

Pacific Journal of Mathematics

AN ASYMPTOTIC PROPERTY OF SOLUTIONS OF

$$y''' + py' + qy = 0$$

GARY DOUGLAS JONES

AN ASYMPTOTIC PROPERTY OF SOLUTIONS OF

$$y''' + py' + qy = 0$$

GARY D. JONES

In this paper, the differential equation

$$(1) \quad y''' + p(x)y' + q(x)y = 0$$

will be studied subject to the conditions that $p(x) \leq 0$, $q(x) > 0$, and $p(x)$, $p'(x)$, and $q(x)$ are continuous for $x \in [0, +\infty)$. A solution of (1) will be said to be oscillatory if it changes signs for arbitrarily large values of x . It will be shown that if (1) has an oscillatory solution then every nonoscillatory solution tends to zero as x tends to infinity.

The above result answers a question that was raised in [1]. The following theorem due to Lazer [1] will be basic in our proof.

THEOREM 1. *Suppose $p(x) \leq 0$ and $q(x) > 0$. A necessary and sufficient condition for (1) to have oscillatory solutions is that for any nontrivial nonoscillatory solution $G(x)$, $G(x)G'(x)G''(x) \neq 0$, $\text{sgn } G(x) = \text{sgn } G''(x) \neq \text{sgn } G'(x)$ for all $x \in [0, +\infty)$, and*

$$\lim_{x \rightarrow \infty} G'(x) = \lim_{x \rightarrow \infty} G''(x) = 0, \quad \lim_{x \rightarrow \infty} G(x) = c \neq \pm \infty.$$

LEMMA 2. *If $G(x)$ is a nonoscillatory solution of (1), where (1) has an oscillatory solution, then*

$$\lim_{x \rightarrow \infty} xG'(x) = 0.$$

Proof. Suppose $G(x) < 0$, $G'(x) > 0$, and $G''(x) < 0$. By Theorem 1, $\int_1^\infty G'(x)dx < \infty$. Let $\varepsilon > 0$. There is an $N > 0$ such that $\int_N^x G'(t)dt < \varepsilon$ for all $x > N$. Thus $\varepsilon > \int_N^x G'(t)dt = G'(\Sigma)[x - N]$ for $N < \Sigma < x$. But $G''(x) < 0$, so $G'(\Sigma)[x - N] \geq G'(x)[x - N] > G'(x) \cdot x - \varepsilon$ for x large since $G'(x) \rightarrow 0$. Thus $2\varepsilon > xG'(x)$ for large x . Hence $\lim_{x \rightarrow \infty} xG'(x) = 0$.

LEMMA 3. *If $G(x)$ is as in Lemma 2, then*

$$\left| \int_1^\infty xG''(x)dx \right| < \infty.$$

Proof. Suppose that $G(x) > 0$, $G'(x) < 0$, and $G''(x) > 0$. Integrating by parts, $\int_1^x tG''(t)dt = xG'(x) - G'(1) - G(x) + G(1)$. Thus $\int_1^\infty xG''(x)dx < \infty$ since $\lim_{x \rightarrow \infty} xG'(x) = 0$ and $\lim_{x \rightarrow \infty} G(x) = K < \infty$.

LEMMA 4. *If $G(x)$ is as in Lemma 2, then*

$$\lim_{x \rightarrow \infty} x^2 G''(x) = 0 .$$

Proof. Suppose $G(x) > 0, G'(x) < 0, G''(x) > 0$. Since

$$\int_1^{\infty} x G''(x) dx < \infty ,$$

for $\varepsilon > 0$ there is an $N > 0$ so that for all $x > N$

$$\varepsilon > \int_N^x t G''(t) dt = G''(\Sigma) \int_N^x t dt$$

for some $N < \Sigma < x$.

