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AN ASYMPTOTIC PROPERTY OF SOLUTIONS OF
 $y''' + py' + qy = 0$

GARY DOUGLAS JONES

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In this paper, the differential equation

$$(1) \quad y''' + p(x)y' + q(x)y = 0$$

will be studied subject to the conditions that $p(x) \leq 0$, $q(x) > 0$, and $p(x)$, $p'(x)$, and $q(x)$ are continuous for $x \in [0, +\infty)$. A solution of (1) will be said to be oscillatory if it changes signs for arbitrarily large values of x . It will be shown that if (1) has an oscillatory solution then every nonoscillatory solution tends to zero as x tends to infinity.

The above result answers a question that was raised in [1].

The following theorem due to Lazer [1] will be basic in our proof.

THEOREM 1. *Suppose $p(x) \leq 0$ and $q(x) > 0$. A necessary and sufficient condition for (1) to have oscillatory solutions is that for any nontrivial nonoscillatory solution $G(x)$, $G(x)G'(x)G''(x) \neq 0$, $\text{sgn } G(x) = \text{sgn } G''(x) \neq \text{sgn } G'(x)$ for all $x \in [0, +\infty)$, and*

$$\lim_{x \rightarrow \infty} G'(x) = \lim_{x \rightarrow \infty} G''(x) = 0, \quad \lim_{x \rightarrow \infty} G(x) = c \neq \pm \infty.$$

LEMMA 2. *If $G(x)$ is a nonoscillatory solution of (1), where (1) has an oscillatory solution, then*

$$\lim_{x \rightarrow \infty} xG'(x) = 0.$$

Proof. Suppose $G(x) < 0$, $G'(x) > 0$, and $G''(x) < 0$. By Theorem 1, $\int_1^\infty G'(x)dx < \infty$. Let $\varepsilon > 0$. There is an $N > 0$ such that $\int_N^x G'(t)dt < \varepsilon$ for all $x > N$. Thus $\varepsilon > \int_N^x G'(t)dt = G'(\Sigma)[x - N]$ for $N < \Sigma < x$. But $G''(x) < 0$, so $G'(\Sigma)[x - N] \geq G'(x)[x - N] > G'(x) \cdot x - \varepsilon$ for x large since $G'(x) \rightarrow 0$. Thus $2\varepsilon > xG'(x)$ for large x . Hence $\lim_{x \rightarrow \infty} xG'(x) = 0$.

LEMMA 3. *If $G(x)$ is as in Lemma 2, then*

$$\left| \int_1^\infty xG''(x)dx \right| < \infty.$$

Proof. Suppose that $G(x) > 0$, $G'(x) < 0$, and $G''(x) > 0$. Integrating by parts, $\int_1^x tG''(t)dt = xG'(x) - G'(1) - G(x) + G(1)$. Thus $\int_1^\infty xG''(x)dx < \infty$ since $\lim_{x \rightarrow \infty} xG'(x) = 0$ and $\lim_{x \rightarrow \infty} G(x) = K < \infty$.

LEMMA 4. *If $G(x)$ is as in Lemma 2, then*

$$\lim_{x \rightarrow \infty} x^2 G''(x) = 0.$$

Proof. Suppose $G(x) > 0$, $G'(x) < 0$, $G''(x) > 0$. Since

$$\int_1^{\infty} x G''(x) dx < \infty,$$

for $\varepsilon > 0$ there is an $N > 0$ so that for all $x > N$

$$\varepsilon > \int_N^x t G''(t) dt = G''(\Sigma) \int_N^x t dt$$

for some $N < \Sigma < x$.

But since $G'''(x) < 0$ by (1), we have

$$G''(\Sigma) \int_N^x t dt \geq [G''(x)/2][x^2 - N^2] \geq [G''(x)/2][x^2] - \varepsilon/2$$

for large x , since $\lim_{x \rightarrow \infty} G''(x) = 0$. Thus

$$3\varepsilon > x^2 G''(x) \text{ for all large } x.$$

Thus $\lim_{x \rightarrow \infty} x^2 G''(x) = 0$.

