ and $X \cong SL_2(5)$ or $L_2(5)$. 

This implies (1) and (2). If \( G = M \) then as \( L_2(q) \) is simple for \( q > 3 \), \( X \) must be isomorphic to \( SL_2(3) \) or \( A_4 \). Therefore, 4.1 of [4] yields (3).

Assume \( p > 3 \). To prove (4) we may assume \( O_p(G) = 1 \). Let \( Q \) be a minimal normal subgroup of \( G \) contained in \( M \). Then \( Q \) is a \( p \)-subgroup for some prime \( q \neq p \). If \( A \) centralizes \( Q \) then \( Q \) is in the center of \( G = \langle D \rangle \), so we can assume \([A, Q] \neq 1\). But then \( [A^q] \leq AQ \) is a solvable \( D \)-subgroup whose order is divisible by \( q \), contradicting (3).

Finally assume \( M = 1 \) and let \( H \) be a minimal normal subgroup of \( G \). If \( A \leq H \) and \( x \in H \) then \( \langle A, A^x \rangle \) has a normal subgroup of index \( p \), so either \( A^x \in A^1 \) or \( \langle A, A^x \rangle \cong SL_2(3) \) or \( A_4 \). If \( A^H \subseteq A^1 \) then \( [H, A] \) is a normal abelian subgroup of \( H \), so \([H, A] = 1\). Thus \( H \) is centralized by \( G = \langle D \rangle \), a contradiction. Therefore, if \( A \leq H \), then \([4] \) implies \( AH \cong PGU_m(2) \). \( PGU_m(2) \) is normal in \( Aut U_m(2) \) so \( G = C_p(H)HA \). By induction on \( |G| \), \( G/H \cong C_p(H)A \cong Z_p \) or \( PGU_m(2) \). But now \([4] \) implies the latter case does not occur.

So we can take \( A \leq H \). So \( G = \langle D \rangle = H \) is simple.

The proof of the following lemma is due to David Wales.

**Lemma 3.2.** Let \( G \cong L_2(q) \) or \( SL_2(q) \), \( q = p^m \) odd, with Sylow \( p \)-subgroup \( P \). Assume \( G \) acts irreducibly on a \( n \)-dimensional vector space over \( GF(p) \), such that \( n = 2 \dim GF(P) \) and \( P \) acts semiregularly on \( V - C(V) \). Then \( G \cong SL_2(q) \), \( n = 2m \), and \( G \) acts in its natural representation on \( V \).

**Proof.** Let \( B \) be a basis of \( V \), and \( GF(r) \) the splitting field for the representation of \( G \) on \( V \). Extend the action of \( G \) to a vector space \( W \) over \( GF(r) \) with basis \( B \). \( W \) is the sum of \( k \) absolutely irreducible \( G \)-invariant subspaces \( W_i \) of \( W \). By inspection of the irreducible representations of \( SL_2(q) \) (e.g. §30, [7]), \( \dim C_{w_i}(P) = 1 \) for all \( i \). Thus as \( n = 2 \dim C_r(P) \) and \( P \) acts semiregularly on \( V - C_r(P) \), \( \dim C_{W_i}(P) = 2 \). Again by inspection of the representations of \( SL_2(q), q = r, G \cong SL_2(q) \), and \( G \) acts in its natural fashion on \( W_i \). Further \( G^{W_i}, 1 \leq i \leq k \), are the \( m \) equivalent representations obtained from \( G^{W_i} \) by \( Aut GF(q) \). Thus \( n = 2m \) and \( G \) acts in its natural fashion on \( V \).

**Lemma 3.3.** Let \( D \) be a class of \( p \)-transvections of \( G, p \) odd, with \( G/O_{\infty}(G) \cong L_2(q) \). Let \( M = O_p(G), A \in D, m = |A^1| \) and \( Z = Z(G) \). Assume \( O_{\infty}(G)/M = Z(G/M) \). Then for some \( B \in D, G = MX \) where \( X = \langle A, B \rangle \cong SL_2(q), Z = [A^1, M] \cap [B^1, M], M = [A, M][B, M], |M/Z| = m^2 \) where \( m = |A^1|, Z = C_x(\alpha) \) for any \( p' \)-element of \( X \), and \( [M, \beta] \) is transitive on \( A^1 \).

**Proof.** As \( G/O_{\infty}(G) \cong L_2(q) \) there exists \( B \in D \) with \( X = \langle A, B \rangle \cong L_2(q) \) or \( SL_2(q) \). Let \( \alpha = A^1 \cap X \), and \( \Omega = \alpha^x \). Let \( K = \prod_o[M, \beta] \).
By 3.1, \([M, \alpha]\) is elementary abelian, \(G = \langle [M, \alpha], X \rangle\) normalizes \(K\) and \([A, M/K] = 1\). So \(M = K\). As \(X^0\) is doubly transitive, \(Z_0 = [M, \alpha] \cap [M, \beta] = [M, \gamma] \cap [M, \delta]\) for all pairs \((\alpha, \beta), (\gamma, \delta)\) from \(\Omega\). So as \([M, \alpha]\) is abelian, \(Z_0 \leq Z\). Thus we can assume \(Z_0 = 1\). Therefore, \(M\) is elementary abelian. \(A\) is in \(m\) groups \(\langle A, C \rangle, C \in B^\nu\), so there are \(m^2\) total \(D\)-subgroups isomorphic to \(L_2(q)\) or \(SL_2(q)\). Set \(G = C^\nu/\upsilon\Gamma\). 

**Lemma 3.2** implies \(X \cong SL_2(q)\) and \(C^\nu(x) = 1\) for all \(p^\nu\)-elements \(x \in X\). So it suffices to show \(Z = 1\). Let \(\langle u \rangle = Z(X)\). Then \(M = Z[M, u]\), so \(D \subseteq X[M, u] \leq G\). Thus \(Z = 1\).

**Lemma 3.4.** Let \(D\) be a class of \(p\)-transvections of \(G, p\) odd, with \(M = O_p(G), X\) a \(D\)-subgroup with \(X/Z(X) \cong U_3(q)\), and \(G = MX\). Let \(Z = Z(G), A \in M\) and \(m = |AM|\). Then \(Z \leq \langle A^\perp, M \rangle\) and \(|M/Z| = m^2\).

**Proof.** Let \(X = \langle A_i, 1 \leq i \leq 3 \rangle, A = A_i\), let \(\alpha_i = A_i \cap X\) and \(\Omega = \alpha^X\). Set \(Z_0 = [\alpha, M] \cap [\alpha_2, M]\). As \(X^0\) is doubly transitive \(Z_0 = [\beta, M] \cap [\gamma, M]\) for \(\beta, \gamma \in \Omega\). \([\alpha, M]\) is abelian so \(G = \langle X, A^\nu \rangle\) centralizes \(Z_0\). Thus we can assume \(Z_0 = 1\).

Set \(N = \prod_{i=1}^3 [M, \alpha_i]\). By 3.3, \([M, \alpha_i]^* \leq [M \alpha_i][M, \alpha_i]\), so \(N\) is normalized by \(G = \langle \alpha, \alpha_2, \alpha_3, M \rangle\). \(A\) centralizes \(M/N\), so \(M = N\). As \(Z_0 = 1\), \(M\) is abelian. Let \(v\) be the involution in \(\langle \alpha, \alpha_2 \rangle\) and \(v\) the involution in \(\langle \alpha_2, \alpha_3 \rangle\). We may assume \([u, v] = 1\). \(M = C_M(u) \times [M, u]\) and by 3.3, \(C_M(u) = C_M(\alpha_i) \cap C_M(\alpha_2)\) and \([M, u] = [M, \alpha_i][M, \alpha_2]\). Therefore, \(C_M(u) \cap C_M(v) = Z\) and as \(X\) has one class of involutions, \(|C_M(u)/Z| = |M/Z| = |C_M(u)/Z| m^2\). So \(|M/Z| = m^2\), and as \(|M| \leq m^3, Z = 1\). That is \(Z = Z_0 \leq \langle A, M \rangle\).

4. Groups with \(\mathcal{O}(D)\) disconnected. This section consists of a proof of the following theorem:

**Theorem 4.1.** Let \(D\) be a conjugacy class of \(p\)-transvections, \(p \geq 5\), of the group \(G\). Assume \(\mathcal{O}(D)\) is disconnected and \(O_p(G) = 1\). Then \(G \cong L_2(q)\) or \(U_3(q)\) for some power \(q\) of \(p\).

Throughout § 4, \(G\) is a counterexample of minimal order to Theorem 4.1. For \(A \in D\) let \(\bar{A}\) be the component of \(\mathcal{O}(D)\) containing \(A\). Let \(\bar{D}\) be the set of components. Write \(A \sim B\) if \(A, B \in D\) and \(\langle A, B \rangle\) is isomorphic to \(L_2(p)\) or \(SL_2(p)\). For \(\bar{A} \neq \bar{B}\) define

\[\Gamma_{\bar{A} \bar{B}} = \{C \in \bar{A}: A \sim E \sim C \text{ for some } E \in \bar{B}\}.\]

Now for \(\bar{A} \neq \bar{B}, A \sim B\) if and only if \(\bar{A} \cup B^A = \bar{B} \cup \bar{A}^B\). Thus if \(A \sim\)
$B$ then $X = \langle \Gamma_{\overline{A}B}, \Gamma_{\overline{B}A} \rangle$ acts on $\Gamma = \overline{A} \cup \overline{B}$ of order $p + 1$, so $Y = \langle \Gamma_{\overline{A}B} \rangle = AY \Gamma$ and $X = \langle Y, B \rangle = \langle A, B \rangle X_\Gamma$. By 3.1, $X_\Gamma = 0_m(X)$ and $Y$ is a $p$-group. Further for fixed $\overline{B} \neq \overline{A}$, the sets $\Gamma_{\overline{C}B}, C \in \overline{A}$, partition $\overline{A}$.

