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ON THE SEMISIMPLICITY OF GROUP RINGS OF LINEAR GROUPS

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ON THE SEMISIMPLICITY OF GROUP RINGS OF LINEAR GROUPS

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In this paper we study the semisimplicity problem for group rings of linear groups. We prove the linear group analog of a result which constitutes part of the solution of the semisimplicity problem for solvable groups. Since all of the necessary group ring lemmas have appeared elsewhere, the work here is strictly group theoretic. We consider the possibility of a linear group being covered by a finite union of root sets of centralizer subgroups.

Let $K[G]$ denote the group ring of G over the field K. Probably the most fascinating and difficult question asked about this ring is when is it semisimple, that is when does its Jacobson radical $JK[G]$ vanish. If K has characteristic 0, then by a result of Amitsur [1] $JK[G] = 0$ for all fields K which are not algebraic over the rationals and in all likelihood $K[G]$ is always semisimple. Thus the real interest is now in characteristic $p > 0$. At this time there is not even a reasonable conjecture as to the answer here and so it is necessary and important that a large number of special cases be studied. So far the only nontrivial family of groups for which this problem has been solved is in fact the solvable groups, that is the groups of finite derived length. It appears that the next family of interest will be the linear groups since some interesting work in this direction has already appeared in [4]. In this paper we study the semisimplicity problem for linear groups.

Let G be a group and let H be a subgroup. We say that H has locally finite index in G and write $[G: H] = \text{l.f.}$ if for all finitely generated subgroups S of G we have $[S: S \cap H] < \infty$. We say that G is a Δ-group if $G = \Delta(G)$, that is if all conjugacy classes of G are finite. The result on solvable groups was proved in a series of three papers [3], [2], and [5] and is as follows.

THEOREM. *(Hampton, Passman, and Zalesskii.) Let K be a field of characteristic $p > 0$ and let G be a solvable group. Then $JK[G] \neq 0$ if and only if there exists an element $x \in \ni(G)$ (a certain characteristic subgroup of G) of order p with $[G: C_G(x)] = l.\text{f.}$*

We remark that $\ni(G)$ is a particular normal Δ-subgroup of a solvable group which is defined in [5] and we propose to call it the Zalesskii subgroup of G.

Now let G be a linear group. That is, G is a subgroup of the group of units of L_u, the ring of $u \times u$ matrices over some field L (which need not be at all related to K). The main results here are.

THEOREM. *Suppose K is a field of characteristic $p > 0$. Let G be a linear group and let H be a normal solvable subgroup. Then $JK[G] \cap K[H] \neq 0$ if and only if there exists an element $x \in \ni(H)$ of order p with $[G\colon C_G(x)] = l.f.$*

COROLLARY. *Suppose K is a field of characteristic $p > 0$. Let G be a linear group and let H be a normal nilpotent subgroup. Then $JK[G] \cap K[H] \neq 0$ if and only if there exists an element $x \in Z(H)$ of order p with $[G\colon C_G(x)] = l.f.$*

Actually the required group ring lemmas have already been proved in [2] and [5]. The work here is strictly group theoretic.

1. Isolated subsets. Let G be a group and let H be a subgroup. We let

$$\sqrt{H} = {}^{G}\!\sqrt{H} = \{g \in G \mid g^m \in H \text{ for some integer } m \geqq 1\}$$

be the root set of H. Thus for example if $H \lhd G$ then $\sqrt{H}$ corresponds to the set of torsion elements of G/H. Certainly $\sqrt{H}$ need not be a subgroup of G. We start by listing three trivial observations.

LEMMA 1. *Let $H, H_1, H_2, \cdots, H_k$ be subgroups of G. We have*
(i) $\sqrt{H_1 \cap H_2 \cap \cdots \cap H_k} = \sqrt{H_1} \cap \sqrt{H_2} \cap \cdots \cap \sqrt{H_k}$.
(ii) $\sqrt{H^g} = (\sqrt{H})^g$ *for any* $g \in G$.
(iii) *if* $H \subseteq \sqrt{H_1} \cup \sqrt{H_2} \cup \cdots \cup \sqrt{H_k}$ *then*
$\sqrt{H} \subseteq \sqrt{H_1} \cup \sqrt{H_2} \cup \cdots \cup \sqrt{H_k}$.

