

Pacific Journal of Mathematics

APOSYNDETTIC PROPERTIES OF HYPERSPACES

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Let X be a compact connected metric space and $2^X(C(X))$ denote the hyperspace of closed subsets (subcontinua) of X . In this paper the hyperspaces are investigated with respect to the property of aposyndesis. The main result states that each of 2^X and $C(X)$ is aposyndetic. If X is semi-aposyndetic, then each of 2^X and $C(X)$ is mutually aposyndetic. An example is given of a non-semi-aposyndetic continuum for which $C(X)$ is not mutually aposyndetic. In an extension of the main result for $C(X)$ it is shown that $C(X)$ is countable closed set aposyndetic. The techniques utilize the partially ordered structure of 2^X and $C(X)$.

A *continuum* will be a compact connected metric space and X will denote a continuum throughout. Each of 2^X and $C(X)$ is endowed with the finite (Vietoris) topology and since X is a continuum each of 2^X and $C(X)$ is also a continuum (see [5]). If A_1, \dots, A_n are subsets of X , then $N(A_1, \dots, A_n) = \{B \in 2^X \mid \text{for each } i = 1, \dots, n, B \cap A_i \neq \emptyset, \text{ and } B \subseteq \bigcup_{i=1}^n A_i\}$. If n is a positive integer, $F_n(X) = \{B \in 2^X \mid B \text{ has at most } n \text{ elements}\}$ and $F(X) = \bigcup_{n=1}^{\infty} F_n(X)$.

For notational purposes, small letters will denote elements of X , capital letters will denote subsets of X and elements of 2^X , and script letters will denote subsets of 2^X . If $A \subseteq X$, then A^* (int A) (bd A) will denote the closure (interior) (boundary) of A in X .

The concept of aposyndesis was introduced by F. Burton Jones [3] and several extensions of this concept have been studied. Let $p, q \in X, p \neq q$. X is *aposyndetic at p with respect to q* provided there exists a continuum M such that $p \in \text{int } M$ and $q \in X - M$. If for each $q \in X - p$, X is aposyndetic at p with respect to q , then X is *aposyndetic at p* . If X is aposyndetic at each of its points then X is *aposyndetic*. X is *semi-aposyndetic* if for each pair (p, q) of distinct elements of X , X is aposyndetic at p with respect to q or X is aposyndetic at q with respect to p (L. E. Rogers [9]). X is *mutually aposyndetic* if for each pair (p, q) of distinct elements of X there exist disjoint continua M and N such that $p \in \text{int } M$ and $q \in \text{int } N$ (Hagopian [2]). Let \mathcal{F} be a collection of closed subsets of X . Then X is *\mathcal{F} -aposyndetic* if for each $x \in X$ and each $F \in \mathcal{F}$ such that $x \notin F$ there exists a continuum M such that $x \in \text{int } M$ and $M \cap F = \emptyset$ (Bennett [1]). If \mathcal{F} is the collection of finite (countable closed) subsets of X and X is \mathcal{F} -aposyndetic then X is said to be *finitely (countable closed set) aposyndetic*.

Let (X, \leq) be a partially ordered space. If $x \in X$, then $S(x) =$

$\{y \in X \mid y \leq x\}$ and $T(x) = \{y \in X \mid x \leq y\}$ are called the *lower and upper sets of x* respectively. Similarly, if $A \subseteq X$, then $S(A) = \cup \{S(a) \mid a \in A\}$ and $T(A) = \cup \{T(a) \mid a \in A\}$ are called the *lower and upper sets of A* respectively. If X is compact and the partial order is closed ($\{(x, y) \mid x \leq y\}$ is closed in $X \times X$), then $S(A)$ and $T(A)$ are closed whenever A is closed (Proposition 4, p. 44 of [6]). A is said to be *decreasing* if $A = S(A)$ and *increasing* if $A = T(A)$.