Let $m = |\Gamma_{\overline{A}B}|$, and let $n$ be the number of classes $\Gamma_{\overline{C}B}$ in $\overline{A}$. If $m > 1$ then applying 3.3 to $X$ we have that $\langle A, B \rangle$ contains a central involution $u = u(A, B)$, and $u$ centralizes only $A$ in $\Gamma_{\overline{A}B}$.

Let $C \in \overline{A}$. $\langle C, B \rangle$ contains $E \in \Gamma_{\overline{A}B}$ and $v = u(E, B)$ is in the center of $\langle C, B \rangle$. Indeed $v = u(C, F)$ where $C \sim F \in \overline{B} \cap \langle C, B \rangle$. As $v$ centralizes a unique member of $\Gamma_{\overline{A}B}$ and $\Gamma_{\overline{C}B}$, each member $C_i$ of $\Gamma_{\overline{C}B}$ determines a distinct member $E_i$ of $\Gamma_{\overline{A}B} \cap \langle C_i, B \rangle$. Thus $m = |\Gamma_{\overline{C}B}|$ for all $C \in \overline{A}$. Further $u = u(C, F_i)$ for some $C_i \in \Gamma_{\overline{C}B}, F_i \in \Gamma_{\overline{F}A}$. So $C_p(u)$ intersects each $\Gamma_{\overline{C}B}$ in $\overline{A}$ in a unique member. Set $K = \langle C_p(u) \rangle$ and $H = \langle K, \overline{A} \rangle$. Minimality of $G$ implies $K \cong SL_2(q)$ for some power $q$ of $p$.

Now there are $m^2$ involutions $u(A_i, B_i), A_i \in \Gamma_{\overline{A}B}, B_i \in \Gamma_{\overline{B}A}$, and $m^2$ pairs $(A_i, C_i), C_i \in \Gamma_{\overline{C}B}$, with $u(A_i, B_i)$ centralizing at most one pair. It follows there exists $u$ with $A, C \in Q$. So as $Q$ is abelian, $\langle \overline{A} \rangle$ is abelian. Notice that if $m = 1$ then $\overline{A} = \Gamma_{\overline{A}B} \cap \langle C, B \rangle$, so again $[A, C] = 1$, and $\langle \overline{A} \rangle$ is abelian. Therefore:

**Lemma 4.2.** $\langle \overline{A} \rangle$ is abelian.

Let $\langle e \rangle = C \in \overline{A}$. We have shown there is an $\langle e \rangle = E \in C_2(u) \cap \Gamma_{\overline{C}B}$, and we can choose $e$ such that $\overline{E} = \overline{E}$. Thus as $\langle \overline{A} \rangle$ is abelian, $\overline{E}^{e} = \overline{E}^{e} = \overline{E}^{e} = \overline{E}^{e}$, so $H$ acts on $A = \overline{A} \cup \overline{B}$, and $H = K \overline{A} = K \overline{B}$ by 3.1.

Summarizing:

**Lemma 4.3.** (1) If $m > 1$ then $\langle A, B \rangle$ contains a central involution $u$. (2) If $\langle A, B \rangle$ contains a central involution $u$ then $\langle \overline{A}, \overline{B} \rangle = H = \langle C_p(u) \rangle 0_p(H)$ with $\langle C_p(u) \rangle \cong SL_2(q)$ for some power $q$ of $p$.

Let $J = N_o(\overline{A}), I = C_o(\overline{A})$. For $X \subseteq G$ let $F(X)$ be the set of points in $\overline{D}$ fixed by $X$.

**Lemma 4.4.** Assume $u$ is an involution in the center of $\langle A, B \rangle$.

Then

(i) If $v$ is an involution in the center of $\langle A, C \rangle$ with $[u, v] = 1$, then $u = v$.

(ii) $J = O(J)C_p(u)$.

**Proof.** Set $H = \langle C_p(u) \rangle$. Let $v$ be as in (i). Then $v$ acts on $H$ and fixes $\overline{A}$. There are $q + 1$ members of $\overline{D}$ intersecting $H$, and $q + 1$ is even, by 4.3. Thus $v$ fixes a second member $\overline{E}$, a $\overline{A}$ of $\overline{D}$
with $\bar{E} \cap H \neq \emptyset$. As $H \cong SL_2(q)$, $v$ centralizes an element $E$ of $\bar{E}$. Thus $\langle u \rangle = Z(\langle A, E \rangle) = \langle \bar{v} \rangle$, yielding (i). (i) and Glauberman's $Z^*$-theorem imply (ii).

Lemma 4.5. Assume $m(\bar{A}, \bar{B}) = 1$ with $A \sim B$. Let $x \in \langle A, B \rangle$ fix $\bar{A}$ and $\bar{B}$. Then

1. $B = \bar{B}(A)$ is the unique element of $\bar{B}$ with $A \sim B$.
2. $x$ acts as scalar multiplication in $GF(p)$ on $Q = \langle \bar{A} \rangle$.
3. Assume $y \in J$ has scalar action on $Q$ and fixes $\bar{B}$. Then $y$ has the same action on $\langle \bar{B} \rangle$ and if $|xI/I| > 2$ then $F(x) = \{\bar{A}, \bar{B}\}$.
4. If $\langle A, C \rangle \cong L_2(p^n)$ or $SL_2(p^n)$, $n$ odd, for all $C \in \bar{B}$, then $\langle \bar{A}, \bar{B} \rangle \cong L_2(q)$ or $SL_2(q)$.
5. If $p = 5$ and $\langle A, C \rangle \cong L_2(p^n)$ or $SL_2(p^n)$, $n$ even, for some $C \in \bar{B}$ then there exists $y$ with $|Iy/I| = 4$ inducing scalar action on $Q$ and $\langle \bar{B} \rangle$.
6. $m(\bar{A}, \bar{C}) = 1$ for all $\bar{C} \neq \bar{A}$.

Proof. (1) is just a restatement of $m(\bar{A}, \bar{B}) = 1$. Let $C \in \bar{A}$. $\langle C, B \rangle$ contains an element $A$, of $D$ centralizing $C$ with $A \sim B$. Thus by (1), $A = \bar{A}(B) = A$. So $x \in \langle A, B \rangle \leq \langle C, B \rangle$ and thus has the same action on $C$ as on $A$. This yields (2). Notice that (2) implies $J = IC_J(x)$.

Assume $y \in J$ is as in the hypothesis of (3). Then for $C \in \bar{A}$, $y$ fixes $C$ and therefore $\bar{B}(C)$. So $y$ acts on $\langle C, B \rangle$ with scalar action on $\bar{B} \cap \langle C, B \rangle$. So $y$ acts on $\bar{B}$ as on $\bar{A}$.

Assume $y$ has order $r^n$ for some prime $r$, $r$ dividing $p - 1$, and $\bar{C} \in F(y) - \{\bar{A}, \bar{B}\}$. Suppose first that $m(\bar{A}, \bar{C}) > 1$. Then by 4.3, $K = \langle A, C \rangle = HM$ where $H = \langle C_0(u) \rangle$, $u = u(A, C)$, and $M = O_{n}(K)$. $y$ fixes $A$ so $y$ fixes $\Gamma_{cA}$ for $A \sim C$. As $|\Gamma_{cA}|$ is a power of $p$ and $p | 1$ mod $r$, $x$ fixes a point $C$ of $\Gamma_{cA}$. As this holds for each $A \in \bar{A}$, we can assume $x$ normalizes $H$. Thus with 4.3, $F(yu) = \{\bar{A}, \bar{C}\}$ and $[y, u] = 1$. Now $J = IC_J(y)$, so $[M, y] \leq M \cap I = [A, M]$ by 3.3. So if $y$ acts by scalar multiplication on $\bar{C}$, then $[M, y] \leq [A, M] \cap [C, M] = Z(K)$ by 3.3, so that $y$ centralizes $M/Z(K)$. But $y$ does not even centralize $[A, M]/Z(K)$. $y$ does not have scalar action on $\bar{C}$.

Set $E = \bar{B}^u$. $y$ has scalar action on $E$ and $\bar{B}$, so as above $m(E, \bar{B}) = 1$. $\langle E, \bar{B} \rangle \cong SL_2(q)$ or $L_2(q)$ so there exists an involution $t$ with cycle $E, \bar{B}$ inverting $y$ mod $C(\bar{B})$. Thus $ut \in N(\bar{B})$ inverts $y$ mod $C(\bar{B})$, while $N(\bar{B}) = C(\bar{B})C(y)$. So $|yC(\bar{B})/C(\bar{B})| = |Iy/I| \leq 2$.

Assume $|Iy/I| > 2$. Then as above $m(E, \bar{F}) = 1$ for all $E, \bar{F} \in F(y)$ and $C_0(y)$ fixes $F(y)$ pointwise. Now if $z$ is an element centralizing $\bar{A}, \bar{B}$, and $y$ then $F(z) = \langle C_0(z) \rangle \cap \bar{D}$ and minimality of $G$ implies $F(z) \cap F(y) = \{\bar{A}, \bar{B}\}$. Thus $z$ moves $\bar{C}$, so $z = 1$. Now there exists an involution $t$ with cycle $\bar{A}$, $\bar{B}$ inverting $y$ modulo $C(\bar{A}) \cap C(\bar{B})$. Thus $y^t = y^{-1}$. Similarly there exists $s$ with cycle $\bar{B}, \bar{C}$ inverting $y$. So $ts$
moves $\bar{A}$ to $\bar{C}$ and centralizes $y$, a contradiction. Thus we have shown (3).

Assume the hypothesis of (4). Let $E \in \bar{A}$, and $C = B(E)$. Then for $\alpha \in Q^f \cap \langle A, C \rangle$, $\langle \alpha \rangle \in \bar{A}$. So $\bar{A} = \{ \langle \alpha \rangle : a \in Q^f \}$. Let $A = \bar{A} \cup B^2$. Clearly $Q$ normalizes $A$. Further for $E = \varepsilon \in \bar{A}$, $B^\varepsilon \subseteq \bar{A} \cup B^{\langle (E, B) \cap Q \rangle}$, so as $\bar{A} = \{ \langle \alpha \rangle : a \in Q^f \}$, $B$ normalizes $A$. Thus $X = \langle \bar{A}, B \rangle$ normalizes $A$. Further $X'$ is 2-transitive with $Q^f \leq X_\uparrow$ and regular on $\Delta - \{ A \}$. Therefore, a result [11] of Kantor and Seitz implies $X' \cong L_2(q)$. This yields (4).