LEMMA 2. *Suppose $G = \bigcup_1^n \sqrt{H_i}$ with $H_i \lhd G$. Then for some j, $G = \sqrt{H_j}$ or equivalently G/H_j is torsion.*

Proof. We proceed by induction on n, the case $n = 1$ being clear. Suppose the result is true for $n - 1$ and we consider n.

Given $H_1, H_2, \cdots, H_n$ we define the parameter r of this situation to be the minimum number of H's whose intersection is equal to $N = H_1 \cap H_2 \cap \cdots \cap H_n$. Clearly r exists, $r \leqq n$ and we prove the n case by induction on the parameter r. Say the numbering is so chosen that $H_1 \cap H_2 \cap \cdots \cap H_r = H_1 \cap H_2 \cap \cdots \cap H_n$. If $r = 1$ then $H_1 \subseteq H_2$ so $\sqrt{H_1} \subseteq \sqrt{H_2}$, $G = \bigcup_2^n \sqrt{H_i}$ and the $n-1$ case yields the result. We may therefore assume that $r > 1$. Set

$$W = H_2 \cap H_3 \cap \cdots \cap H_r$$

so by definition $W > N$ and $H_1 \cap W = N$. There are two cases to consider according to whether W/N is torsion or not.

Suppose first that W/N is torsion. Let $\bar{G} = G/W$ and let $\bar{H}_i = H_i W/W \subseteq \bar{G}$. Clearly $\bar{G} = \bigcup_1^n \sqrt{\bar{H}_i}$. Also for $i = 2, 3, \cdots, r$, $\bar{H}_i = H_i/W$ so $\bar{H}_2 \cap \bar{H}_3 \cap \cdots \cap \bar{H}_r = \langle 1 \rangle$. This clearly implies that

$$\bigcap_1^n \bar{H}_i = \langle 1 \rangle$$

and that the parameter of this situation is less than r. By induction for some j, $\bar{G} = \sqrt{\bar{H}_j}$ or equivalently $G/H_j W$ is torsion. Now $H_j W/H_j \cong W/(H_j \cap W)$ and this is a homomorphic image of the torsion group W/N. Thus $H_j W/H_j$ is torsion and hence so is G/H_j. The result follows in this case.

Finally suppose W/N is not torsion and choose $x \in W$ to correspond to an element of infinite order in W/N. Let $h \in H_1$ and consider the $n+1$ elements $x, hx, h^2x, \cdots, h^n x$. Then for each i some power of $h^i x$ is contained in some H_k. Since there are $n+1$ elements and only n subgroups it follows that for two different i, j we have $h^i x, h^j x \in \sqrt{H_k}$ for some k. By choosing a sufficiently high power m we can assume that $(h^i x)^m$, $(h^j x)^m \in H_k$. Note that

$$H_1 \cap W = N \quad \text{so} \quad (H_1/N)(W/N) = (H_1/N) \times (W/N)$$

is a direct product. We show first that $k \neq 1$. For suppose $k = 1$. Then from $(h^i x)^m \in H_1$ and $h \in H_1 \lhd G$ we have easily $x^m \in H_1$. Thus $x^m \in W \cap H_1 = N$ a contradiction since x has infinite order modulo N. Thus $k \neq 1$. Now clearly $(h^i x)^m = h^{im} x^m g$ for some $g \in N \subseteq H_k$ so we have $h^{im} x^m \in H_k$ and similarly $h^{jm} x^m \in H_k$ so if $i > j$ then

$$h^{(i-j)m} = (h^{im} x^m)(h^{jm} x^m)^{-1} \in H_k$$

and $h \in \sqrt{H_k}$. We have therefore shown that $H_1 \subseteq \bigcup_2^n \sqrt{H_i}$ so $\sqrt{H_1} \subseteq \bigcup_2^n \sqrt{H_i}$ and $G = \bigcup_2^n \sqrt{H_i}$. The result follows from the $n-1$ case.