But since $G'''(x) < 0$ by (1), we have

$$G''(\Sigma) \int_N^x t dt \geq [G''(x)/2][x^2 - N^2] \geq [G''(x)/2][x^2] - \varepsilon/2$$

for large x , since $\lim_{x \rightarrow \infty} G''(x) = 0$. Thus

$$3\varepsilon > x^2 G''(x) \text{ for all large } x .$$

Thus $\lim_{x \rightarrow \infty} x^2 G''(x) = 0$.

THEOREM 5. *If $G(x) > 0, G'(x) < 0, G''(x) > 0$ is a solution of (1) which has oscillatory solutions then two linearly independent oscillatory solutions of*

$$(2) \quad y''' + p(x)y' + (p'(x) - q(x))y = 0$$

satisfy the differential equation

$$(3) \quad (y'/G(x))' + [(G''(x) + p(x)G(x))/G^2(x)]y = 0 .$$

Proof. Let $u(x)$ and $v(x)$ be two solutions of (1) defined by $u(1) = u'(1) = 0, u''(1) = 1, v(1) = v''(1) = 0, v'(1) = 1$. By [1], $u(x)$ and $v(x)$ are linearly independent oscillatory solutions of (1). Let

$$U(x) = u(x)G'(x) - G(x)u'(x)$$

$$V(x) = v(x)G'(x) - G(x)v'(x) .$$

Then $U(x)$ and $V(x)$ are linearly independent oscillatory solutions of (2). Now

$$\begin{vmatrix} V(x) & U(x) \\ V'(x) & U'(x) \end{vmatrix} = G(x) \begin{vmatrix} G(x) & v(x) & u(x) \\ G'(x) & v'(x) & u'(x) \\ G''(x) & v''(x) & u''(x) \end{vmatrix}$$

$$= G(x) \begin{vmatrix} G(1) & 0 & 0 \\ G'(1) & 1 & 0 \\ G''(1) & 0 & 1 \end{vmatrix} = G(1)G(x) .$$

Thus

$$G(1)G'(x) = \begin{vmatrix} V(x) & U(x) \\ V''(x) & U''(x) \end{vmatrix}$$

and

$$G(1)G''(x) = \begin{vmatrix} V'(x) & U'(x) \\ V''(x) & U''(x) \end{vmatrix} + \begin{vmatrix} V(x) & U(x) \\ V'''(x) & U'''(x) \end{vmatrix} .$$

Now $U(x)$ and $V(x)$ are solutions of the differential equation

$$(4) \quad \begin{vmatrix} V(x) & U(x) & y \\ V'(x) & U'(x) & y' \\ V''(x) & U''(x) & y'' \end{vmatrix} = 0 .$$

But

$$\begin{aligned} \begin{vmatrix} V(x) & U(x) \\ V'''(x) & U'''(x) \end{vmatrix} &= V(x)[-p(x)U'(x) - p'(x)U(x) + q(x)U(x)] \\ &- U(x)[-p(x)V'(x) - p'(x)V(x) + q(x)V(x)] = -p(x)G(1)G(x) . \end{aligned}$$

Thus (4) becomes

$$(5) \quad G(1)G(x)y'' - G(1)G'(x)y' + [G(1)G''(x) + p(x)G(1)G(x)]y = 0$$

or

$$(y'/G(x))' + [(G''(x) + p(x)G(x))/G^2(x)]y = 0 .$$

Our main result now follows.

THEOREM 6. *If $G(x)$ is as in Theorem 5, then $\lim_{x \rightarrow \infty} G(x) = 0$.*

Proof. Suppose not. By Theorem 1, $\lim_{x \rightarrow \infty} G(x) = K < \infty$. Suppose without loss of generality that $K = 1$. Now for large x , $G(x) < 2$, hence

$$1/G(x) > 1/2 .$$

Also

$$G''(x) \geq G''(x)/G^2(x) \geq G''(x)/G^2(x) + p(x)G(x)/G^2(x) .$$

Since (3) is oscillatory, by the Sturm-Picone Theorem [2]

$$(6) \quad (y'/2)' + G''(x)y = 0$$

is oscillatory. Letting $y = x^{1/2}z$, (6) becomes

$$(7) \quad (xz)' + (2x^2G''(x) - 1/4)x^{-1}z = 0.$$

But since $\lim_{x \rightarrow \infty} x^2G''(x) = 0$, $(2x^2G'' - 1/4)$ is eventually negative and so (7) is clearly nonoscillatory. From this contradiction, we conclude $\lim_{x \rightarrow \infty} G(x) = 0$.