Assume the hypothesis of (5). Then there exists $y \in \langle A, C \rangle$ with $\lvert y / I \rvert = 4$ inducing scalar action on $Q \cap \langle A, C \rangle$ and $\langle B \rangle \cap \langle A, C \rangle$. By (2), $x = y^a$ inverts $Q$ and $\langle B \rangle$, so orbits of $x$ on $\bar{A}$ have order at most two. Suppose $(A_1, A_2)$ is such an orbit. Let $B_2 = \bar{B}(A_2)$ and set $X = \langle A_1, B_2 \rangle$. Then $y$ normalizes $X$ with $x$ inverting $Q \cap X$, so $y$ induces scalar action on $Q \cap X$ and fixes $A_1$, a contradiction. Thus $y$ fixes $\bar{A}$ pointwise and induces scalar action on $Q$. This yields (5).

It remains to show (6). Assume $m(\bar{A}, \bar{C}) > 1$ and let $u = u(A, C)$. By 4.4, $J = \binom{0(J)}{C}(u)$. As $J = IC_\uparrow(y)$, $[u, y] \leq 0(I)$. Thus some conjugate $v$ of $u$ centralizes $y$. Now if $p > 5$ or $p = 5$ and $\langle A, E \rangle \cong L_2(5^*)$ or $SL_2(5^*)$, $n$ even, for some $E \in \bar{B}$, then we can choose $y$ with $\lvert y / I \rvert > 2$. So by (3), $F(y) = \{ \bar{A}, \bar{B} \}$. As $[v, y] = 1$ and $v$ fixes $\bar{A}$, $v$ fixes $\bar{B}$. So $v$ centralizes some $B \in \bar{B}$, and by 4.3, as $m(\bar{A}, \bar{B}) = 1$, $v \in I$. But this is impossible as $u \not\in I$.

It follows from (4) that $\langle \bar{A}, \bar{B} \rangle \cong L_2(q)$ or $SL_2(q)$ with $q = p^e$, $n$ odd. So $\bar{A} = \{ \langle a \rangle : a \in Q^f \}$. But by 4.3, $\langle \bar{A}, \bar{C} \rangle = H = \langle C_\uparrow(u) \rangle O_p(H)$ with $O_p(H) \neq Z(H)$. Thus there exists $a \in Q^f \cap O_p(H)$ with $\langle a \rangle \not\in \bar{A}$, a contradiction.

**Lemma 4.6.** $m(\bar{A}, \bar{B}) = 1$ for all $\bar{B} \neq \bar{A}$.

**Proof.** Assume not. Then by 4.5.6, $m(\bar{A}, \bar{B}) > 1$ for all $\bar{B} \neq \bar{A}$. Let $u = u(A, B)$, $v = u(A, C)$. By 4.4, $u$ is conjugate to $v$ under $J$, so $J$ takes $\bar{C}$ to a point of $F(u)$. But by 4.3 and 4.4, $C_\uparrow(u)^{F(u)}$ is 2-transitive. Thus $J$ is transitive on $\bar{D} - \{ \bar{A} \}$. Let $K = \langle \bar{A}, \bar{B} \rangle$, $H = \langle C_\uparrow(u) \rangle$ and $M = O_p(K)$. Let $\Omega = \bigcup_{x \in J} C_\uparrow(u^k)$. Suppose $w \in u'$ inverts $x \not\in \Omega$. Then $wu^k$ inverts $x$ while by 4.4, $wu^k$ has odd order. So $X = [Q, u'] \leq \langle Q - \Omega \rangle \leq M \cap Q$ by 3.3. But $X \leq \leq J$, $J$ is transitive on $\bar{D} - \{ \bar{A} \}$ and $M \cap Q$ fixes $\bar{B}$, so $X$ fixes $\bar{D}$ pointwise, contradicting 3.1.5.

**Lemma 4.7.** (1) There exists a prime $r$ such that for all $\bar{B} \neq \bar{A}$, $J = IN_\lambda(R)$ for some $r$-group with $F(R) = \{ \bar{A}, \bar{B} \}$.

(2) $\bar{G}^D$ is doubly transitive.
Proof. (1) implies that there exists a prime $r$ such that for any $B \neq \bar{A}$, a Sylow $r$-subgroup of $G_{\bar{A}B}$ fixes only two points. This implies $G^\nu$ is doubly transitive. So it suffices to prove (1). But unless $p = 5$ there exists a prime $r$ dividing $p - 1$ and an $r$-element $y \in \langle A, B \rangle$ fixing $\bar{A}$ and $\bar{B}$ with $|I_y| > 2$. So 4.5 implies (1) unless $p = 5$ and $\langle \bar{A}, \bar{B} \rangle = H \cong L_2(5^n)$ or $SL_2(5^n)$, $n$ odd. As $5^n = |Q| = |\langle \bar{A} \rangle|$, this holds for all $\bar{B} \neq \bar{A}$.

Suppose $u$ is an involution in $I$ and let $(\bar{C}, \bar{E})$ be a cycle in $u$ and $X = \langle \bar{C}, \bar{E} \rangle$. As $u$ does not centralize $X$, $u$ acts fixed point free on $X \cap \bar{D}$, so as $n$ is odd, $u$ induces an outer automorphism in $PGL_2(5^n)$ on $X$, and thus there exists a 2-element $y \in X$ inducing scalar action in $GF(5)$ on $\langle \bar{C} \rangle$ and $\langle \bar{E} \rangle$ with $y^2$ not centralizing $\langle \bar{C} \rangle$. Thus by 4.5, $|F(y)| = 2$, so $|\bar{D}| = m$ is even.

Assume $m$ is odd. Then $I$ has odd order. Let $x$ be the involution in $\langle A, B \rangle \cap J$. By 4.5, $J = IC_J(x)$. But as $m$ is odd $J$ contains a Sylow 2-subgroup of $G$, so the $Z^*$-theorem contradicts $O_\infty(G) = 1$. Therefore, $m$ is even.

If a Sylow 2-subgroup of $G_{\bar{A}\bar{B}}$ fixes exactly two points for every $\bar{B} \neq \bar{A}$, then $G^\nu$ is doubly transitive. So choose $\bar{B}$ such that a Sylow group of $G_{\bar{A}\bar{B}}$ fixes more than two points. Then $H = \langle \bar{A}, \bar{B} \rangle \cong L_2(5^n)$, $C_J(H)$ has odd order and the involution $x \in H_{\bar{A}\bar{B}}$ fixes three or more points. Suppose $y^2 = x$ for some $y \in G$. If $(\bar{C}, \bar{E})$ is a cycle of $y$ in $F(x)$ then $y$ normalizes $X = \langle \bar{C}, \bar{E} \rangle$ so as $y^2 = x$ and $n$ is odd, $y$ fixes two points in $X \cap \bar{D}$, which must be $\bar{C}$ and $\bar{E}$. This is a contradiction, so $x$ is not rooted in this manner.

Suppose $I$ has odd order. Then by 4.5, $J = IC_J(y)$ for any involution $y \in \langle \bar{A}, \bar{C} \rangle$ and any $\bar{C} \neq \bar{A}$. So $y \in x'$. Let $u$ be an involution. We may assume $u$ has cycle $(\bar{A}, \bar{B})$. So $u$ normalizes $H$, and as $I$ has odd order and $x$ is not rooted in $\langle u, H \rangle$, $u \in H$. Thus $u \in x'$. Thus $G$ has one class of involutions, so as $x$ is not rooted, a Sylow 2-subgroup of $G$ is elementary abelian. Walter’s classification of such groups [13] implies $G \cong L_2(5^n)$, a contradiction. So $I$ has even order. Thus $x$ centralizes some involution $u \in I$; as $|\bar{D}|$ is even, there exists $\bar{R} \in F(x) \cap F(u) - \{\bar{A}\}$; minimality of $G$ implies $\langle C_I(u) \rangle \cong L_2(5^n)$, $SL_2(5^n)$ or $U_2(5^n)$, so $F(x) \cap F(u) = \{\bar{A}, \bar{R}\}$.

Consider $C_G(x)^{F(x)}$. Arguments such as in 4.5.3 and in the last paragraph show that nontrivial elements of $C(x)^{F(x)}$ fix at most two points. Let $(\bar{C}, \bar{E})$ be a cycle of $u$ in $F(x)$. We have shown $x$ is rooted modulo $C(\bar{C}) \cap C(\bar{E})$, while $x$ is not rooted. So $C(\bar{C}) \cap C(\bar{E})$ has even order and there exists an involution $v \in C(x)^{F(x)}$, fixing $\bar{C}$ and $\bar{E}$, and centralizing $u$. $v$ acts on $F(x) \cap F(u) = \{\bar{A}, \bar{R}\}$. Let $L = C_{A, R}^F(v)$. $L$ acts semiregularly on $F(x) - \{\bar{A}, \bar{R}\}$ and $C_L(v)$ acts on $F(v) \cap F(x) = \{\bar{C}, \bar{E}\}$, so $\langle v \rangle = C_L(u)$. So a Sylow 2-subgroup $S$ of $\langle L, v \rangle = L^*$ is semidihedral or dihedral, and there are one or two classes of involu-
tions in $L^* - L$, respectively. But if $T \in F(x) - \{\bar{A}, \bar{R}\}$ let $t$ be the involution in $C(x)^{F(z)}$ fixing $T$ and $T^*$ and centralizing $u$. Then $t \in \psi_i$, $i = 1$ or 2, one of the (at most) two classes of involutions in $L^* - L$. So $L$ takes $F(t) \cap F(x) = \{\bar{T}, T^*\}$ to $F(x) \cap F(\psi_i)$. Thus $L$ has one orbit, or two orbits of equal length, on $F(x) - \{\bar{A}, \bar{R}\}$, for $S$ semidihedral or dihedral, respectively. It now follows easily that $C(x)^{F(z)}$ is 2-transitive. But $J$ and therefore $C_J(x)$ cannot take $B$ to $R$ as there is no involution in $I$ fixing $B$. This last contradiction completes the proof of 4.7.

