AN ASYMPTOTIC PROPERTY OF SOLUTIONS OF
\[ y''' + py' + qy = 0 \]

GARY DOUGLAS JONES
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In this paper, the differential equation

$$y''' + p(x)y' + q(x)y = 0$$  \(1\)

will be studied subject to the conditions that $p(x) \leq 0$, $q(x) > 0$, and $p(x)$, $p'(x)$, and $q(x)$ are continuous for $x \in [0, +\infty)$. A solution of (1) will be said to be oscillatory if it changes signs for arbitrarily large values of $x$. It will be shown that if (1) has an oscillatory solution then every nonoscillatory solution tends to zero as $x$ tends to infinity.

The above result answers a question that was raised in [1]. The following theorem due to Lazer [1] will be basic in our proof.

**Theorem 1.** Suppose $p(x) \leq 0$ and $q(x) > 0$. A necessary and sufficient condition for (1) to have oscillatory solutions is that for any nontrivial nonoscillatory solution $G(x)$, $G(x)G'(x)G''(x) \neq 0$, $\text{sgn} G(x) = \text{sgn} G'(x) \neq \text{sgn} G''(x)$ for all $x \in [0, +\infty)$, and

$$\lim_{x \to \infty} G'(x) = \lim_{x \to \infty} G''(x) = 0, \lim G(x) = c \neq \pm \infty .$$

**Lemma 2.** If $G(x)$ is a nonoscillatory solution of (1), where (1) has an oscillatory solution, then

$$\lim_{x \to \infty} xG'(x) = 0 .$$

**Proof.** Suppose $G(x) < 0$, $G'(x) > 0$, and $G''(x) < 0$. By Theorem 1, $\int_1^\infty G'(x)dx < \infty$. Let $\varepsilon > 0$. There is an $N > 0$ such that $\int_N^x G'(t)dt < \varepsilon$ for all $x > N$. Thus $\varepsilon > \int_N^x G'(t)dt = G'(\Sigma)[x - N]$ for $N < \Sigma < x$.

But $G''(x) < 0$, so $G'(\Sigma)[x - N] \geq G'(x)[x - N] > G'(x)\cdot x - \varepsilon$ for $x$ large since $G'(x) \to 0$. Thus $2\varepsilon > xG'(x)$ for large $x$. Hence $\lim_{x \to \infty} xG'(x) = 0$.

**Lemma 3.** If $G(x)$ is as in Lemma 2, then

$$\left| \int_1^\infty xG''(x)dx \right| < \infty .$$

**Proof.** Suppose that $G(x) > 0$, $G'(x) < 0$, and $G''(x) > 0$. Integrating by parts, $\int_1^\infty tG''(t)dt = xG'(x) - G'(1) - G(x) + G(1)$. Thus $\int_1^\infty xG''(x)dx < \infty$ since $\lim_{x \to \infty} xG'(x) = 0$ and $\lim_{x \to \infty} G(x) = K < \infty$. 

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Lemma 4. If $G(x)$ is as in Lemma 2, then
\[ \lim_{x \to \infty} x^2 G''(x) = 0. \]

Proof. Suppose $G(x) > 0, G'(x) < 0, G''(x) > 0$. Since
\[ \int_1^x xG''(x)dx < \infty, \]
for $\varepsilon > 0$ there is an $N > 0$ so that for all $x > N$
\[ \varepsilon > \int_N^x tG''(t)dt = G''(\Sigma)\int_N^x tdt \]
for some $N < \Sigma < x$.
But since $G''''(x) < 0$ by (1), we have
\[ G''(\Sigma)\int_N^x tdt \geq [G''(x)/2][x^2 - N^2] \geq [G''(x)/2][x^2] - \varepsilon/2 \]
for large $x$, since $\lim_{x \to \infty} G''(x) = 0$. Thus
\[ 3\varepsilon > x^2 G''(x) \] for all large $x$.
Thus $\lim_{x \to \infty} x^2 G''(x) = 0$.

