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Relationships between a topological space (more generally a convergence space) and its associated function space $C(X)$ are investigated. The algebra of all continuous real-valued functions on a space X together with the continuous convergence structure is denoted by $C_c(X)$. After appropriate generalizations of the axioms of countability to convergence spaces, it is shown: 1. A completely regular topological space X is Lindelöf if and only if $C_c(X)$ is first countable. 2. A completely regular topological space X is separable and metrizable if and only if $C_c(X)$ is second countable. Generalizations of (1) and (2) are introduced, and results and examples which justify the use of axioms of countability in convergence space theory are presented.

1. Preliminaries. We wish to investigate the interplay between a convergence space (Limesraum, [1]) X and $C(X)$, the algebra of all continuous real-valued functions on X. Since the algebraic properties of $C(X)$ are not, in general, sufficient to determine the space X, we are led to consider additional structures on $C(X)$. The algebra $C(X)$ endowed with the *continuous convergence structure* (see [1]), which we denote by $C_c(X)$, proves to be particularly well suited for our work. We note that in the case of a locally compact topological space X, the continuous convergence structure on $C(X)$ coincides with the compact-open topology. Thus Theorem 3 ((1) in the above paragraph) generalizes a result proved by Warner for locally compact spaces. (See [5], Theorem 7.)

We will study the largest class of convergence spaces with the property that $C_c(X)$ determines the space X. Specifically, let $\mathrm{Hom}_c\, C_c(X)$ denote the collection of all continuous homomorphisms from $C_c(X)$ onto the reals together with the continuous convergence structure. A convergence space X is said to be *c-embedded* if $i_X: X \to \mathrm{Hom}_c\, C_c(X)$ is a homomorphism, where i_X maps each $x \in X$ to the homomorphism of point evaluation by x (i.e., $i_X(x) i_X(x)(f) = f(x) = f(x)$ for every $f \in C(X)$). Indeed, two c-embedded spaces X and Y are homeomorphic if and only if $C_c(X)$ and $C_c(Y)$ are bicontinuously isomorphic. (See [2], Satz 5.) Furthermore, Binz has shown in [2] that the c-embedded spaces are the largest class of convergence spaces with this property. It is evident that every completely regular topological space X is c-embedded. In a convergence space X, we will use the notation "$\phi \to x$" to indicate that a filter ϕ converges to x in X.

2. The aim of this section is to characterize Lindelöf and more generally upper $\aleph$-compact spaces.

We will first generalize a few topological concepts. By a *covering system* $\mathscr{S}$ of a convergence space X, we mean a collection of subsets of X with the property that for every convergent filter ϕ on X, there exists an $S \in \mathscr{S}$ such that $S \in \phi$. A *basic subcovering* of a covering system $\mathscr{S}$ is a subfamily $\mathscr{S}'$ of $\mathscr{S}$ with the property that for every convergent filter ϕ on X, there exists a finite number of elements in $\mathscr{S}'$, $\{S_i\}_{i=1}^n$, such that $\bigcup_{i=1}^n S_i \in \phi$.

Let $\aleph$ denote an arbitrary infinite cardinal number.

DEFINITION 1. A convergence space X is said to be *upper $\aleph$-compact* if every covering system of X has a basic subcovering of cardinal number less than $\aleph$. In particular, X is *Lindelöf* if it is upper $\aleph_1$-compact.

DEFINITION 2. A convergence space X is said to be *first countable* (respectively *$\aleph$-countable*) if for any point $x \in X$ and any filter ϕ convergent to x in X, there exists a coarser filter ϕ' such that $\phi' \to x$ and ϕ' has a countable basis (respectively a basis of cardinal number less than $\aleph$).

It is evident that our definitions correspond to the usual definitions in the case of topological spaces.

Given a convergence group G (see [1]), we note that G is $\aleph$-countable if and only if the condition in Definition 2 holds for filters convergent to the identity element in G.

We will need the following two technical results. Given a c-embedded convergence space X, let X' denote the underlying set X together with the weak topology induced by $C(X)$. We call X' the *associated completely regular space* of X and note that X' is homeomorphic to $\mathrm{Hom}_s\, C_c(X)$, where the subscript s denotes the topology of pointwise convergence.