The following definition is due to L. Nachbin [6]. A partially ordered space is *normally ordered* if for each pair A, B of disjoint closed subsets of X such that $A = S(A)$ and $B = T(B)$, there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$ and such that $U = S(U)$ and $V = T(V)$. It is known that any compact Hausdorff space with closed partial order is normally ordered (Theorem 4, p. 48 of [6]).

In a partially ordered space, a *chain* is a totally ordered subset. An arc which is also a chain is called an *order arc*.

It is easy to establish that the natural partial order on $2^X(C(X))$ induced by inclusion is a closed partial order. If $A, B \in 2^X(C(X))$, then there exists an order arc in $2^X(C(X))$ from A to B if and only if $A \leq B$ and each component of B intersects A (Lemma 2.3 and Lemma 2.6 of [4]).

LEMMA 1. *Let $A \in 2^X$ and $M \in C(X)$. Then $T(A), T(M) \cap C(X)$, and $S(M) \cap C(X)$ are continua. Consequently, if \mathcal{A} is a closed set in $2^X(C(X))$, then $T(\mathcal{A})$ is a continuum in $2^X(C(X))$.*

Proof. $T(A)$ is a continuum in 2^X because $T(A) = \{B \in 2^X \mid A \subseteq B\}$ is the continuous image of 2^X under the function $C \rightarrow A \cup C$. If $N \in T(M) \cap C(X)$, there exists an order arc \mathcal{L} in $C(X)$ from N to X , and $\mathcal{L} \subset T(M) \cap C(X)$. It follows that $T(M) \cap C(X)$ is connected, and since the partial order is closed, $T(M) \cap C(X)$ is a continuum. $S(M) \cap C(X) = C(M)$, so $S(M) \cap C(X)$ is a continuum. For each $A \in \mathcal{A}$, $T(A)$ is a continuum in $2^X(C(X))$ and $X \in T(A)$. It follows that $T(\mathcal{A}) = \cup \{T(A) \mid A \in \mathcal{A}\}$ is a continuum in $2^X(C(X))$.

LEMMA 2. *$2^X(C(X))$ is locally connected at X .*

Proof. Let \mathcal{U} be an open set containing X and $N(U_1, \dots, U_n)$ be a basic open set such that $X \in N(U_1, \dots, U_n) \subset \mathcal{U}$. If $A \in N(U_1, \dots, U_n)$, then there exists an order arc \mathcal{L} in $2^X(C(X))$ from A to X . Since $A, X \in N(U_1, \dots, U_n)$, each element of \mathcal{L} is in $N(U_1, \dots, U_n)$. It follows that $N(U_1, \dots, U_n)$ is connected. Hence $2^X(C(X))$ is locally connected at X .

The following theorem of Nachbin [6] will yield a useful method for constructing continua in the hyperspaces.

THEOREM A. *Let X be a normally ordered space with closed partial order and A be a compact subset of X . Then every continuous, order-preserving, real-valued function on A can be extended to X in such a way as to remain continuous and order-preserving.*

We now have the necessary equipment to prove the main result.

THEOREM 1. *Each of 2^X and $C(X)$ is aposyndetic.*

Proof. Let $A \in 2^X(C(X))$. We will show that $2^X(C(X))$ is aposyndetic at A with respect to each of the other points of $2^X(C(X))$. If $A = X$, then by Lemma 2, $2^X(C(X))$ is locally connected at A and hence aposyndetic at A . So we will assume that A is a proper closed subset (subcontinuum) of X .

Let $B \in 2^X(C(X))$, $B \neq A$. If $B \subset A$ or if B and A are not related under inclusion, then there exists $x \in A$ such that $x \notin B$. Let U be an open set containing x such that $U^* \cap B = \emptyset$. Let V be an open set containing $A - U$ such that $x \notin V^*$. Then $A \in N(U, V)$ and $B \notin N(U^*, V^*) = N(U, V)^*$ (Lemma 2.3.2 of [5]). By Lemma 1, $T(N(U, V)^*)$ is a continuum, and $A \in \text{int } T(N(U, V)^*)$. Furthermore, $B \notin T(N(U, V)^*)$. Hence $2^X(C(X))$ is aposyndetic at A with respect to B .