Set $L = G_{\bar{A}B}$, $H = \langle \bar{A}, \bar{B} \rangle$, $K = C_G(H)$, and $Q = \langle \bar{A} \rangle$.

**Lemma 4.8.** (1) $J = IL$ and $K \neq 1$.

(2) $H \cong L_2(q)$ or $SL_2(q)$.

**Proof.** By 4.7.1 there exists a prime $r$ such that a Sylow $r$-subgroup $R$ of $L$ fixes only $\bar{A}$ and $\bar{B}$, and $J = IN_r(R)$. $N_r(R)$ acts on $F(R) = \{\bar{A}, \bar{B}\}$; so $N_r(R) \leq L$. If $K = I \cap L = 1$ then $I$ is regular on $\bar{D} - \{\bar{A}\}$ by 4.7.2, so [11] implies $G \cong L_2(q)$ or $U_3(q)$. Thus $K \neq 1$. Minimality of $G$ implies $H = \langle C_D(K) \rangle \cong SL_2(q)$ or $L_2(q)$.

**Lemma 4.9.** Suppose $x \in L^*$ with $|C_Q(x)| = q_0 > 1$. Then $\langle C_D(x) \rangle \cong L_2(q_0)$, $SL_2(q_0)$ or $U_3(q_0)$ and $|F(x)| = q_0 + 1$ or $q_0^3 + 1$.

**Proof.** Minimality of $G$ yields the desired form for $\langle C_D(x) \rangle$. If $\bar{C} \in F(x)$ then $[x, C] = 1$ where $C = \bar{C}(A)$, $A \in C_T(x)$. Thus $|F(x)| = q_0 + 1$ or $q_0^3 + 1$.

**Lemma 4.10.** Set $n = |\bar{D}|$. Then $(n - 1, |K|)$ is a power of $p$.

**Proof.** Let $r$ be a prime divisor of $|K|$, and $R$ a Sylow $r$-subgroup of $K$. By 4.9, $F(R) = q + 1$ or $q^2 + 1$, so if $r \neq p$ then a Sylow $r$-subgroup $R_i$ of $N_r(R)$ fixes a second point $\bar{B}$ of $F(R)$; that is $R_i = R$. So $R$ is Sylow in $I$ and $r$ does not divide $n - 1 = |I: K|$.

**Lemma 4.11.** $|\bar{D}| = n$ is even. If $u$ is an involution then $n \equiv |F(u)| \mod 4$. $|L|$ is even.

**Proof.** Results of Bender on doubly transitive groups [5.6] imply $L$ has even order. By 3.1, $G$ is simple, so any involution $u$ must act as an even permutation on $\bar{D}$. Thus $n \equiv |F(u)| \mod 4$. If $n$ is odd, 2-elements fix an odd number of points. So by 4.8 and 4.9, $|K|$ and $|L/\bar{H}K|$ are odd. And by 4.5.3, $|H \cap L| \neq 0 \mod 4$. As $L$ has even order, $|H \cap L| \equiv |L| = 2 \mod 4$. Thus $p \equiv q \equiv 5 \mod 8$. Let $u$ be the involution in $H \cap L$, and $S$ a $u$-invariant Sylow 2-subgroup of $I$. As
n is odd and $J = IL, S\langle u \rangle$ is Sylow in $G$. As $G$ has no subgroup of index two, $S \neq 1$. Let $s$ be an involution in $S$, and $(\bar{B}, \bar{C})$ a cycle in $s$. Then $s$ normalizes $X = \langle \bar{B}, \bar{C} \rangle$ and as $|F(s)| = 1$, $s$ acts fixed point free on $D \cap X$. So as $p \equiv q \equiv 5 \pmod{8}$, $\langle s, X \rangle \cong PGL_2(q)$ and there exists $y \in \langle s, X \rangle$ of order 4 inducing scalar multiplication on $\langle \bar{B} \rangle$ and fixing $\bar{B}$ and $\bar{C}$. By 4.5.3, $|F(y)| = 2$, contradicting $n$ odd.

**Lemma 4.12.** If $J = O(I)\Lambda$ then $J = O_\pi(I)\Lambda$, where $\pi$ is the set of primes dividing $n - 1$. Also $O_\pi(K) \neq 1$, and $O_\pi(I)$ is not nilpotent.

**Proof.** Set $P = O_\pi(I)$. If $P \neq O(I)$ let $R/P$ be minimal normal in $J/P, R < O(I)$. $R/P$ is an $r$-group for some prime $r$ and by a Frattini argument, $J = PN_j(R)$ where $R_j$ is a Sylow $r$-subgroup of $R$ contained in $K$. By 4.9, $N_j(R) = LP_j$ where $|P_j| = q$ or $q^n$, and $P_j \leq N_j(R)$. Thus $PP_j \leq J$, so $P_j \leq P$ and $J = PL$. Results of Kantor and Seitz on doubly transitive groups [11, 12] imply $P$ is not nilpotent or regular on $D - \{\bar{\Lambda}\}$. Thus $1 \neq P \cap L = P \cap K = O_\pi(K)$ by 4.10.

**Lemma 4.13.** Let $X \subseteq L$ fix 3 or more points of $D$. Then $C_\sigma(X)^{F(X)}$ is doubly transitive.

**Proof.** It suffices to show there exists a prime $r$ such that a Sylow $r$-subgroup of $C_\sigma(X)$ fixes only $\bar{\Lambda}$ and $\bar{B}$. Thus with 4.5 we can assume $q = 5^m$ with $m > 1$ odd. Thus there is an $r$-element $1 \neq y \in H \cap L, r > 2$, and as $m$ is odd $y$ is not inverted in $J/I$ by 4.8. Thus arguing as in 4.5, $F(y) = \langle \bar{\Lambda}, \bar{B} \rangle$. $[y, X] = 1$ unless $C_\sigma(X) \neq 1$, in which case 4.9 implies $C_\sigma(X)^{F(X)}$ is doubly transitive.

**Lemma 4.14.** Assume $q \equiv -1 \pmod{4}$ and $x$ is an involution in $L$ inverting $Q$ with $|F(x)| > 2$. Then $|F(x)| = q + 1$.

**Proof.** As $q \equiv -1 \pmod{4}$, $q$ is an odd power of $p$, so no element in $H \cap L$ is inverted in $J/I$. Thus if $y \in H \cap L$ with $|y| > 2$ then $|F(y)| = 2$. Therefore, with 4.9 and 4.13, $C_\sigma(x)^{F(x)}$ is a Zassenhaus group. So $C_\sigma(x)^{F(x)}$ has a normal subgroup isomorphic to $L_\sigma(m)$, of index at most two, with $|F(x)| = m + 1$. Now if $m \equiv 1 \pmod{4}$ then by 4.9 and 4.11, $K$ has odd order, and $\langle x \rangle$ is Sylow in $L$, so that $|C_\sigma(x)^{F(x)}|$ is odd, contradicting $m \equiv 1 \pmod{4}$. So $m \equiv -1 \pmod{4}$. Thus $C_\sigma(x)^{F(x)}$ is cyclic and inverted by any $t \in C_\sigma(x)$ with cycle $(\bar{\Lambda}, \bar{B})$. As we can choose $t \in H$, and $[K, t] = 1$, it follows that $|C_\sigma(x)| = \varepsilon \leq 2$. Further $\varepsilon(m - 1)/2 = |C_\sigma(x)^{F(x)}| = \varepsilon |H \cap L| = \varepsilon(q - 1)/2$, so $m = q$.

**Lemma 4.15.** Suppose $u$ is an involution in $Z^*(L)$ fixing 3 or more points. Then $u \in Z^*(J)$.
Proof. \( u \in Z^*(L) \) so \( u^c \cap L = u^c \). Further as \( |D| \) is even, if \( v \) is a conjugate of \( u \) in \( J \) centralizing \( u \) then we can assume \( v \in L \), so \( v \in u^c \cap C_L(u) = u^L \cap C_L(u) = \{u\} \). Thus by the \( Z^* \)-theorem, \( u \in Z^*(J) \).

**Lemma 4.16.** If \( H \cong L_2(q) \) then \( H \cap D = F(X) \) for any \( 1 \neq X \leq K \).

**Proof.** If \( F(X) \neq H \cap D \) then by 4.9, \( H \leq \langle C_D(X) \rangle \cong U_3(q) \), so \( H \cong SL_2(q) \).

**Lemma 4.17.** Assume \( u \) is an involution in \( L \) fixing \( m + 1 \geq 3 \) points, let \( c = |L:C_L(u)| \) and let \( e \) be the number of conjugates of \( u \) with cycle \((\bar{A}, \bar{B})\). Then \( |D| - 1 = m(m + 1)e/c + m \).

**Proof.** Let \( \Omega \) be the set of pairs \((v, \alpha)\) where \( v \in u^c \) and \( \alpha \) is a cycle in \( v \). Then \( |u^c|(n - m - 1)/2 = |\Omega| = n(n - 1)e/2 \) where \( n = |D| \). Further by 4.13, \( |u^c| = n(n - 1)e/m(m + 1) \).

**Lemma 4.18.** (1) Let \( S \) be a 2-group such that \( C_S(S) \neq 1 \). Then \( S \) has rank at most one.

(2) \( J = O(I)L \).

**Proof.** Suppose \( 1 \neq \langle u \rangle = H \cap L \). Then by 4.15, \( u \in Z^*(I) \), so \( J = O(I)L \). Define \( P = O_s(I) \) as in 4.12, and assume \( S \) has 2-rank at least two. Then \( P = \prod s \in S^* C_p(s) \), while by 4.9, \( C_p(s) \) is a p-group for \( s \in S^* \). Thus \( P \) is a p-group, contradicting 4.12.