THEOREM 5. *If $G(x) > 0$, $G'(x) < 0$, $G''(x) > 0$ is a solution of (1) which has oscillatory solutions then two linearly independent oscillatory solutions of*

$$(2) \quad y''' + p(x)y' + (p'(x) - q(x))y = 0$$

satisfy the differential equation

$$(3) \quad (y'/G(x))' + [(G''(x) + p(x)G(x))/G^2(x)]y = 0.$$

Proof. Let $u(x)$ and $v(x)$ be two solutions of (1) defined by $u(1) = u'(1) = 0$, $u''(1) = 1$, $v(1) = v'(1) = 0$, $v''(1) = 1$. By [1], $u(x)$ and $v(x)$ are linearly independent oscillatory solutions of (1). Let

$$U(x) = u(x)G'(x) - G(x)u'(x)$$

$$V(x) = v(x)G'(x) - G(x)v'(x).$$

Then $U(x)$ and $V(x)$ are linearly independent oscillatory solutions of (2). Now

$$\begin{vmatrix} V(x) & U(x) \\ V'(x) & U'(x) \end{vmatrix} = G(x) \begin{vmatrix} G(x) & v(x) & u(x) \\ G'(x) & v'(x) & u'(x) \\ G''(x) & v''(x) & u''(x) \end{vmatrix}$$

$$= G(x) \begin{vmatrix} G(1) & 0 & 0 \\ G'(1) & 1 & 0 \\ G''(1) & 0 & 1 \end{vmatrix} = G(1)G(x).$$

Thus

$$G(1)G'(x) = \begin{vmatrix} V(x) & U(x) \\ V''(x) & U''(x) \end{vmatrix}$$

and

$$G(1)G''(x) = \begin{vmatrix} V'(x) & U'(x) \\ V'''(x) & U'''(x) \end{vmatrix} + \begin{vmatrix} V(x) & U(x) \\ V'''(x) & U'''(x) \end{vmatrix}.$$

Now $U(x)$ and $V(x)$ are solutions of the differential equation

$$(4) \quad \begin{vmatrix} V(x) & U(x) & y \\ V'(x) & U'(x) & y' \\ V''(x) & U''(x) & y'' \end{vmatrix} = 0.$$

But

$$\begin{vmatrix} V(x) & U(x) \\ V'''(x) & U'''(x) \end{vmatrix} = V(x)[-p(x)U'(x) - p'(x)U(x) + q(x)U(x)] \\ - U(x)[-p(x)V'(x) - p'(x)V(x) + q(x)V(x)] = -p(x)G(1)G(x).$$

Thus (4) becomes

$$(5) \quad G(1)G(x)y'' - G(1)G'(x)y' + [G(1)G''(x) + p(x)G(1)G(x)]y = 0$$

or

$$(y'/G(x))' + [(G''(x) + p(x)G(x))/G^2(x)]y = 0.$$

Our main result now follows.

THEOREM 6. *If $G(x)$ is as in Theorem 5, then $\lim_{x \rightarrow \infty} G(x) = 0$.*

Proof. Suppose not. By Theorem 1, $\lim_{x \rightarrow \infty} G(x) = K < \infty$. Suppose without loss of generality that $K = 1$. Now for large x , $G(x) < 2$, hence

$$1/G(x) > 1/2.$$

Also

$$G''(x) \geq G''(x)/G^2(x) \geq G''(x)/G^2(x) + p(x)G(x)/G^2(x).$$

Since (3) is oscillatory, by the Sturm-Picone Theorem [2]

$$(6) \quad (y'/2)' + G''(x)y = 0$$

is oscillatory. Letting $y = x^{1/2}z$, (6) becomes

$$(7) \quad (xz)' + (2x^2G''(x) - 1/4)x^{-1}z = 0.$$

But since $\lim_{x \rightarrow \infty} x^2G''(x) = 0$, $(2x^2G'' - 1/4)$ is eventually negative and so (7) is clearly nonoscillatory. From this contradiction, we conclude $\lim_{x \rightarrow \infty} G(x) = 0$.

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