Now suppose $A, B \in C(X)$ and $A \subset B$. Let $\mathcal{C} = \{A\} \cup \{B\} \cup F_1(X)$. Then \mathcal{C} is a compact subset of $C(X)$. Define $f: \mathcal{C} \rightarrow [0, 1]$ by

$$f(C) = \begin{cases} 0 & \text{if } C = A \text{ or } C \in F_1(X) \\ 1 & \text{if } C = B. \end{cases}$$

Since A is a proper subset of B , f is continuous and order-preserving, so by Theorem A, f has a continuous extension $\hat{f}: C(X) \rightarrow R$ (reals) which is also order-preserving. Since $C(X)$ is a continuum, $\hat{f}(C(X))$ is a closed interval, and since \hat{f} is order-preserving, for some $b \geq 1$, $\hat{f}(C(X)) = [0, b]$. Let $t \in (0, 1)$ and consider $\hat{f}^{-1}([0, t])$. If $L \in \hat{f}^{-1}([0, t])$, then $S(L) \subset \hat{f}^{-1}([0, t])$. By Lemma 1, $S(L)$ is a continuum. Moreover, $S(L) \cap F_1(X) \neq \emptyset$. Since $F_1(X) \subset \hat{f}^{-1}([0, t])$ and $\hat{f}^{-1}([0, t]) = \bigcup \{S(L) \mid L \in \hat{f}^{-1}([0, t])\}$, it follows that $\hat{f}^{-1}([0, t])$ is connected, and since \hat{f} is continuous, it follows that $\hat{f}^{-1}([0, t])$ is a continuum. Furthermore, $A \in \text{int } \hat{f}^{-1}([0, t])$ and $B \notin \hat{f}^{-1}([0, t])$. Hence $C(X)$ is aposyndetic at A with respect to B . This concludes the proof that $C(X)$ is aposyndetic.

Finally, suppose $A, B \in 2^X$ and $A \subset B$. Let U be an open set such that $A \subset U$ and $B - U^* \neq \emptyset$. Let $\mathcal{H} = N(U)^* \cup N(X - U)$. Observe that $A \in \text{int } \mathcal{H}$ and $B \notin \mathcal{H}$. Now $F(X) \cap \mathcal{H}$ is dense in \mathcal{H} .

We will show that $F(X) \cap \mathcal{K}$ is connected.

Let $\{x_1, \dots, x_n\} \in N(X - U)$. Then for each $i = 1, \dots, n$, $x_i \in X - U^*$ or $x_i \in \text{bd } U$. Let

$$C_i = \begin{cases} \text{the component of } X - U^* \text{ containing } x_i & \text{if } x_i \in X - U^* \\ x_i & \text{if } x_i \in \text{bd } U. \end{cases}$$