Theorem 5. If $G(x) > 0, G'(x) < 0, G''(x) > 0$ is a solution of (1) which has oscillatory solutions then two linearly independent oscillatory solutions of
\[ y''' + p(x)y' + (p'(x) - q(x))y = 0 \]
satisfy the differential equation
\[ (y'/G(x))' + [(G''(x) + p(x)G(x))/G''(x)]y = 0. \]

Proof. Let $u(x)$ and $v(x)$ be two solutions of (1) defined by $u(1) = u'(1) = 0, u''(1) = 1, v(1) = v''(1) = 0, v'(1) = 1$. By [1], $u(x)$ and $v(x)$ are linearly independent oscillatory solutions of (1). Let
\[ U(x) = u(x)G'(x) - G(x)u'(x) \]
\[ V(x) = v(x)G'(x) - G(x)v'(x). \]
Then $U(x)$ and $V(x)$ are linearly independent oscillatory solutions of (2). Now
\[ \begin{vmatrix} V(x) & U(x) \\ V'(x) & U'(x) \end{vmatrix} = \begin{vmatrix} G(x) & v(x) & u(x) \\ G'(x) & v'(x) & u'(x) \\ G''(x) & v''(x) & u''(x) \end{vmatrix} \]
AN ASYMPTOTIC PROPERTY OF SOLUTIONS OF \( y'' + px' + qx = 0 \)

\[
\begin{pmatrix}
G(1) & 0 & 0 \\
G'(1) & 1 & 0 \\
G''(1) & 0 & 1 \\
\end{pmatrix} = G(x) 
\]

Thus

\[
G(1)G'(x) = \begin{vmatrix}
V(x) & U(x) \\
V'(x) & U'(x) \\
V''(x) & U''(x) \\
\end{vmatrix}
\]

and

\[
G(1)G''(x) = \begin{vmatrix}
V'(x) & U'(x) \\
V''(x) & U''(x) \\
V'''(x) & U'''(x) \\
\end{vmatrix}
\]

Now \( U(x) \) and \( V(x) \) are solutions of the differential equation

\[
\begin{vmatrix}
V(x) & U(x) & y \\
V'(x) & U'(x) & y' \\
V''(x) & U''(x) & y'' \\
\end{vmatrix} = 0 .
\]

But

\[
\begin{vmatrix}
V(x) & U(x) \\
V'(x) & U'(x) \\
V''(x) & U''(x) \\
\end{vmatrix} = V(x)[-p(x)U'(x) - p'(x)U(x) + q(x)U(x)]
\]

\[-U(x)[-p(x)V'(x) - p'(x)V(x) + q(x)V(x)] = -p(x)G(1)G(x) .
\]

Thus (4) becomes

\[
(5) \quad G(1)G(x)y'' - G(1)G'(x)y' + [G(1)G''(x) + p(x)G(1)G(x)]y = 0
\]

or

\[
(y'/G(x))' + [(G''(x) + p(x)G(x))/G^2(x)]y = 0 .
\]

Our main result now follows.

**Theorem 6.** If \( G(x) \) is as in Theorem 5, then \( \lim_{x \to \infty} G(x) = 0. \)

**Proof.** Suppose not. By Theorem 1, \( \lim_{x \to \infty} G(x) = K < \infty \). Suppose without loss of generality that \( K = 1 \). Now for large \( x \), \( G(x) < 2 \), hence

\[
1/G(x) > 1/2 .
\]

Also

\[
G''(x) \geq G''(x)/G^2(x) \geq G''(x)/G^2(x) + p(x)G(x)/G^2(x) .
\]

Since (3) is oscillatory, by the Sturm-Picone Theorem [2]
(6) \( (y'/2)' + G''(x)y = 0 \)

is oscillatory. Letting \( y = x^{1/2}z \), (6) becomes

(7) \( (xz)' + (2x^2G''(x) - 1/4)x^{-1}z = 0 \).

But since \( \lim_{x \to \infty} x^2G''(x) = 0 \), \( (2x^2G'' - 1/4) \) is eventually negative and so (7) is clearly nonoscillatory. From this contradiction, we conclude \( \lim_{x \to \infty} G(x) = 0 \).

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