So \( H \cap L = 1 \) and by 4.16, \( N_J(H) = QK \) is strongly embedded in \( I \). As \( Q \leq O(I) \) and \([K, H \cap L] = 1 \), Bender's classification of groups with a strongly embedded subgroup [6] implies \( J = O(I)N_J(H \cap L) \). By 4.5, augmented by arguments such as in 4.13 for the case \( q = 5^m \), \( m \) odd, \( N_J(H \cap L) = L \). Now arguing as above, \( S \) has 2-rank at most one.

Define \( P = O_s(I) \) as in 4.12. Set \( P_0 = O_p(K) \). \( P_0 \neq 1 \) by 4.12 and 4.18.

**Lemma 4.19.** (1) \( F(X) = H \cap D \) for \( 1 \neq X \leq P_0 \).

(2) \( H \cap K = 1 \).

(3) Assume \( u \) is an involution in \( K \) and let \( v \in u^c \) have cycle \((\bar{A}, \bar{B})\). Let \( P_i \) be a \( \langle u, v \rangle \) invariant Sylow p-group of \( O(K) \). Then \([v, P_i] = P_i \) and \([u, P_i] \neq 1 \).

**Proof.** Assume \( 1 \neq X \leq P_0 \) with \( F(X) \neq H \cap D \). Then \( Y = \langle C_D(X) \rangle \cong U_3(q) \) by 4.9. So \( H \cap K = \langle u \rangle \neq 1 \). Further as \( N_K(X)^{F(X)} \) is a p'-group, \( X = P_0 \). Let \((\bar{C}, \bar{E})\) be a cycle in \( u \) and \( v \in u^c \) fix \( \bar{C} \) and \( \bar{E} \). Then \([u, v] = 1 \) so \( v \) acts on \( \langle C_D(u) \rangle = H \) and thus also on \( P_0 \).
\(v\) induces an automorphism on \(Y \cong U_3(q)\) and therefore fixes points \(A_1 \in F(P_0)\). So \(C \in \langle \widetilde{A}_1, \widetilde{A}_2 \rangle \leq Y\) and therefore \(F(P_0) = \widetilde{D}\), a contradiction. This yields (1).

Assume \(1 \neq \langle u \rangle = H \cap K\). Then in particular \([u, P_0] = 1\). Let \(v \in u^o\) have cycle \((\widetilde{A}, \widetilde{B})\). \(v\) acts on \(P_0\) and \(F(v) \cap F(x) = F(v) \cap F(u) = \emptyset\) for \(x \in P_0^*\). Thus \(C_{P_0}(v)\) acts fixed point free on \(F(v)\) of order \(q + 1\), so \(C_{P_0}(v) = 1\). Define \(e\) and \(c\) as in 4.17. It follows that \(c = 1\) and \(e \equiv 0 \mod p\) for \(q + 1 = q[(q + 1)e/c + 1] \equiv q \mod pq\). So \(P_0Q\) is Sylow in \(P\) and \(u\) centralizes \(P_0Q\), and inverts a Hall \(p'\)-group \(P_i\) of \(P\). Thus \(P = P_1 \times (P_0Q)\) is nilpotent, contradicting 4.12. This yields (2).

Assume the hypothesis of (3) and define \(c\) and \(e\) as in 4.17. Arguing as above, \([v, P_1] = P_i\), so \(p\) divides \(e\). By 4.18, \(L = O(K)C_{L}(u)\), so if \([P, u] = 1\), then \(p\) does not divide \(c\). But then arguing as above we have a contradiction.

**Lemma 4.20.** \(q \equiv 1 \mod 4\).

**Proof.** Assume \(q \equiv -1 \mod 4\). By 4.9, 4.10, and 4.14, \(C_p(x)\) is a \(p\)-group for any involution \(x \in L\), while by 4.12, \(P\) is not a \(p\)-group. Thus \(L\) has 2-rank one. Suppose \(K\) has odd order. By 4.11, \(L\) has even order so there exists an involution \(x \in L\) and \(\langle x \rangle\) is Sylow in \(J\). If \(|F(x)| = 2\), then by 4.11, \(n = |\widetilde{D}| \equiv 2 \mod 4\), and [2] implies \(G \cong L_2(q)\). Thus by 4.14, \(|F(x)| = q + 1\). Let \(v\) be a conjugate of \(x\) with cycle \((\widetilde{A}, \widetilde{B})\). We may choose \(v = t\) or \(tx\) where \(t \in H\). By 4.16, \(F(P_0) = H \cap \widetilde{D}\), so \(|F(P_0) \cap F(v)| = 0\) or 2. Thus if \(C_{P_0}(v) \neq 1\) then \(1 \equiv q + 1 = |F(x)| \equiv 0 \mod 2p\), so \(v\) inverts \(P_0\). Thus \(v = tx\), and \(x\) inverts \(P_0\). Define \(e\) and \(c\) as in 4.17. Then \(e = (q - 1)c/2\), so by 4.17, \(n - 1 = q(q^2 + 1)/2\). In particular \(Q_{P_0}\) is Sylow in \(P\) and inverted by \(x\). As \(|F(x)| = q + 1\), \(x\) inverts an \(x\)-invariant Sylow \(r\)-subgroup of \(P\) for \(r \neq P\), with 4.10. Thus \(x\) inverts \(P\), and \(P\) is abelian, contradicting 4.12.

So \(K\) contains an involution \(w\). Let \(v \in u^o\) have cycle \((\widetilde{A}, \widetilde{B})\), with \([v, u] = 1\). As \(H \cap K = 1\) and \(v\) acts fixed point free on \(F(u) = H \cap \widetilde{D}\), \(v = t\) or \(ut\) where \(t \in H\). By 4.19 \([v, P_0] \neq 1\), so \(v = ut\). Thus defining \(e\) and \(c\) as in 4.17, \(e = (q - 1)c/2\), so by 4.17, \(n - 1 = q[(q + 1)e/c + 1] = q(q^2 + 1)/2\). Let \(R\) be a \(\langle u \rangle(H \cap L)\) invariant \(r\)-Sylow group of \(P\), where \(r \neq p\). Then \(\langle u \rangle(H \cap L)\) acts semiregularly on \(R\), \(|R| > q\). As a \(p'\)-Hall group of \(P\) has order \((q^2 + 1)/2, (q^2 + 1)/2\) is a prime power. Thus \(q\) is a prime (e.g. Lemma 3.1, [1]). \(P_0\) acts semiregularly on \(\widetilde{D} - F(P_0)\) of order \(q(q + 1)/2 - q = q(q^2 - 1)/2\), so \(|P_0| = q\). Thus \(Q = C_p(u) \leq Z(P)\), or \([P, u]\) is a Hall \(p'\)-group of \(P\). In either event \(P\) is nilpotent, contradicting 4.12.

Proof. Assume $K$ has even order and let $u$ be an involution in $K$ and $v$ a conjugate of $u$, with cycle $(\overline{A}, \overline{B})$. By 4.1, $[v, P] = P$ and $[u, P] \neq 1$. So $C_{P}(uv) \neq 1$, $|F(uv)| \equiv 0 \bmod p$ and $uv \not\in u^{g}$. So by 4.11 and 4.18, $uv \in x^{2}$ or $(ux)^{q}$ where $x \in H$. Now $[x, P] = 1$ so $|F(x)| \equiv 2 \bmod p$. Thus $uv \in (ux)^{g}$ and as $|F(uv)| \equiv 0 \bmod p$ and $|F(P_{0}) \cap F(ux)| = 2$, $C_{P_{0}}(ux) = C_{P_{0}}(u) = 1$. So $Q = C_{P}(u)$, yielding a contradiction as in 4.20.

LEMMA 4.22. $L$ has 2-rank one.

Proof. Assume not. Then as $|K|$ is odd by 4.21, there exists an involution $x \in H \cap L$ and an involution $u \in L$ with $|C_{Q}(u)| = r$, $q = r^{2}$, and $Q = C_{Q}(u) \times C_{Q}(ux)$. Notice $P = C_{P}(x)C_{P}(u)C_{P}(ux) = C_{P}(x)Q$. Set $m + 1 = |F(x)|$. As $P_{0}$ acts semi-regularly on $F(x) - \{\overline{A}, \overline{B}\}$, $m \equiv 1 \bmod p$. Let $P_{2}$ be a subgroup of $C_{P}(x)$ maximal with respect to being normal in $C_{P}(x)$ and semiregular on $F(x) - \{\overline{A}\}$. Let $M/P_{2}$ be a minimal subgroup of $C_{P}(x)/P_{2}$ contained in $C_{P}(x)$. By 4.10, $M/P_{2}$ is a $p$-group and as $P_{2}$ is semi-regular on $F(x) - \{\overline{A}\}$ of order $m \equiv 1 \bmod p$, $P_{2}$ is a $p'$-group. Thus $M = P_{2}(P_{0} \cap M) = P_{2}M_{0}$ and $C_{P}(x) = P_{2}(N(M_{0}) \cap C_{P}(x)) = P_{2}C_{P}(x)$ as $F(x) \cap F(M_{0}) = \{\overline{A}, \overline{B}\}$. So $|P_{2}| = m$ and $P_{2} \leq QC_{P}(x) = P$. Thus $P_{2}Q$ is regular on $\overline{D} - \{\overline{A}\}$. As $u$ inverts $P_{2}$, $P_{2}Q$ is nilpotent and thus contained in $\text{Fit}(P)$, the Fitting subgroup of $P$. So $\text{Fit}(P)$ is transitive on $\overline{D} - \{\overline{A}\}$ and nilpotent, contradicting 4.12.

LEMMA 4.23. $|\overline{D}| \equiv 2 \bmod 4$.