Then $C_i^* \cap \text{bd } U \neq \emptyset$, because C_i is a component of the open set $X - U^*$ and hence meets $\text{bd } (X - U^*) \subseteq \text{bd } U$. Let $x'_i \in C_i^* \cap \text{bd } U$. Let $\mathcal{D}_i = \{\{x_1, \dots, x_{i-1}, y, x'_{i+1}, \dots, x'_n\} \mid y \in C_i^*\}$. Now \mathcal{D}_i is the continuous image of the continuum C_i^* , so \mathcal{D}_i is a continuum in $N(X - U)$ containing $\{x_1, \dots, x_{i-1}, x_i, x'_{i+1}, \dots, x'_n\}$ and $\{x_1, \dots, x_{i-1}, x'_i, x'_{i+1}, \dots, x'_n\}$. Then for each $i = 2, \dots, n$, $\mathcal{D}_{i-1} \cap \mathcal{D}_i \neq \emptyset$. So $\bigcup_{i=1}^n \mathcal{D}_i$ is a continuum in $N(X - U)$ containing $\{x_1, \dots, x_n\}$ and $\{x'_1, \dots, x'_n\}$. For each $i = 1, \dots, n - 1$, let $\mathcal{L}_i = \{\{x'_1, \dots, x'_i, y\} \mid y \in X\}$. Then \mathcal{L}_i is a continuum in \mathcal{K} containing $\{x'_1, \dots, x'_i, x'_{i+1}\}$ and $\{x'_1, \dots, x'_i\}$, and for each $i = 2, \dots, n - 1$, $\mathcal{L}_{i-1} \cap \mathcal{L}_i \neq \emptyset$. Hence $\bigcup_{i=1}^{n-1} \mathcal{L}_i$ is a continuum in \mathcal{K} containing $\{x'_1, \dots, x'_n\}$ and $\{x'_1\}$. Let $\mathcal{M} = \{\{y\} \mid y \in C_1^*\}$. Then \mathcal{M} is a continuum in $N(X - U)$ containing $\{x'_1\}$ and $\{x_1\}$. So

$$\left(\bigcup_{i=1}^n \mathcal{D}_i \right) \cup \left(\bigcup_{i=1}^{n-1} \mathcal{L}_i \right) \cup \mathcal{M}$$

is a continuum in \mathcal{K} containing $\{x_1, \dots, x_n\}$ and $\{x_1\}$.

If $\{x_1, \dots, x_n\} \in N(U)^*$, we can use a similar construction to show that \mathcal{K} contains a continuum containing $\{x_1, \dots, x_n\}$ and $\{x_1\}$. So if $C \in F(X) \cap \mathcal{K}$ and $c \in C$, there exists a continuum in $F(X) \cap \mathcal{K}$ containing C and $\{c\}$. Hence $F(X) \cap \mathcal{K}$ can be written as a union of continua, each of which meets $F_1(X)$. Since $U^* \cup (X - U) = X$, it follows that $F_1(X) \subset F(X) \cap \mathcal{K}$, and since $F_1(X)$ is connected, it follows that $F(X) \cap \mathcal{K}$ is connected. Furthermore, $F(X) \cap \mathcal{K}$ is dense in \mathcal{K} , so \mathcal{K} is connected. Hence \mathcal{K} is a continuum containing A in its interior which misses B . This concludes the proof that 2^X is aposyndetic.

A continuum X is said to be *unicoherent* provided that whenever A and B are proper subcontinua such that $X = A \cup B$, then $A \cap B$ is connected. S. B. Nadler, Jr. [7] has proved that each of 2^X and $C(X)$ is unicoherent. D. E. Bennett [1] has shown that a unicoherent aposyndetic continuum is finitely aposyndetic. These results and Theorem 1 imply the following corollary.

COROLLARY 1. *Each of 2^X and $C(X)$ is finitely aposyndetic.*

THEOREM 2. *Let X be a semi-aposyndetic continuum. Then each of 2^X and $C(X)$ is mutually aposyndetic.*

Proof. Let $M, N \in C(X)$, $M \neq N$. Suppose $M \subset N$ or that M and N are not related under inclusion and that $N \notin F_1(X)$. Let $\mathcal{A} = \{M\} \cup \{N\} \cup F_1(X)$ and define $f: \mathcal{A} \rightarrow [0, 1]$ by

$$f(A) = \begin{cases} 0 & \text{if } A = M \text{ or } A \in F_1(X) \\ 1 & \text{if } A = N. \end{cases}$$