This completes the r induction step and the n case is true for all r. Thus the lemma follows by induction on n.

Now let G be a linear group so that $G \subseteq L_u$ for some field L and some integer u and we fix this notation throughout the remainder of this section. If H is a subgroup of G we let $\hat{H}$ denote its L-linear span. Thus clearly $\hat{H}$ is a subalgebra of L_u. We say that H is a pure subgroup of G if $H = \hat{H} \cap G$ and we let $\mathscr{P}(G)$ denote the set of all such H. If $H \in \mathscr{P}(G)$ we use $\dim H$ to denote the L-dimension of $\hat{H}$. We list a number of trivial observations.

LEMMA 3. *Let G be as above.*

(i) *$H \in \mathscr{P}(G)$ if and only if $H = G \cap R$ for some L-subalgebra R of L_u.*

(ii) *$\mathscr{P}(G)$ is closed under conjugation by G and arbitrary intersections.*

(iii) *If $H_1, H_2 \in \mathscr{P}(G)$ and $H_1 > H_2$ then* $\dim H_1 > \dim H_2$.

Let H be a pure subgroup of G. We say that H is root reduced if

$$\sqrt{H} \subseteq \sqrt{H_1} \cup \sqrt{H_2} \cup \cdots \cup \sqrt{H_k}$$

for finitely many pure subgroups H_i of G implies that $H_i \supseteq H$ for some i.

LEMMA 4. *Let $H \in \mathscr{P}(G)$. Then*

$$\sqrt{H} = \sqrt{H_1} \cup \sqrt{H_2} \cup \cdots \cup \sqrt{H_k}$$

for finitely many pure root reduced subgroups $H_i \subseteq H$.

Proof. We proceed by induction on $\dim H$. If $\dim H = 1$ then clearly H is contained in all pure subgroups of G. Thus H is certainly root reduced and the result follows here. Suppose now that $\dim H > 1$ and that the result is true for all pure subgroups of smaller dimension. If H is root reduced the result is clear so we may suppose not. Then

$$\sqrt{H} \subseteq \sqrt{H_1} \cup \sqrt{H_2} \cup \cdots \cup \sqrt{H_k}$$

with H_i pure and $H \not\subseteq H_i$ for all i. Since $\sqrt{H} \cap \sqrt{H_i} = \sqrt{H \cap H_i}$ we have

$$\sqrt{H} = \sqrt{H \cap H_1} \cup \sqrt{H \cap H_2} \cup \cdots \cup \sqrt{H \cap H_k}$$

and $H > H \cap H_i$. Now $\dim H > \dim(H \cap H_i)$ so by induction each $\sqrt{H \cap H_i}$ is a finite union of root sets of root reduced pure subgroups of $H \cap H_i$. Therefore, if we replace each $\sqrt{H \cap H_i}$ by its corresponding union, then the result follows.

If H is a subgroup of a group G we let $\operatorname{core} H = \bigcap_{g \in G} H^g$ be the intersection of all G-conjugates of H. Thus $\operatorname{core} H$ is the largest normal subgroup of G contained in H.

LEMMA 5. *Suppose G is a linear group and*

$$G = \sqrt{H_1} \cup \sqrt{H_2} \cup \cdots \cup \sqrt{H_n}$$

for pure subgroups H_i. Then for some subscript j, $G = \sqrt{\operatorname{core} H_j}$, that is $G/\operatorname{core} H_j$ is torsion.