LEMMA 1. *Let X be a c-embedded convergence space and X' its associated completely regular space. If ϕ is a convergent filter in X, then the filter $\bar{\phi}$ generated by*

$$\{\bar{M}^{X'} : M \in \phi\} ,$$

where $\bar{M}^{X'}$ is the closure of M in X', is also convergent in X.

Let $\phi \to x$ in X for some $x \in X$. We can consider ϕ convergent to x in $\mathrm{Hom}_c\, C_c(X)$. This means that for every convergent filter θ in $C_c(X)$, say $\theta \to f$, and for every $\varepsilon > 0$, there exists a $T \in \theta$ and an

$M \in \phi$ such that

$$w(T \times M) \subset \{f(x) + [-\varepsilon, \varepsilon]\} ,$$

where w is the evaluation map sending each (f, p) to $f(p)$ (i.e., $|g(y) - f(x)| \leqq \varepsilon$ for every $g \in T$ and every $y \in M$). Since X' carries the weak topology induced by all the functions in $C(X)$,

$$w(T \times \bar{M}^{X'}) \subset \{f(x) + [-\varepsilon, \varepsilon]\} .$$

Hence $\bar{\phi}$ converges to x in X.

We say that $\mathscr{R}$ is a *refinement* of a covering system $\mathscr{S}$, if $\mathscr{R}$ is a covering system with the property that each $R \in \mathscr{R}$ is contained in some element of $\mathscr{S}$.

LEMMA 2. *Let X be a c-embedded convergence space. Every covering system of X has a refinement consisting of sets closed in the associated completely regular space.*

Let $\mathscr{S}$ be a covering system of X and let Φ denote the collection of all convergent filters in X. For $\phi \in \Phi$, Lemma 1 implies $\bar{\phi} \in \Phi$. Therefore, there exists an $S \in \mathscr{S}$ such that $S \in \bar{\phi}$. Since $\bar{\phi}$ has a basis consisting of sets closed in X', we can choose a set $B_\phi \in \bar{\phi}$ such that B_ϕ is closed in X' and $B_\phi \subset S$. Of course $\bar{\phi}$ is coarser than ϕ and hence $\{B_\phi\}_{\phi \in \Phi}$ is indeed a refinement of $\mathscr{S}$.

THEOREM 1. *A c-embedded convergence space X is upper $\aleph$-compact (respectively Lindelöf) if and only if $C_c(X)$ is $\aleph$-countable (respectively first countable).*

Proof. Assume X is upper $\aleph$-compact. Again, denote by Φ the collection of all convergent filters in X. Let θ be an arbitrary filter in $C_c(X)$ convergent to $\mathbf{0}$, the zero function. This means that for every $1/n$, where $n \in N$, and every $\phi \in \Phi$ there exists a $T_{1/n,\phi} \in \theta$ and an $M_{1/n,\phi} \in \phi$ so that

$$w(T_{1/n,\phi} \times M_{1/n,\phi}) \subset \left[\frac{-1}{n}, \frac{1}{n}\right].$$

For a fixed $n \in N$, the collection

$$\{M_{1/n,\phi} \colon \phi \in \Phi\}$$

is a covering system of X and by assumption admits a basic subcovering

$$\mathscr{S}_n = \{M_\alpha : \alpha \in \mathscr{A}_n\}$$

of cardinal number less than $\aleph$. Let T_α be the element of θ that corresponds to M_α as above. That is,

$$w(T_\alpha \times M_\alpha) \subset \left[\frac{-1}{n}, \frac{1}{n}\right].$$

It follows that

$$\left\{T_\alpha : \alpha \in \bigcup_{n=1}^{\infty} \mathscr{A}_n\right\}$$

generates a filter θ' coarser than θ. Obviously θ' has a basis of cardinal number less than $\aleph$. It only remains to verify that $\theta' \to \mathbf{0}$. Given $1/n$ for $n \in N$ and $\phi \in \Phi$ there exists a finite subset of $\mathscr{A}_n$, $\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$, such that $\bigcup_{i=1}^k M_{\alpha_i} \in \phi$. Now $T = \bigcap_{i=1}^k T_{\alpha_i}$ is an element of θ' with the property that

$$w\left(T \times \bigcup_{i=1}^{k} M_{\alpha_i}\right) \subset \left[\frac{-1}{n}, \frac{1}{n}\right],$$

and hence θ' converges to $\mathbf{0}$ in $C_c(X)$.