Then f is continuous and order-preserving, so by Theorem A f has a continuous extension $\hat{f}: C(X) \rightarrow R$ which is also order-preserving. Let $\hat{f}(C(X)) = [0, b]$ ($b \geq 1$) and $t_1, t_2 \in (0, 1)$, $t_1 < t_2$. As in the proof of Theorem 1, $\hat{f}^{-1}([0, t_1])$ is a continuum containing M in its interior which misses N . Now suppose $L \in \hat{f}^{-1}([t_2, b])$. Observe that $T(L) \subset \hat{f}^{-1}([t_2, b])$ and $X \in T(L)$. By Lemma 1, $T(L)$ is a continuum. Since $\hat{f}^{-1}([t_2, b]) = \bigcup \{T(L) \mid L \in \hat{f}^{-1}([t_2, b])\}$, $\hat{f}^{-1}([t_2, b])$ is a union of continua, each of which contains X . Hence $\hat{f}^{-1}([t_2, b])$ is a continuum containing N in its interior which misses M . Since $\hat{f}^{-1}([0, t_1]) \cap \hat{f}^{-1}([t_2, b]) = \emptyset$, it follows that $C(X)$ is mutually aposyndetic at (M, N) .

Let $A, B \in 2^X$, $A \neq B$. Suppose that $A \subset B$ or that A and B are not related under inclusion and $B \notin F_1(X)$. Let $x \in B - A$ and $y \in B$, $x \neq y$. Let U be an open set containing A and y such that $x \notin U^*$. Let V_y be an open set containing y such that $V_y^* \subset U$. Let V_x be an open set containing x such that $V_x^* \subset \text{int}(X - U)$. Let W be an open set containing $B - (V_x \cup V_y)$ such that $x, y \notin W^*$. (If $B - (V_x \cup V_y) = \emptyset$, replace $N(V_x, V_y, W)$ by $N(V_x, V_y)$ in the remainder of the argument.) Then $B \in N(V_x, V_y, W)$ and $N(V_x, V_y, W)$ is disjoint from the sets $N(U)^*$ and $N(X - U)$. As in the proof of Theorem 1, $N(U)^* \cup N(X - U)$ is a continuum containing A in its interior which misses B . By Lemma 1, $T(N(V_x, V_y, W)^*)$ is a continuum, and $B \in \text{int } T(N(V_x, V_y, W)^*)$. If $C \in T(N(V_x, V_y, W)^*)$, then C meets V_x^* and V_y^* , so C meets U and $\text{int}(X - U)$. Therefore $C \notin N(U)^* \cup N(X - U)$. So $T(N(V_x, V_y, W)^*) \cap (N(U)^* \cup N(X - U)) = \emptyset$. Hence 2^X is mutually aposyndetic at (A, B) .

Finally, suppose $A, B \in F_1(X)$, $A \neq B$. We will write $A = \{p\}$, $B = \{q\}$. Since X is semi-aposyndetic, we assume that X is aposyndetic at p with respect to q . Then there exists a subcontinuum M of X such that $p \in \text{int } M$ and $q \in X - M$. Let V be an open set containing q such that $V^* \cap M = \emptyset$. By Lemma 1, $T(N(V)^*)$ is a continuum in $2^X(C(X))$ and $\{q\} \in \text{int } T(N(V)^*)$. Now $2^M(C(M))$ is a subcontinuum of $2^X(C(X))$ and $\{p\} \in \text{int } 2^M(C(M))$. Moreover, $2^M(C(M))$ and $T(N(V)^*)$ are disjoint, since M and V^* are disjoint. Hence $2^X(C(X))$ is mutually aposyndetic at $(\{p\}, \{q\})$. This concludes the proof.

In the preceding theorem we have shown that if X is any continuum and at least one of A and B is not an element of $F_1(X)$, then

each of 2^X and $C(X)$ is mutually aposyndetic at the pair (A, B) . We now give an example of a non-semi-aposyndetic continuum for which $C(X)$ fails to be mutually aposyndetic at certain pairs of elements belonging to $F_1(X)$.