Proof. By Lemma 4 we may replace each $\sqrt{H_i}$ by a finite union of root sets of root reduced pure subgroups of H_i. Thus from the above we get

$$G = \sqrt{W_1} \cup \sqrt{W_2} \cup \cdots \cup \sqrt{W_t} \tag{*}$$

where for each i, W_i is a root reduced pure subgroup of G and W_i is contained in $H_{i'}$ for some i'. Furthermore, by successively eliminating unnecessary W_i's in (*) we may assume that this union is irredundant. If $g \in G$, then conjugating (*) by g yields for each i

$$\sqrt{W_i} \subseteq G = \sqrt{W_1^g} \cup \sqrt{W_2^g} \cup \cdots \cup \sqrt{W_t^g} .$$

Thus since W_i is root reduced we conclude that for each i and g there exists a subscript s with $W_i \subseteq W_s^g$.

We show now by inverse induction on $\dim W_i$ that all G-conjugates of W_i occur in the union (*). Suppose first that $\dim W_i$ is as large as possible. Then from $W_i \subseteq W_s^g$ and the maximality of $\dim W_i$ we have $W_i = W_s^g$ and $W_s = W_i^{g^{-1}}$ occurs in (*) for all $g \in G$. This starts the induction. Now suppose the result to be true for all W_k with $\dim W_k > \dim W_i$. Let $g \in G$ and let s be given by $W_i \subseteq W_s^g$. If $W_s^g > W_i$ then $\dim W_s > \dim W_i$ so by induction W_s^g occurs in (*). Thus since $\sqrt{W_s^g} \supseteq \sqrt{W_i}$ we see that the W_i term in (*) is redundant, a contradiction. Therefore, we must have $W_s^g = W_i$ so $W_s = W_i^{g^{-1}}$ occurs in (*) and this fact follows.

Now (*) is a finite union so obviously each W_i has only finitely many conjugates in G. Thus we can find a normal subgroup $\bar{G}$ of G of finite index which normalizes each W_i. If $\bar{W}_i = \bar{G} \cap W_i$ then $\bar{W}_i \lhd \bar{G}$ and we have clearly

$$\bar{G} = \sqrt[\bar{G}]{\bar{W}_1} \cup \sqrt[\bar{G}]{\bar{W}_2} \cup \cdots \cup \sqrt[\bar{G}]{\bar{W}_t} .$$

Thus by Lemma 2 applied to the abstract group $\bar{G}$ we have $\bar{G} = \sqrt[\bar{G}]{\bar{W}_i}$ for some i and hence since $G/\bar{G}$ is finite and $W_i \supseteq \bar{W}_i$ we obtain $G = \sqrt{W_i}$. Now W_i has only finitely many conjugates and certainly for each W_i^g, $G = \sqrt{W_i^g}$. Thus since the intersections below are all really finite we have

$$G = \bigcap_{g \in G} \sqrt{W_i^g} = \sqrt{\bigcap_g W_i^g} = \sqrt{\operatorname{core} W_i} .$$

Finally $W_i \subseteqq H_j$ for some j so $\operatorname{core} W_i \subseteqq \operatorname{core} H_j$ and $G = \sqrt{\operatorname{core} H_j}$. The result follows.

2. **Centralizer subgroups.** Let G be a linear group so that $G \subseteqq L_u$. If H is a subgroup of G, then we say that H is a centralizer subgroup if $H = C_G(T)$ for some nonempty subset $T \subseteqq L_u$.

LEMMA 6. *Let H be a centralizer subgroup of G. Then H is a pure subgroup and $G/\operatorname{core} H$ is a linear group.*

Proof. Let $H = C_G(T)$. Then clearly $G \cap \hat{H}$ centralizes T so $G \cap \hat{H} \subseteqq H$ and H is pure. If $g \in G$ then $H^g = C_G(T^g)$ so certainly $\operatorname{core} H = C_G(S)$ where $S = \bigcup_{g \in G} T^g$. Let $\hat{S}$ denote the L-linear span of S. Then $\hat{S}$ is a finite dimensional L-vector space and G acts on $\hat{S}$ by conjugation. Since clearly $\operatorname{core} H = C_G(\hat{S})$ we see that $G/\operatorname{core} H$ is a linear group.

We now complete our group theoretic work.