Conversely, assume $C_c(X)$ is $\aleph$-countable. Let

$$\mathscr{S} = \{S_\alpha : \alpha \in \mathscr{A}\}$$

be an arbitrary covering system of X. Because of Lemma 2, we can assume that the elements of $\mathscr{S}$ are closed in the associated completely regular space. We will prove that $\mathscr{S}$ has a basic subcovering of cardinal number less than $\aleph$. For each $S_\alpha \in \mathscr{S}$, set

$$T_\alpha = \{f \in C(X) : f(S_\alpha) = \{0\}\}.$$

Clearly, the collection of all sets T_α for $\alpha \in \mathscr{A}$ generates a filter θ that converges to $\mathbf{0}$ in $C_c(X)$. By assumption, there exists a filter θ' coarser than θ, convergent to $\mathbf{0}$ in $C_c(X)$, and having a base of cardinal number less than $\aleph$. Let

$$\{D_\beta : \beta \in \mathscr{B}\}$$

be a basis for θ', where the cardinal number of the index set $\mathscr{B}$ is less than $\aleph$. Since $\theta' \to \mathbf{0}$, for every $\phi \in \Phi$ there exists a $D_\beta \in \theta'$ and an $L_\phi \in \phi$ such that

$$w(D_\beta \times L_\phi) \subset [-1, 1]. \tag{I}$$

For a fixed $\beta \in \mathscr{B}$, let the union of all sets L_ϕ that correspond to

D_β in the sense of (I) be denoted by R_β. It follows that

$$R = \{R_\beta : \beta \in \mathscr{B}\}$$

is a covering system for X. Since $\theta' \leqq \theta$, for a given $\beta \in \mathscr{B}$, there exists a finite subset $\mathscr{A}_\beta$ of $\mathscr{A}$ such that

$$D_\beta \supset \bigcap_{\alpha \in \mathscr{A}_\beta} T_\alpha .$$

We claim that

$$\text{(II)} \qquad R_\beta \subset \bigcup_{\alpha \in \mathscr{A}_\beta} S_\alpha .$$

Assume to the contrary, that there exists a point $x \in R_\beta \backslash \bigcup_{\alpha \in \mathscr{A}_\beta} S_\alpha$, where "$\backslash$" denotes the set theoretic difference. The fact that $\bigcup_{\alpha \in \mathscr{A}_\beta} S_\alpha$ is closed in the associated completely regular space X' implies that there exists a function $f \in C(X')$ such that

$$f(x) = 2 \quad \text{and} \quad f\Big(\bigcup_{\alpha \in \mathscr{A}_\beta} S_\alpha\Big) = \{0\} .$$

Because of the natural isomorphism from $C(X')$ onto $C(X)$, we can assume $f \in C(X)$. Clearly $f \in \bigcap_{\alpha \in \mathscr{A}_\beta} T_\alpha$ but, in view of (I), the function $f \notin D_\beta$. This contradicts the fact that $D_\beta \supset \bigcap_{\alpha \in \mathscr{A}_\beta} T_\alpha$, and hence our claim is established. Now, it follows from the inclusion (II) that the collection

$$\mathscr{S}' = \Big\{S_\alpha : \alpha \in \bigcup_{\beta \in \mathscr{B}} \mathscr{A}_\beta\Big\}$$

is a basic subcovering of $\mathscr{S}$. Furthermore, the cardinality of $\mathscr{S}'$ is less than $\aleph$, and thus X is upper $\aleph$-compact.

COROLLARY. *Let X be a c-embedded convergence space. If $C_c(X)$ is Lindelöf, then X is first countable.*

If $C_c(X)$ is Lindelöf then $C_c(C_c(X))$ is first countable. Since X is c-embedded, it is homeomorphic to a subspace of $C_c(C_c(X))$, and thus first countable.

In §4 we will provide examples of Lindelöf convergence algebras $C_c(X)$.

3. Here, we obtain a characterization of separable metrizable topological spaces.

Let X be a convergence space. By a *basis* for X, we mean a collection $\mathscr{T}$ of subsets of X with the following property: For any convergent filter ϕ on X, say $\phi \rightarrow x$, there exists a coarser filter ϕ' such that $\phi' \rightarrow x$ and ϕ' has a basis consisting of sets in $\mathscr{T}$.