EXAMPLE 1. Let X be the planar continuum which is the union of $S^1 = \{(r, \theta) | r = 1\}$ and $S = \{(r, \theta) | \theta \geq 0 \text{ and } r = 1 + 1/(1 + \theta)\}$. J. T. Rogers, Jr. [8] has shown that for this X , $C(X)$ is homeomorphic to the cone over X . Moreover, the homeomorphism carries $F_1(X)$ onto the base of the cone.

Observe that if $p, q \in S^1$, then X is not semi-aposyndetic at (p, q) . To show that $C(X)$ is not mutually aposyndetic at $(\{p\}, \{q\})$ it will suffice to show that $X \times I$ ($I = [0, 1]$) is not mutually aposyndetic at (p', q') where $p' = (p, 0)$ and $q' = (q, 0)$.

Suppose $M_{p'}$ and $M_{q'}$ are disjoint continua containing p' and q' respectively in their interiors. Let $N_{p'}$ be the component of $(S^1 \times I) \cap M_{p'}$ which contains p' . Let U_1, \dots, U_n be a finite cover of $N_{p'}$ by spherical open sets such that $(\bigcup_{i=1}^n U_i) \cap M_{q'} = \emptyset$. Using the fact that each component of $(S^1 \times I) - (\bigcup_{i=1}^n U_i)$ is arcwise connected, it can be established that no component of $(S^1 \times I) - (\bigcup_{i=1}^n U_i)$ meets both $S^1 \times \{0\}$ and $S^1 \times \{1\}$. It follows that $\bigcup_{i=1}^n U_i$ contains a simple closed curve C which separates $S^1 \times I$ between $S^1 \times \{0\}$ and $S^1 \times \{1\}$. Furthermore, for some $c \in C$, $\bigcup_{i=1}^n U_i$ contains an arc $[p', c]$ such that $[p', c] \cap C = \{c\}$.

Let $N_{q'}$ be the component of $(S^1 \times I) \cap M_{q'}$ which contains q' and let V_1, \dots, V_m be a finite cover of $N_{q'}$ by spherical open sets disjoint from $\bigcup_{i=1}^n U_i$. In the analogous manner, $\bigcup_{i=1}^m V_i$ contains a simple closed curve D which separates $S^1 \times I$ between $S^1 \times \{0\}$ and $S^1 \times \{1\}$ and for some $d \in D$, $\bigcup_{i=1}^m V_i$ contains an arc $[q', d]$ such that $D \cap [q', d] = \{d\}$. It can now be shown (this involves a consideration of some properties of S^2) that $([p', c] \cup C) \cap ([q', d] \cup D) \neq \emptyset$, a contradiction. Hence $C(X)$ is not mutually aposyndetic at $(\{p\}, \{q\})$.

The final theorem extends the main result and Corollary 1 for $C(X)$. First we need the following lemma.

LEMMA 3. Let $M \in C(X)$ and \mathcal{A} be a countable closed set in $C(X)$. Then there exists a decreasing open set \mathcal{U} in $C(X)$ such that $M \in \mathcal{U}$ and $(\text{bd } \mathcal{U}) \cap \mathcal{A} = \emptyset$.

Proof. Let $\varepsilon > 0$ and let d denote the metric on X . For each $x \in M$ let $S_\varepsilon(x) = \{y \in X | d(x, y) < \varepsilon\}$. Let $U_\varepsilon = \bigcup_{x \in M} S_\varepsilon(x)$ and $\mathcal{U}_\varepsilon = N(U_\varepsilon)$. Then \mathcal{U}_ε is a decreasing open set in $C(X)$ which contains M . If $L \in \text{bd } \mathcal{U}_\varepsilon$, then there exists $y \in L$ such that

$$y \in \left(\bigcup_{x \in M} S_\varepsilon(x) \right)^* - \left(\bigcup_{x \in M} S_\varepsilon(x) \right).$$

It follows that for each $x \in M$, $d(x, y) \geq \varepsilon$ and for some $x_0 \in M$, $d(x_0, y) = \varepsilon$. Therefore if $\varepsilon_1 \neq \varepsilon_2$ and $L_1 \in \text{bd}(\mathcal{U}_{\varepsilon_1})$ and $L_2 \in \text{bd}(\mathcal{U}_{\varepsilon_2})$, then $L_1 \neq L_2$. Since \mathcal{A} is countable there exists $\varepsilon > 0$ such that $\text{bd}(\mathcal{U}_\varepsilon) \cap \mathcal{A} = \emptyset$.