Proof. Assume not. Let $x$ be the involution in $H \cap L$. By 4.11, $|F(x)| \equiv 0 \bmod 4$. As in 4.14, $C_{Q}(x)^{F(x)}$ is a Zassenhaus group and $t$ inverts $L^{F(x)}$ where $t \in H$ has cycle $(\overline{A}, \overline{B})$. But $[t, P_{0}] = 1$ and $P_{0} \cong P_{0}^{F(x)}$, a contradiction.

4.22 and 4.23 together with [2] imply $G \cong L_{2}(q)$ or $U_{3}(q)$. Thus the proof of Theorem 4.1 is complete.

5. Examples.

Hypothesis 5.1. Let $V$ be a 2m dimensional space over $GF(q)$, $q$ a power of the odd prime $p$, with nondegenerate skew symmetric bilinear form $(,)$. For $u \in V^{*}$ the transvection $u^{*}$ determined by $u$ is the map $u^{*}: \langle x \rangle \longrightarrow \langle x + (x, u)u \rangle$
considered as a projective transformation of $V$. Let $D = \{\langle u^* \rangle: u \in V^t\}$ and $G = \langle D \rangle$.

$G$ is the $2m$ dimensional projective symplectic group $SP_{2m}(q)$ over $GF(q)$.

**Lemma 5.2.** Assume hypothesis 5.1. Let $A = \langle a^* \rangle$ and $B = \langle b^* \rangle$ lie in $D$ with $[A, B] \neq 1$. Set $L = \langle D_A \cap D_B \rangle$. Then

1. $D$ is a class of $p$-transvections of $G$.
2. $L/Z(L) \cong SP_{2m-2}(q)$ for $m > 1$.

**Proof.** Let $\langle c^* \rangle = C \in D$. Then $[A, C] = 1$ if and only if $(a, c) = 0$. So $(,)\text{ restricted to } \langle a, b \rangle$ is a nondegenerate skew symmetric bilinear form and therefore $\langle A, B \rangle$ is a homomorphic image of a subgroup of $SL_2(q)$. This yields (1). Similarly $L$ acts as a symplectic group on $\langle a, b \rangle^{-1}$ yielding (2).

**Hypothesis 5.3.** Let $V$ be a $n$-dimensional vector space over $GF(q^2)$ with nondegenerate semibilinear form $(,)$. For nonsingular vector $u$ let $u^*$ be the transvection determined by $u$ considered as a projective transformation of $V$. Let $D = \{u^*: (u, u) = 0\}$, and $G = \langle D \rangle$.

$G$ is the $n$ dimensional projective special unitary groups, $U_n(q)$.

**Lemma 5.4.** Assume hypothesis 5.3. Let $A = \langle a^* \rangle$ and $B = \langle b^* \rangle$ lie in $D$ with $[A, B] \neq 1$. Set $L = \langle D_A \cap D_B \rangle$ then

1. $D$ is a class of $p$-transvections of $G$.
2. $L/Z(L) \cong U_n(q)$ for $n \geq 4$.
3. $G$ contains a unique class of $D$-subgroups $K^o$ with $K/Z(K) \cong U_{n-1}(q)$.

**Proof.** The proofs of (1) and (2) are as in 5.2. Assume $K$ is a $D$-subgroup of $G$ with $K/Z(K) \cong U_{n-1}(q)$. As $[a^*, c^*] = 1$ if and only if $(a, c) = 0$, $\langle u: \langle u^* \rangle \in K \cap D \rangle$ is a nonsingular hyperplane of $V$ preserved by $K$. As $G$ is transitive on such hyperplanes, (3) follows.

6. Proof of main theorem. For the remainder of this paper $G$ is a counter example of minimal order to the main theorem. Lemma 3.1 implies:

**Lemma 6.1.** $G$ is simple.

Theorem 4.1 implies:
**Lemma 6.2.** \( \mathcal{D}(D) \) is connected.

Let \( A \in D \). By 2.4, \( A \) is contained in a unique maximal set of imprimitivity \( \alpha \) of \( G^p \). Set \( H = \langle D_\alpha \rangle, M = O_\omega(H) \), and \( \Omega = \alpha^g \). By 2.4, \( H \) is \( D^*_\omega \)-simple. Minimality of \( G \) implies \( H/M \equiv Sp_n(q) \) or \( U_n(q) \), for some power \( q \) of \( p \).

**Lemma 6.3.** Let \( \beta \in D_\alpha, \gamma \in D_\beta \cap A_\alpha \). Set \( \Gamma = D_\beta \cap D_\gamma \) and \( L = \langle \Gamma \rangle \). Then \( LM = H, M \neq Z(H) \) and \( \alpha^* \beta = \{\alpha\} \cup \beta^w \).

**Proof.** Let \( B \in \beta \). \( H/M \equiv Sp_n(q) \) or \( U_n(q) \) has \( V_{BM/M} \) as a set of imprimitivity on \( D^*_\omega \) or \( \beta \), so \( \mathcal{D}(L \cap D) \) is disconnected. Thus \( L/O_\omega(L) \equiv L(q) \times L(q) \) or \( U_\delta(q) \times U_\delta(q) \). As \( U_\delta(q) \) contains no \( D \)-subgroup of the latter type, that case is eliminated. As \( \beta = B^\perp \cap D_\delta, \beta = B^\perp \cap D^*_\alpha \). Now let \( \gamma \in \gamma \) with \( X = \langle A, C \rangle \equiv SL_n(q) \), and \( x \in X \) fix \( \alpha \) and \( \gamma \) with \( |x| \geq 4 \). \( x \) centralizes \( L \) and normalizes \( H \). Suppose \( L \neq \langle C_{D^*_\alpha}(x) \rangle = Y \). Then there exists \( \delta \in A_\delta \cap Y \). Minimality of \( G \) implies \( \mathcal{D}(Y \cap D) \) is connected so we can choose \( \delta \in D_\sigma \) for some \( \sigma \subseteq L \). Let \( Z = \langle \lambda, \delta \rangle \). As \( \gamma, \delta \in D_\delta, Z/O_\sigma(Z) \equiv SL_n(q) \). So as \( |x| = 1 \), we get \( [x, \lambda] = 1 \), a contradiction. So \( L = Y \) and as \( x \) induces an automorphism on \( H/M \equiv Sp_n(q) \) or \( U_n(q) \), with \( Y/O_\alpha(Y) \equiv L_{q^2} \times L_n(q) \), this automorphism has order two. As \( |x| > 2, 1 \neq x^2 \) centralizes \( H/M \). As \( [x^2, B^\perp \cap D^*_\alpha] = 1, H, x^2 \), so \( \langle x^2 \rangle = Z(X) \) and \( X \equiv SL_n(5) \). But now \( C_{D^*}(x^2) \) is a component of \( \mathcal{D}(D) \), contradicting 6.2.

So \( L \) is \( D \)-simple. Therefore, minimality of \( G \) implies \( L/O_\omega(L) \equiv H/M \) and \( O_\omega(K) = M_1 \cap K \neq Z(K) \). As \( D_\Gamma \cap (\alpha^* \beta) = \{\beta\}, \alpha^* \beta = \{\alpha\} \cup \beta^w \).

Thus we may assume \( n \leq 3 \). Suppose \( X = \langle A, E \rangle \equiv SL_n(q) \) for \( E \in D^*_\beta \). Then we may choose \( C \in \gamma \cap X \). Let \( \langle u \rangle = Z(X) \). Then \( u \in \langle A, C \rangle \), so \( [u, L] = 1 \). \( u \) acts on \( H/M \) and centralizes \( \beta \), so \( J = \langle C_{D^*_\alpha}(u) \rangle \) contains a \( D \)-subgroup isomorphic to \( SL_n(q_0) \) for some \( q_0 \) dividing \( q \). Let \( \langle v \rangle \) be the center of that subgroup. If \( J \neq L \) then considering \( \langle J, X \rangle \), minimality of \( G \) yields a contradiction. So \( J = L \) and \( [v, X] = 1 \). \( \langle C_{D^*_\alpha}(v) \rangle = X_0 \equiv SL_n(q) \), so arguing on \( v \) in place of \( u \) we get \( X_0 = L \) and \( q_0 = q \). If \( H = LM \), then \( D_\alpha = D_\gamma, M \neq Z(H) \), and as above \( \alpha^* \beta = \{\alpha\} \cup \beta^w \). So we may assume \( H/M \equiv U_n(q) \). Define \( x \) as above with \( u \in \langle x \rangle \). \( [x, L] = 1 \) and \( x \) acts on \( H/M \equiv U_n(q) \), so as \( 2 < |x| \) divides \( q - 1 \), \( u \in \langle x \rangle \) centralizes \( H/M \), contradicting \( LM \neq H \).

So \( X \) does not exist. Thus \( H \equiv L_\alpha(q) \). Claim \( \beta = B^\perp \cap D^*_\alpha = \alpha^* \beta \). For if not \( \beta \equiv \langle \alpha^* \beta \rangle \) whereas \( \alpha \not\equiv \langle \alpha^* \beta \rangle \).
Choose $1 \neq x \in H$ fixing $\alpha$ and $\lambda$. $x$ acts on $H$ and centralizes $\beta$, so $[x, H] = 1$. Let $E \in D^*_\alpha - L$ and $C \in \gamma$. The action of $x$ on $\langle C, E \rangle$ yields a contradiction.

**Lemma 6.4.** Let $(\alpha, \gamma, \beta)$ be a triangle in $\Omega$. Then there exists $\sigma$ with $\alpha, \beta,$ and $\gamma$ in $D_\sigma$.

**Proof.** Claim $\mathcal{D}(\Omega)$ has diameter two. For if not $\alpha\beta\gamma\delta$ be a chain with $d(\alpha, \delta) = 3$. Let $H_i = \langle D_i \rangle$, $M_i = O_\alpha(H_i)$, $\Gamma = D_\alpha \cap D_\gamma$ and $L = \langle \Gamma \rangle$. Then by 6.3, $H_i = LM_i$, so $\delta M_i = \sigma M_i$ for some $\sigma \in \Gamma$. Thus $\sigma \in D_\alpha \cap D_\gamma$, contradicting $d(\alpha, \delta) = 3$. Thus $\mathcal{D}(\Omega)$ has diameter two, so if $(\alpha, \gamma, \beta)$ is a triangle, by 6.3, $LM = H$. So again there exists $\sigma \in \Gamma$ with $\sigma M = \beta M$. $\alpha, \beta,$ and $\gamma$ are in $D_\sigma$.