PROPOSITION 7. *Suppose G is a linear group and*

$$G = \sqrt{H_1} \cup \sqrt{H_2} \cup \cdots \cup \sqrt{H_n}$$

where each H_i is a centralizer subgroup of G. Then for some subscript j, $G/\operatorname{core} H_j$ is locally finite and hence $[G: H_j] = l.f.$

Proof. By the above lemma each H_i is a pure subgroup of G and thus Lemma 5 implies that for some j, $G/\operatorname{core} H_j$ is torsion. Then again by the above lemma $G/\operatorname{core} H_j$ is a periodic linear group and hence, as is well known, it is locally finite. Finally let S be a finitely generated subgroup of G. Then $S(\operatorname{core} H_j)/\operatorname{core} H_j$ is a finitely generated subgroup of $G/\operatorname{core} H_j$ so

$$S/(S \cap \operatorname{core} H_j) \cong S(\operatorname{core} H_j)/\operatorname{core} H_j$$

is finite. Since $S \supseteqq S \cap H_j \supseteqq S \cap \operatorname{core} H_j$ we have $[S: S \cap H_j] < \infty$ and hence $[G: H_j] = \text{l.f.}$

Now we consider group rings.

Proof of the Theorem. Let G, H, and K be as given and suppose first that $I = JK[G] \cap K[H] \neq 0$. Since H is solvable and I is a nonzero ideal of $K[H]$, results of [5] imply that $I \cap K[\ni(H)] \neq 0$. Thus

$$JK[G] \cap K[\ni(H)] = I \cap K[\ni(H)] \neq 0 .$$

Now $\ni(H)$ is a normal solvable Δ-subgroup of G so by Theorem A of [2] there exist finitely many nonidentity elements $h_1, h_2, \cdots, h_n$ of $\ni(H)$ of order a power of p with

$$G = \bigcup_1^n \sqrt{C_G(h_i)} .$$

Since G is a linear group and $C_G(h_i)$ is a centralizer subgroup we conclude from Proposition 7 that for some j, $[G: C_G(h_j)] = \text{l.f.}$ Now $h_j \neq 1$ has order a power of p so we can choose $x \in \langle h_j \rangle$ to have order p. Since $C_G(x) \supseteqq C_G(h_j)$ we have $[G: C_G(x)] = \text{l.f.}$ and this half is proved.

Conversely suppose $x \in \ni(H)$ is given with $[G: C_G(x)] = \text{l.f.}$ and suppose x has order p. Then since $\ni(H)$ is a normal Δ-subgroup of G we see easily that the proof of the converse part of Theorem B of [2] carries over to prove the result that $JK[G] \cap K[\ni(H)] \neq 0$. Thus certainly $JK[G] \cap K[H] \neq 0$ and the result follows.

Proof of the Corollary. Let G, H, and K be as above. If H is nilpotent then by [5] the same result holds with $\ni(H)$ replaced by $\Delta(H)$. Thus we need only show that if $\Delta(H)$ has an element y of order p with $[G: C_G(y)] = \text{l.f.}$ then we can also find such an element in $Z(H)$.

Now $y \in \Delta(H)$ has finite order so $\langle y \rangle^H$, the normal closure of y in H has finite order. Moreover, since $\langle y \rangle^H$ is a finite nilpotent group generated by p-elements, it is a p-group. Since H is nilpotent $\langle y \rangle^H \cap Z(H) \neq \langle 1 \rangle$ and we can choose x to be an element of order p in this intersection. Now $C_G(y)$ normalizes $\langle y \rangle$ and H so it normalizes the finite group $\langle y \rangle^H$. Thus some subgroup of $C_G(y)$ of finite index centralizes $\langle y \rangle^H$ and hence $[C_G(y): C_G(y) \cap C_G(x)] < \infty$. By Lemma 1 (v) (i) of [3] we conclude first that

$$[G: C_G(y) \cap C_G(x)] = \text{l.f.}$$

and then

$$[G: C_G(x)] = \text{l.f.}$$

The Corollary is proved.

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