THEOREM 3. *$C(X)$ is countable closed set aposyndetic.*

Proof. Let $M \in C(X)$ and \mathcal{A} be a countable closed set in $C(X)$ such that $M \notin \mathcal{A}$. If $M = X$, then by Lemma 2, $C(X)$ is locally connected at M and hence $C(X)$ is countable closed set aposyndetic at M .

Suppose M is a nondegenerate proper subcontinuum of X . Let $\mathcal{A}_S = \mathcal{A} \cap S(M)$. By Lemma 3, for each $L \in \mathcal{A}_S$ there exists a decreasing open set \mathcal{U}_L such that $L \in \mathcal{U}_L$ and $\text{bd}(\mathcal{U}_L) \cap (\mathcal{A} \cup \{M\}) = \emptyset$. Since \mathcal{A}_S is compact there exist $\mathcal{U}_{L_1}, \dots, \mathcal{U}_{L_n}$ such that $\mathcal{A}_S \subset \bigcup_{i=1}^n \mathcal{U}_{L_i}$. Let $\mathcal{A}_F = F_1(X) \cap \mathcal{A}$. For each $x \in \mathcal{A}_F$ let \mathcal{V}_x be a decreasing open set such that $\{x\} \in \mathcal{V}_x$ and $(\text{bd} \mathcal{V}_x) \cap (\mathcal{A} \cup \{M\}) = \emptyset$. Since \mathcal{A}_F is compact there exist $\mathcal{V}_{x_1}, \dots, \mathcal{V}_{x_m}$ such that $\mathcal{A}_F \subset \bigcup_{i=1}^m \mathcal{V}_{x_i}$.

Let

$$\mathcal{A}_0 = \mathcal{A} \cap \left[\left(\bigcup_{i=1}^n \mathcal{U}_{L_i} \right) \cup \left(\bigcup_{i=1}^m \mathcal{V}_{x_i} \right) \right] = \mathcal{A} \cap \left[\left(\bigcup_{i=1}^n \mathcal{U}_{L_i}^* \right) \cup \left(\bigcup_{i=1}^m \mathcal{V}_{x_i}^* \right) \right]$$

and let $\mathcal{A}_1 = \mathcal{A} - \mathcal{A}_0$. Then \mathcal{A}_0 and \mathcal{A}_1 are disjoint closed subsets of \mathcal{A} . Define $f: \mathcal{A} \cup \{M\} \cup F_1(X) \rightarrow [0, 1]$ by

$$f(A) = \begin{cases} 0 & \text{if } A \in \mathcal{A}_0 \text{ or if } A \in F_1(X) \\ 1/2 & \text{if } A = M \\ 1 & \text{if } A \in \mathcal{A}_1. \end{cases}$$

Then f is continuous and order-preserving, so by Theorem A, f has a continuous extension $\hat{f}: C(X) \rightarrow [0, b]$ ($b \geq 1$) which is also order-preserving. \hat{f} has the property that if $t \in [0, b]$, then $\hat{f}^{-1}([0, t])$ and $\hat{f}^{-1}([t, b])$ are subcontinua of $C(X)$. Since $C(X)$ is unicoherent, $\hat{f}^{-1}([0, 3/4]) \cap \hat{f}^{-1}([1/4, b])$ is a continuum containing M in its interior which misses \mathcal{A} .