**Lemma 6.5.** Let $\gamma \in A_\alpha$. Then $\langle \alpha, \gamma \rangle \cong SL_2(q)$ and $|\langle \alpha \rangle| = q$.

**Proof.** Set $X = \langle \alpha, \gamma \rangle$. By 6.4, there exists $\beta \in D_\alpha \cap D_\gamma$. Let $H_i = \langle D_i \rangle$, $M_i = O_\alpha(H_i)$. Suppose $A \neq E \in \alpha$ with $A \equiv E \bmod M_i$. Then $A = \langle a \rangle$, $E = \langle e \rangle$ with $x = ae^{-1} \in M_i$. Thus $x$ fixes every singular line $\beta^*\delta = \{\beta\} \cup \delta^{M_i}$ through $\beta$. As $H \leq C_\phi(x)$ is transitive on $D_\alpha$, $x$ fixes all singular lines through any $\beta \in D_\alpha$. Let $\sigma \in A_\alpha$. By 6.3, there are distinct singular lines $\beta^*_i\sigma$, $i = 1, 2$, with $\beta_i \in D_\alpha$. Then $x$ fixes $(\beta^*_i\sigma) \cap (\beta^*_i\sigma) = \{\sigma\}$. Thus $x$ fixes $\Omega$ pointwise. But this contradicts 6.1.

So $|\langle \alpha \rangle| = |\langle \alpha \rangle M/M| = q$ by 6.3. By 6.3, $X/O_{\phi}(X) \cong SL_2(q)$, so $|\langle \alpha \rangle | = q, O_{\phi}(X) = 1$.

**Lemma 6.6.** $\Omega$ is locally conjugate in $G$, $\langle \alpha^\perp \rangle$ is transitive on $A_\alpha$, and $G^0$ is rank 3.

**Proof.** By 6.5, $\Omega$ is locally conjugate in $G$. Therefore, to show $\langle \alpha^\perp \rangle$ is transitive on $A_\alpha$ and thus that $G^0$ is rank 3, it suffices to show (*) of 2.7. But if $(\alpha, \gamma, \beta)$ is a triangle in $\Omega$, set $X = \langle \alpha, \gamma, \beta \rangle$. Then by 6.3, $X/O_{\phi}(X) \cong SL_2(q)$ with $\alpha^\perp \cap X = \alpha^{\phi(X)}$. So 3.3 yields (*).

Following the notation of D. Higman let $k = |D_\alpha|$, $l = |A_\alpha|$, $\lambda = |D_\alpha \cap D_\beta|$ for $\beta \in D_\alpha$, and $\mu = |D_\alpha \cap D_\gamma|$ for $\gamma \in A_\alpha$. Let $m = |\beta^\perp|$. [10] implies:

**Lemma 6.7.** $l = k(k - \lambda - 1)/\mu$ and either

1. $k = l$ and $\mu = (\lambda + 1)/2 = k/2$ or
2. $d^2 = (\lambda - \mu)^2 + 4(k - \mu)$ is a square and $d$ divides $2k + (\lambda - \mu)(k + l)$.

**Lemma 6.8.** $O_\omega(L) = Z(L)$.
Proof. Assume not. Then there exists \( x \in O_\omega(L) = L \cap M \) with \( B^x \neq B \). By 6.5, \( \beta^x \neq \beta \), so \( \beta^x \in (\alpha^* \beta) \cap D_r = \{ \beta \} \), a contradiction.

**Lemma 6.9.** \( \alpha^* \gamma = \langle \alpha, \gamma \rangle \cap \Omega \) has order \( q + 1 \). If \( H/M \cong U_s(q) \) then \( m = q^2 \).

Proof. Assume \( n \geq 4 \). Then a hyperbolic line \( \beta \delta \) is as claimed. But \( \beta^* \delta \subseteq \beta \delta \) while clearly \( \langle \beta, \delta \rangle \cap \Omega \subseteq \beta^* \delta \). Next assume \( n = 2 \). Then by 6.3, \( D_\alpha \cap D_\gamma = \langle \beta, \delta \rangle \cap \Omega \) for \( \beta, \delta \in D_\alpha \cap D_\gamma \), and \( D_\beta \cap D_\gamma = \langle \alpha, \gamma \rangle \cap \Omega \), so \( \alpha^* \gamma \) is as claimed. Finally assume \( H/M \cong U_s(q) \). Let \( Z = Z(\langle \alpha \rangle) \). \( Z \) acts semiregularly on \( \alpha^* \gamma \). So if \( |\alpha^* \gamma| = q + 1 \) then \( |Z| = q \). If \( |\alpha^* \gamma| \neq q + 1 \) then \( \alpha^* \gamma = D_\beta \cap D_\delta \), for \( \beta, \delta \in D_\alpha \cap D_\gamma \). So \( |\alpha^* \gamma| = q^3 \) and \( N_\alpha(\alpha^* \gamma) \) acts as a subgroup of \( Aut(U_s(q)) \).

Thus by 3.4, \( Z \) is elementary abelian, while an elementary subgroup of \( Aut(U_s(q)) \) acting semiregularly on \( q^3 \) letters has order at most \( q \). Further \( |\alpha^* \gamma| - 1 = |N_{M(\alpha)}(\alpha^* \gamma)| = |C_{M(\alpha)}(L)| = |Z| = q \) by 3.4. So \( |\alpha^* \gamma| = q + 1 \).

Finally \( \mu = |\Gamma| = q^2 + 1 \), \( \lambda = m - 1 \), and \( k = \mu m \) by 6.3 and 6.8. Thus by 6.7, \( q^3 m^2 = l \), while by 6.6, \( l = |\langle \alpha \rangle| = N_{M(\alpha)}(\alpha^* \gamma) \) by 3.4. Thus \( m = q^2 \).

**Lemma 6.10.** If \( H/M \cong L_s(q) \) then \( m = q \) or \( q^2 \). If \( H/M \cong Sp_s(q) \) or \( U_s(q), n \geq 3 \), then \( m = q \) or \( q^2 \) respectively.

Proof. Assume \( H/M \cong L_s(q) \). Then \( \mu = q + 1 \), \( k = \mu m \) and \( \lambda = m - 1 \). So by 6.7, \( l = m^2 q \) and \( \mu + \lambda = m + q \) divides \( 2k + (\lambda - \mu)(k + l) \equiv -2(q^2 - 1)q \mod (m + q) \). By 3.3, an element of order \( q - 1 \) in \( L \) acts semiregularly on \( ([A, M]/Z)^g \) of order \( m - 1 \), so \( q - 1 \) divides \( m - 1 \). Thus \( q \) divides \( m = q^r + 1 \). So \( q^r + 1 \) divides \( 2(q^2 - 1) \) and therefore \( r \leq 1 \). That is \( m = q \) or \( q^2 \).

So with 6.9 we can assume \( n \geq 4 \). Therefore, singular lines in \( L \) have order \( q \) or \( q^2 \), respectively. Thus as \( \alpha^* \beta = \{ \alpha \} \cup \beta^\mu \) these lines are also lines in \( G \).

**Lemma 6.11.** \( H/M \cong U_s(q) \) and \( m = q^2 \).

Proof. If not \( \mu = \lambda + 2 \), so \( \mathcal{B}(\Omega) \) is a symmetric block design. Further all lines have order \( q + 1 \). Thus a result of Dembowski and Wager [8] implies \( \mathcal{B}(\Omega) \) is \( (n + 1) \)-dimensional projective space over \( GF(q) \). As \( G \) is generated by the set of elations of \( \mathcal{B}(\Omega) \) commuting with the symplectic polarity \( \alpha \leftrightarrow \alpha^\perp \), \( G \cong Sp_{s+2}(q) \).

The case \( n = 2 \) must be treated differently since in this case the existence of \( D \)-subgroups isomorphic to \( U_s(q) \) are not assured. The following lemma treats this special case.
Lemma 6.12. \( n \geq 3 \).

Proof. Assume \( n = 2 \). Let \( \beta, \delta \in \Gamma \), and set \( X = L_{\beta} \). We first determine the fixed point sets of elements of \( L \).

If \( x \in \langle \beta \rangle^2 \) then \( F(x) = \beta' \). If \( x \in X - Z(L) \), then \( F(x) = \{ \beta, \delta \} \cup \alpha^* \gamma \). For if \( \sigma \in F(x) \) is not as claimed, then by 3.3, \( \sigma \in A_x \). \( x \) normalizes \( \langle \beta, \alpha \rangle \cong SL_2(q) \) and centralizes \( \alpha \), so \( x \) centralizes \( \sigma \). Thus a similar argument on \( \langle \sigma, \beta \rangle \) and \( \langle \sigma, \delta \rangle \) shows \( \sigma \in D_\beta \cap D_\delta = \alpha^* \gamma \). If \( \langle x \rangle = Z(L) \) then \( F(x) = \Gamma \cup (\alpha^* \gamma) \). For arguing as above \( F(x) = C_\alpha(x) \), and minimality of \( G \) implies \( \langle C_\alpha(x) \rangle / Z(\langle C_\alpha(x) \rangle) \cong L_2(q) \times L_2(q) \); that is \( C_\alpha(x) = \langle \sigma \rangle = \alpha^* \gamma \). Finally let \( x \in L \) act fixed point free on \( \Gamma \). As above \( F(x) = C_\alpha(x) \) and as \( D_\alpha \cap C_\alpha(x) \) is empty, \( \langle C_\alpha(x) \rangle = Y_{F(x)} \cong SL_2(q) \) or \( U_2(q) \). And if \( Y \cong U_2(q) \) then \( Y \) is doubly transitive so \( x \in \langle D_\alpha \cap D_\delta \rangle \) for \( \sigma \in F(x) - \{ \alpha \} \). Thus \( x \) is in \( q \) distinct conjugates of \( L \) in \( H \). However, with 3.3, \( C_M(x) = \langle \alpha \rangle \), so there are \( m^2 q(q - 1)/2 \) conjugates of \( \langle x \rangle \) in \( H \). On the other hand there are \( m^2 \) conjugates of \( L \), each containing \( q(q - 1)/2 \) conjugates of \( \langle x \rangle \), so \( \langle x \rangle \) is in a unique conjugate of \( L \). So \( F(x) = \alpha^* \gamma \).