DEFINITION 3. The least infinite cardinal number of a basis for X is called the *weight* of X. In particular, X is *second countable* if it has weight $\aleph_0$.

It is easy to verify that our definitions of basis, weight, and second countable coincide with the usual concepts in the case of topological spaces.

The following generalization of a topological result is evident.

REMARK. (a) Let X be a convergence space having weight $\aleph$. Then any subspace of X has weight less than or equal to $\aleph$.

(b) Any subspace of a second countable convergence space is second countable.

(c) A second countable convergence space is first countable.

THEOREM 2. *A c-embedded convergence space X has weight $\aleph$ (respectively is second countable) if and only if $C_c(X)$ has weight $\aleph$ (respectively is second countable).*

Proof. Assume X has weight $\aleph$. Let

$$\mathscr{T} = \{U_\alpha : \alpha \in \mathscr{A}\}$$

be a basis for X of cardinal number $\aleph$. Given $\alpha \in \mathscr{A}$, $r \in \boldsymbol{Q}$ (the rational numbers), and $n \in N$, we define the following subset of $C(X)$:

$$M_{\alpha,r,n} = \left\{f \in C(X) : f(U_\alpha) \subset \left[r - \frac{1}{n}, r + \frac{1}{n}\right]\right\}.$$

Denote by $\mathscr{M}$ the collection of all finite intersections of sets of the form $M_{\alpha,r,n}$, for $\alpha \in \mathscr{A}$, $r \in \boldsymbol{Q}$, and $n \in N$. Clearly, the cardinality of $\mathscr{M}$ is still $\aleph$. We now show that $\mathscr{M}$ is indeed a basis for $C_c(X)$. Let θ be an arbitrary convergent filter in $C_c(X)$. Say $\theta \rightarrow f$. Our assumption implies that for any convergent filter ϕ in X, say $\phi \rightarrow x$, there exists a convergent filter ϕ' which is coarser than ϕ, and has a base consisting of sets in $\mathscr{T}$. Thus, we can find a $U_\alpha \in \phi$, and a $T \in \theta$ such that

$$w(T \times U_\alpha) \subset \left\{f(x) + \left[\frac{-1}{2n}, \frac{1}{2n}\right]\right\}.$$

Now choose as $r \in \boldsymbol{Q}$ so that

$$|f(x) - r| \leqq \frac{1}{2n}.$$

Because of our construction, there exists an $M_{\phi,n} \in \mathscr{M} (M_{\phi,n} = M_{\alpha,r,n})$ such that for every $g \in M_{\phi,n}$ and every $y \in U_\alpha$,

$$|g(y) - f(x)| \leqq |g(y) - r| + |r - f(x)| \leqq \frac{2}{n}$$

or

$$w(M_{\phi,n} \times U_\alpha) \subset \left\{f(x) + \left[\frac{-2}{n}, \frac{2}{n}\right]\right\}.$$

We observe that $M_{\phi,n} \supset T$, since

$$|g(y) - r| \leqq |g(y) - f(x)| + |f(x) - r| \leqq \frac{1}{n}$$

for every $g \in T$ and every $y \in U_\alpha$. Therefore, the collection of all $M_{\phi,n}$, for ϕ a convergent filter on X and $n \in N$, generates a filter θ' coarser than θ with a basis consisting of sets in $\mathscr{M}$. It is also clear that θ' converges to f. Further, there can exist no basis $\mathscr{M}'$ for $C_c(X)$ of cardinality strictly less than $\aleph$. If such an $\mathscr{M}'$ existed, then, as we have just proved, $C_c(C_c(X))$ would have a basis of cardinality strictly less than $\aleph$. Because of the preceding remark and the fact that X is homeomorphic to a subspace of $C_c(C_c(X))$, it follows that X would have weight unequal to $\aleph$.

Conversely, assume $C_c(X)$ has weight $\aleph$. Then, as above, X must have weight less than or equal to $\aleph$. The necessity of the theorem implies that X has weight exactly $\aleph$.

Since a completely regular topological space is separable and metrizable if and only if it is second countable (see [4], p. 187 and p. 195), we have the following result.