REFERENCES

1. A. C. Lazer, *The behavior of solutions of the differential equation $y''' + p(x)y' + q(x)y = 0$* , Pacific J. Math., **17** (1966), 435-466.
2. Walter Leighton, *Ordinary Differential Equations*, Wadsworth Publishing Company, Belmont, California, 1967.

Received March 23, 1972.

MURRAY STATE UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

D. GILBARG AND J. MILGRAM
Stanford University
Stanford, California 94305

J. DUGUNDJI*
Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT
University of Washington
Seattle, Washington 98105

RICHARD ARENS
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

* C. DePrima will replace J. Dugundji until August 1974.

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 47, No. 1

January, 1973

K. Adachi, Masuo Suzuki and M. Yoshida, <i>Continuation of holomorphic mappings, with values in a complex Lie group</i>	1
Michael Aschbacher, <i>A characterization of the unitary and symplectic groups over finite fields of characteristic at least 5</i>	5
Larry Eugene Bobisud and James Calvert, <i>Energy bounds and virial theorems for abstract wave equations</i>	27
Christer Borell, <i>A note on an inequality for rearrangements</i>	39
Peter Southcott Bullen and S. N. Mukhopadhyay, <i>Peano derivatives and general integrals</i>	43
Wendell Dan Curtis, Yu-Lee Lee and Forrest Miller, <i>A class of infinite dimensional subgroups of $\text{Diff}^r(X)$ which are Banach Lie groups</i>	59
Paul C. Eklof, <i>The structure of ultraproducts of abelian groups</i>	67
William Alan Feldman, <i>Axioms of countability and the algebra $C(X)$</i>	81
Jack Tilden Goodykoontz, Jr., <i>Aposyndetic properties of hyperspaces</i>	91
George Grätzer and J. Płonka, <i>On the number of polynomials of an idempotent algebra. II</i>	99
Alan Trinler Huckleberry, <i>The weak envelope of holomorphy for algebras of holomorphic functions</i>	115
John Joseph Hutchinson and Julius Martin Zelmanowitz, <i>Subdirect sum decompositions of endomorphism rings</i>	129
Gary Douglas Jones, <i>An asymptotic property of solutions of $y''' + py' + qy = 0$</i>	135
Howard E. Lacey, <i>On the classification of Lindenstrauss spaces</i>	139
Charles Dwight Lahr, <i>Approximate identities for convolution measure algebras</i>	147
George William Luna, <i>Subdifferentials of convex functions on Banach spaces</i>	161
Nelson Groh Markley, <i>Locally circular minimal sets</i>	177
Robert Wilmer Miller, <i>Endomorphism rings of finitely generated projective modules</i>	199
Donald Steven Passman, <i>On the semisimplicity of group rings of linear groups</i>	221
Bennie Jake Pearson, <i>Dendritic compactifications of certain dendritic spaces</i>	229
Ryōtarō Satō, <i>Abel-ergodic theorems for subsequences</i>	233
Henry S. Sharp, Jr., <i>Locally complete graphs</i>	243
Harris Samuel Shultz, <i>A very weak topology for the Mikusinski field of operators</i>	251
Elena Stroescu, <i>Isometric dilations of contractions on Banach spaces</i>	257
Charles W. Trigg, <i>Versum sequences in the binary system</i>	263
William L. Voxman, <i>On the countable union of cellular decompositions of n-manifolds</i>	277
Robert Francis Wheeler, <i>The strict topology, separable measures, and paracompactness</i>	287