Now suppose that for some $x_0 \in X$, $M = \{x_0\}$. Let $\mathcal{A}_F = \mathcal{A} \cap F_1(X)$. For each $\{x\} \in \mathcal{A}_F$ let \mathcal{U}_x be a decreasing open set such that $\{x\} \in \mathcal{U}_x$ and $(\text{bd} \mathcal{U}_x) \cap (\mathcal{A} \cup \{M\}) = \emptyset$. Since \mathcal{A}_F is compact there exist x_1, \dots, x_n such that $\mathcal{A}_F \subset \bigcup_{i=1}^n \mathcal{U}_{x_i}$. Let $\mathcal{A}_0 = \mathcal{A} \cap \left(\bigcup_{i=1}^n \mathcal{U}_{x_i} \right) = \mathcal{A} \cap \left(\bigcup_{i=1}^n \mathcal{U}_{x_i}^* \right)$ and let $\mathcal{A}_1 = \mathcal{A} - \mathcal{A}_0$. Then \mathcal{A}_0 and \mathcal{A}_1 are disjoint closed subsets of \mathcal{A} . Define $f: \left(\bigcup_{i=1}^n \mathcal{U}_{x_i}^* \right) \cup \mathcal{A} \cup F_1(X) \rightarrow [0, 1]$ by

$$f(A) = \begin{cases} 0 & \text{if } A \in \left(\bigcup_{i=1}^n \mathcal{U}_{x_i}^* \right) \cup F_1(X) \\ 1 & \text{if } A \in \mathcal{A}_1. \end{cases}$$

Then f is continuous and order-preserving, so f has a continuous, order-preserving extension $\hat{f}: C(X) \rightarrow [0, b]$ ($b \geq 1$). Let $t \in (0, 1)$. Then $\hat{f}^{-1}([0, t])$ is a continuum containing M in its interior which misses \mathcal{A}_1 .

Let $\mathcal{M} = C(X) - \bigcup_{i=1}^n \mathcal{U}_{x_i}$. \mathcal{M} is a closed set containing M in its interior which misses \mathcal{A}_0 . Since each of $\mathcal{U}_{x_1}, \dots, \mathcal{U}_{x_n}$ is decreasing, it follows that \mathcal{M} is increasing. So $\mathcal{M} = T(\mathcal{M})$ and by Lemma 1, $T(\mathcal{M})$ is a continuum. Observe that $C(X) = \hat{f}^{-1}([0, t]) \cup \mathcal{M}$. Since $C(X)$ is unicoherent, $\hat{f}^{-1}([0, t]) \cap \mathcal{M}$ is a continuum containing M in its interior which misses \mathcal{A} . Hence $C(X)$ is countable closed set aposyndetic.

REFERENCES

1. D. E. Bennett, *Aposyndetic properties of unicoherent continua*, Pacific J. Math., **37** (1971), 585-589.
2. C. L. Hagopian, *Mutual aposyndesis*, Proc. Amer. Math. Soc., **23** (1969), 615-622.
3. F. B. Jones, *Aposyndetic continua and certain boundary problems*, Amer. J. Math., **63** (1941), 545-553.
4. J. L. Kelley, *Hyperspaces of a continuum*, Trans. Amer. Math. Soc., **52** (1942), 23-36.
5. E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc., **71** (1951), 152-182.
6. L. Nachbin, *Topology and Order*, D. Van Nostrand Co., New York, 1965.
7. S. B. Nadler, Jr., *Inverse limits and multicoherence*, Bull. Amer. Math. Soc., **76** (1970), 411-414.
8. J. T. Rogers, Jr., *The cone equals hyperspace property*, Canad. J. Math., **24** (1972), 279-285.
9. L. E. Rogers, *Concerning n -mutual aposyndesis in products of continua*, Trans. Amer. Math. Soc., **162** (1971), 239-251.

Received March 9, 1972 and in revised form June 28, 1972. The main results of this paper are part of the author's 1971 doctoral dissertation written under the direction of Professor Carl Eberhart at the University of Kentucky.

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Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 47, No. 1

January, 1973

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