Let \( \bar{G} = U_4(q) \), let \( \bar{D} \) be the class of subgroups generated by transvections in \( \bar{G} \), let \( \bar{\alpha} \) consists of the members of \( \bar{D} \) whose center is a given singular point of the associated projective space, and let \( \bar{\sigma} = \bar{\alpha}^\beta \). Let \( \gamma \in A_\bar{\alpha} \) and \( \bar{L} = \langle D_{\gamma} \cap D_{\alpha} \rangle \). The discussion above implies \( \bar{L} \) is permutation isomorphic to \( L^2 \).

Lemma 6.3 implies that every \( \sigma \) in \( \Omega - (\alpha^* \beta) \) appears in a unique \( D_{\beta_i}, \beta_i \in \alpha^* \beta \). Set \( K = L_{\beta} \), and let \( t \in L \) have cycle \( (\beta, \delta) \). Let \( \sum_{i=0}^{\beta_i^K} \beta_i^K \) be a partition of \( \alpha^* \beta \) with \( \beta_0 = \alpha \) and \( \beta_i = \beta \). Set \( \Lambda_i = (\beta_i - (\alpha^* \beta)) \cup \{ \beta_0 \} \), and \( \Lambda = U \Lambda_i \). Then \( L \) maps the edge set of \( \mathcal{D}(A) \) onto the edge set of \( \mathcal{D}(\Omega) \), except for edges in \( \mathcal{D}(\alpha^* \beta) \).

Let \( T \) be permutation isomorphism of \( L \) and \( \bar{L} \), and let \( \bar{\beta} = \beta T \). Let \( \bar{\beta}_i^K \) be orbits of \( KT \) on \( \bar{\alpha}^* \bar{\beta} \) and define \( \bar{L} \) as above with respect to these \( \bar{\beta}_i \). There exists an isomorphism \( S \) of \( \mathcal{D}(A) \) and \( \mathcal{D}(\bar{L}) \) such that \( S \) restricted to \( \mathcal{D}(\Lambda_i) \) commutes with \( T \) restricted to \( N_\Lambda(\Lambda_i) \) and \( N_\bar{L}(\sigma S) = (N_\bar{L}(\sigma)) T \) for \( \sigma \in \Lambda_i \). For \( \sigma \in \Lambda_i \) there exists \( \bar{\sigma} \in \bar{\Lambda_i} \) with \( N_\bar{L}(\bar{\sigma}) = (N_\bar{L}(\bar{\sigma})) T \) from the discussion above, so \( \bar{S} \) can be defined in the obvious manner.

So we can apply 2.6 to show \( \mathcal{D}(\Omega) \cong \mathcal{D}(\bar{\Omega}) \) and thus \( G \cong \bar{G} \), if we show condition (ii) of 2.6 is satisfied.

Clearly (ii) holds on \( \Lambda_0 \). Suppose \( \sigma, \sigma' \in \Lambda_i, x \in L \). Claim \( \sigma^x = \sigma^y \) for \( y \in K \). As \( L = K \cup KtK \) we can assume \( x = t \). Thus \( \sigma^x \in D_\beta \cap D_\delta = \alpha^* \gamma \), so \( \sigma = \sigma' \) is fixed by \( t \). But \( K = N_\Lambda(\Lambda_i) \), so (ii) holds here.

Suppose \( \sigma, \sigma^x \in \Lambda_i, i \geq 2 \). We consider the case \( |\sigma^x| = q^2 - 1 \); the case \( |\sigma^x| = q(q^2 - 1) \) is analogous. Now \( \langle \beta \rangle = N_\Lambda(\Lambda_i) \) and \( q^2 = |\Lambda_i \cup \bigcup_{\sigma \in D_{\omega}} D_{\omega}| \) in \( q \) orbits of length \( q \) under \( \langle \beta \rangle \). These are the points in orbits of length \( q^2 - 1 \) under \( L \). Let \( \theta \) be the set of edges \( (\beta_i^x, \omega) \) with \( y \in L \).
and $|\omega^k| = q^2 - 1$. Let $N$ be the number of orbits of $L$ on $\theta$. Then $q(q^2 - 1)N = |(\beta_i, \sigma)|N = |\theta| = |\beta_i^*|q^2 = (q^2 - 1)q^2$, so $N = q$. Thus $(\beta_i, \sigma^*) = (\beta_i, \omega^*)$ for some $\omega \in A_i, y \in \langle \beta \rangle$. That is condition (ii) holds on $A_i$.

This completes the proof of 6.12.

A unitary $(\alpha, \beta, \gamma)$ in $\Omega$ is a triple with $\beta \in A_\alpha$ and

$$\gamma \in \bigcap_{\delta \in \alpha^*} A_\delta.$$ 

**Lemma 6.13.** If $(\alpha, \beta, \gamma)$ is a unitary triple then $\langle \alpha, \beta, \gamma \rangle/Z(\langle \alpha, \beta, \gamma \rangle) \cong U_3(q)$.

**Proof.** We can choose a unitary triple $(\beta_1, \beta_2, \beta_3)$ in $H$. Set $X = \langle \beta_1, \beta_2, \beta_3 \rangle$. As $H/M \cong U_n(q), X/Z(X) \cong U_3(q)$. If $n = 3$ we can count the number of unitary triples and the number of such triples centralizing some $\alpha \in \Omega$. These two numbers are equal. So assume $n \geq 4$, and let $(\sigma_1, \sigma_2, \sigma_3)$ be a unitary triple. Choose $\beta \in D_{\sigma_1} \cap D_{\sigma_2}$. If $\sigma_3 \in D_\delta$ set $\beta = \alpha$. If not let $\alpha^* \beta$ be a singular line in $D_{\sigma_1} \cap D_{\sigma_2}$. By 6.3, we can assume $\alpha \in D_{\sigma_3}$. Thus as above we are through.

Let $(\alpha, \gamma, \delta)$ be a unitary triple in $D_\beta$. Set $J = \langle D_\delta \cap \Gamma \rangle$.

**Lemma 6.14.** $J/Z(J) \cong U_{n-1}(q)$.

**Proof.** If $n = 3$, $\langle \alpha, \gamma, \delta \rangle = D_\delta \cap D_{\sigma}$ for suitable $\sigma \in A_{D_\delta}$ and $J = \langle \beta^* \sigma \rangle$. If $n = 4$, $J$ has width one and a counting argument shows $|J \cap \Omega| = q^3 + 1$. Thus by minimality of $G, J/Z(J) \cong U_3(q)$. Finally if $n > 4$, then arguing as in 6.3, $J$ is transitive on $J \cap D$ and $\langle D_\delta \cap J \rangle/O_\omega(\langle D_\delta \rangle) \cong U_{n-3}(q)$, so minimality of $G$ implies the desired result.

**Lemma 6.15.** Let $\theta = \Gamma \cup \delta^L$ and $K = \langle \theta \rangle$. Then $K \cong SU_{n+1}(q)$ and $\Omega = \theta \cup \alpha^L$.

**Proof.** Claim $\theta^\delta = \theta$. Clearly $L$ normalizes $\theta$, so it suffices to show $\delta$ normalizes $\theta$. Let $\sigma \in \Gamma \cap A_\delta$. Then $\langle \sigma, \delta \rangle \cong SL_n(q)$, so $\delta^\sigma = \delta^\sigma \subseteq \theta$. Thus $\Gamma^\delta \subseteq \theta$. Using the fact that 6.15 is true in $U_{n+1}(q)$, one can check that

$$L = J(\bigcup_{\sigma \in \Omega} \langle \sigma_1^* \sigma_2 \rangle)$$

where $\Omega$ is the set of lines in $L - J$. Thus it suffices to show $X \cap \Omega \subseteq \theta$ when $X = \langle \sigma_1, \sigma_2, \delta \rangle$. But if $(\sigma_1, \sigma_2, \delta)$ is unitary, 6.13 implies $X \cap \Omega = \sigma_1^* \sigma_2 \cup \delta^{(\sigma_1, \sigma_2)} \subseteq \theta$ and if $(\sigma_1, \delta, \sigma_2)$ is a triangle then $X/O_\sigma(X) \cong SL_2(q)$ and 3.3 yields the same equality.

So $\theta^\delta = \theta$. $\alpha \in \theta$, so $K \neq G$. $Y = \langle D_\delta \cap \theta \rangle = \langle D_\delta \cap \Gamma, \delta \rangle$, so
$Y/O_n(Y) \cong U_{n-1}(q)$. \([L, \alpha] = 1\) and $\delta \in A_n$, so $I = D_n \cap \theta$. Arguing as above $\theta \cup \alpha^k$ is self normalizing, so $\Omega = \theta \cup \alpha^k$.

Let $Z = Z(K)$. $Z$ fixes $\theta$ pointwise and $K \subseteq C_\circ(Z)$ is transitive on $\Omega - \theta$, so $Z$ does not fix $\alpha$. $|SU_{n+1}(q)|/|SU_n(q)| = |\alpha^k| = |K: N_K(\alpha)|$ and $LZ/Z \cong SU_n(q)$, so $|Z| = (n + 1, q)$. Considering the covering group of $U_{n+1}(q)$ we get $K \cong SU_{n+1}(q)$.

Put $K$ and $D_\delta$ in the roles of $H$ and $\Lambda$ in 2.6. Then 6.15 and 5.4 together with 2.6 imply $G \cong U_{n+2}(q)$.

This completes the proof of the main theorem.

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