THEOREM 3. *A completely regular topological space X is separable and metrizable if and only if $C_c(X)$ is second countable.*

COROLLARY. *Let X be a completely regular topological space. $C_c(X)$ is a separable and metrizable topological space if and only if X is separable, metrizable and locally compact.*

For a completely regular topological space X, one can verify that $C_c(X)$ is topological space if and only if X is locally compact. (See [3], p. 329.) Thus, in view of the discussion preceding the last theorem, the proof is immediate.

4. We will extend two results that are well known for topological spaces to the class of convergence spaces.

THEOREM 4. *Let X be a convergence space that has weight less than $\aleph$ (respectively is second countable). Then any subspace of X is upper $\aleph$-compact (respectively Lindelöf).*

Because of the remark in §3, it suffices to show that X itself is upper $\aleph$-compact. Consider $\mathscr{T} = \{T_\alpha\}$ to be a basis for X of cardinal number less than $\aleph$. Let $\mathscr{S}$ be an arbitrary covering system for X. For each $T_\alpha \in \mathscr{T}$ choose S_α to be a fixed element in $\mathscr{S}$ such that $S_\alpha \supset T_\alpha$ if such an element S_α exists. Denote by $\mathscr{S}'$ the collection of these S_α. Clearly $\mathscr{S}'$ is collection of cardinal number less than $\aleph$. We will verify that $\mathscr{S}'$ is actually a basic subcovering of $\mathscr{S}$. Let ϕ be an arbitrary convergent filter in X, say $\phi \to x$. By assumption, there exists a filter ϕ' coarser than ϕ such that $\phi' \to x$ and ϕ' has a basis consisting of sets in $\mathscr{T}$. Since $\mathscr{S}$ is a covering system, there exists an S in $\mathscr{S}$ with $S \in \phi'$. Because S must contain some element $T_{\alpha_0} \in \mathscr{T}$, where T_{α_0} is also in ϕ', we can find an $S_{\alpha_0} \in \mathscr{S}'$ such that $S_{\alpha_0} \supset T_{\alpha_0}$. Thus S_{α_0} is an element of both ϕ' and ϕ.

EXAMPLES. It is now easy to demonstrate that there exist convergence spaces that are upper $\aleph$-compact (respectively Lindelöf) and not topological, namely, $C_c(X)$ for X a completely regular topological space having weight less than $\aleph$ (respectively second countable) and not locally compact. Moreover, such a $C_c(X)$ has weight less than $\aleph$ (respectively is second countable) but is not topological.

For an example of a first countable convergence space that is neither second countable nor topological, consider $C_c(X)$ where X is a completely regular topological space which is Lindelöf and neither second countable nor locally compact.

In analogy with topological spaces, we say a subset S is *dense* in a convergence space Y if the adherence of S is Y. The space Y is said to be *separable* if it contains a countable dense subset.

THEOREM 5. *Any subspace of a second countable convergence space is separable.*

Let Y be a second countable convergence space with

$$\mathscr{T} = \{T_i\}_{i=1}^{\infty}$$

a countable basis. In light of the remark in §3, it is sufficient to prove that Y is separable. For each $T_i \in \mathscr{T}$, pick a $y_i \in Y$ such that $y_i \in T_i$. We claim that $\{y_i\}_{i=1}^{\infty}$ is dense in Y. Given $y \in Y$, there exists a filter ϕ convergent to y in Y with the property that ϕ has a basis consisting of sets in $\mathscr{T}$. Hence ϕ has a trace on $\{y_i\}_{i=1}^{\infty}$, which completes the proof.

REMARK. We have shown (Theorems 3, 4, and 5) that if X is a separable and metrizable topological space, then $C_c(X)$ is second countable, first countable, Lindelöf, and separable.

REFERENCES

1. E. Binz and H. H. Keller, *Funktionenräme in der Ketegorie der Limesräume*, Ann. Acad. Scie. Fenn. A, I, **383** (1966), 1-21.

2. E. Binz, *Zu den Beziehungen zwischen c-cinbettbaren Limesräumen und ihrer limitierten Funktionenalgebren*, Math. Ann., **181** (1969), 45-52.

3. N. Bourbaki, *General Topology*, Part II, Addison and Wesley, (1966).

4. J. Dugundji, *Topology*, Allyn and Bacon, Boston, (1966).

5. S. Warner, *The topology of compact convergence on continuous function spaces*, Duke Math. J., **25** (1958), 265-282.

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