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A NOTE ON AN INEQUALITY FOR REARRANGEMENTS

CHRISTER BORELL

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If $a = (a_1, \dots, a_n)$ is a n -tuple of positive real numbers let $a^+ = (a_1^+, \dots, a_n^+)$ denote the rearrangement of a in increasing order and $a^- = (a_1^-, \dots, a_n^-)$ the rearrangement of a in decreasing order. In this note a characterization is given for those functions $f: R_+ \rightarrow R$ such that

$$(1) \quad \sum_1^n f\left(\frac{a_k^+}{b_k^+}\right) \leq \sum_1^n f\left(\frac{a_k}{b_k}\right) \leq \sum_1^n f\left(\frac{a_k^+}{b_k^-}\right)$$

holds for all $a, b \in R_+^n$. Here n is a fixed integer > 1 .

In [3] Minc proved that

$$\prod_1^n (a_k^+ + b_k^+) \leq \prod_1^n (a_k + b_k) \leq \prod_1^n (a_k^+ + b_k^-).$$

This inequality was generalized by London [1, Theorem 1], who (in an equivalent form) proved that (1) holds if the function $f(e^x - 1)$, $x \geq 0$, is convex and $f(x) \geq f(0)$, $x > 0$.

London [1, Theorem 2] also proved (1) if f is convex and $f(x) \geq f(0)$, $x > 0$, which in fact is contained in the previous case.

The proofs in [1] are based on an interesting representation theorem of Mirsky.

The purpose of this note is to characterize those functions f , for which (1) holds.

The left inequality in (1) is in fact a special case of a theorem of Lorentz [2, Theorem 1]. This theorem especially gives, that if $\Phi = \Phi(u, v)$ belongs to $C^{(2)}(R_+ \times R_+)$ then

$$(2) \quad \sum_1^n \Phi(a_k^+, b_k^+) \leq \sum_1^n \Phi(a_k, b_k) \quad \text{all } a, b \in R_+^n \quad (n > 1)$$

if and only if

$$(3) \quad \frac{\partial^2 \Phi}{\partial u \partial v} \leq 0.$$

From this it is fairly easy to deduce that (2) and (3) are both equivalent to

$$(4) \quad \sum_1^n \Phi(a_k, b_k) \leq \sum_1^n \Phi(a_k^+, b_k^-) \quad \text{all } a, b \in R_+^n \quad (n > 1).$$

We shall give an independent proof of these equivalences, which differs from that given in [2].

Let $a, b \in \mathbf{R}_+^n$ and choose $\varepsilon \in \mathbf{R}_+$ such that $\varepsilon < \min(\min a_k, \min b_k)$. If A is a finite set, let $|A|$ denote the number of distinct elements in A .

By adding the identities

$$\begin{aligned} \Phi(a_k, b_k) &= \Phi(\varepsilon, \varepsilon) - \Phi(a_k, \varepsilon) - \Phi(\varepsilon, b_k) + \int_{\varepsilon}^{a_k} \int_{\varepsilon}^{b_k} \frac{\partial^2 \Phi}{\partial u \partial v} du dv, \\ k &= 1, \dots, n, \end{aligned}$$

we have

$$(5) \quad \begin{aligned} \sum_1^n \Phi(a_k, b_k) &= n\Phi(\varepsilon, \varepsilon) - \sum_1^n \Phi(a_k, \varepsilon) - \sum_1^n \Phi(\varepsilon, b_k) \\ &+ \int_{\varepsilon}^{\infty} \int_{\varepsilon}^{\infty} |\{k \mid a_k \geq u, b_k \geq v\}| \frac{\partial^2 \Phi}{\partial u \partial v} du dv. \end{aligned}$$

Set

$$N(a, b; u, v) = |\{k \mid a_k \geq u, b_k \geq v\}|.$$

Then

$$N(a^+, b^-; u, v) \leq N(a, b; u, v) \leq N(a^+, b^+; u, v)$$

and so (2) and (4) follow from (3) and (5).

To prove, (4) implies (3), set

$$D(u, v) = N(a, b; u, v) - N(a^+, b^-; u, v).$$

In view of (5), (4) is then equivalent to

$$(6) \quad \int_0^{\infty} \int_0^{\infty} D(u, v) \frac{\partial^2 \Phi}{\partial u \partial v} du dv \leq 0.$$

If (4) holds for some $n > 1$, then it also holds for $n = 2$. To see this we only need to apply (4) to the n -tuples $a = (a_1, a_2, \varepsilon, \dots, \varepsilon)$ and $b = (b_1, b_2, \delta, \dots, \delta)$, where $\varepsilon > \max(a_1, a_2)$ and $0 < \delta < \min(b_1, b_2)$.

Choose $0 < u_1 < u_2, 0 < v_1 < v_2$ and set $a_1 = u_1, a_2 = u_2, b_1 = v_1$ and $b_2 = v_2$. Then $D(u, v) = 1$ if $u_1 < u \leq u_2, v_1 < v \leq v_2$ and $= 0$ elsewhere. Therefore (3) follows from (6).

Analogously we can prove that (2) implies (3).

In [2] there are also necessary and sufficient conditions on Φ in order for (2) ((4)) to hold, when Φ is only continuous. It is easy to see that (2) \Leftrightarrow (3) \Leftrightarrow (4) in this case too, if (3) is interpreted in the distribution sense, that is the left-hand side is a negative measure. A formal proof goes via regularization of Φ .

We now return to the inequality (1). This corresponds to

$$\Phi(u, v) = f\left(\frac{u}{v}\right),$$

which gives

$$\frac{\partial^2 \Phi}{\partial u \partial v} = -\frac{1}{v^2} \left\{ f''\left(\frac{u}{v}\right) \cdot \frac{u}{v} + f'\left(\frac{u}{v}\right) \right\} \leq 0,$$

that is

$$f''(x) \cdot x + f'(x) \geq 0, \quad x > 0.$$

This means that (1) holds if and only if the function $f(e^x)$, $-\infty < x < +\infty$, is convex.

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