Pacific Journal of Mathematics

A VERY WEAK TOPOLOGY FOR THE MIKUSINSKI FIELD OF OPERATORS

HARRIS SAMUEL SHULTZ

Vol. 47, No. 1 January 1973

PACIFIC JOURNAL OF MATHEMATICS
Vol, 47, No. 1, 1973

A VERY WEAK TOPOLOGY FOR THE MIKUSINSKI FIELD OF OPERATORS

HARRIS S. SHULTZ

Using a generalized Laplace transformation the Mikusinski field is given a topology T such that sequences which converge in the sense of Mikusinski converge with respect to T, such that the mapping $q \to q^{-1}$ is continuous and such that the series $\sum (-\lambda)^n s^n/n!$ converges to the translation operator $e^{-\lambda s}$.

In [3] it is shown that the notion of convergence defined in [8] for the Mikusinski field of operators is not topological. Topologies for the Mikusinski field are given in [1], [3], and [9]. In the present paper we endow this field with a topology T such that sequences which converge in the sense of Mikusinski converge with respect to T, such that the identity

$$e^{-\lambda s} = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} s^k \qquad (\lambda > 0)$$

holds and such that the mapping $q \to q^{-1}$ is continuous. The author wishes to acknowledge that this paper constitutes proofs of assertions proposed by Gregers Krabbe [7].

Let L denote the family of complex-valued functions which are locally integrable on $[0, \infty)$. Under addition and convolution L is an integral domain. If Q denotes the quotient field of L then Q is the Mikusinski field of operators. Elements of Q will be denoted $\{f(t)\}$: $\{g(t)\}$ and the injection of L into Q will be denoted $f \to \{f(t)\}$. We define S to be the set of all f in L for which the integral

$$\int_0^{\infty} e^{-zt} f(t)dt$$

converges for some z. For f in S let

$$\bar{f}(z) = \int_0^{\infty} e^{-zt} f(t)dt$$

and $\bar{S} = \{\bar{f}: f \in S\}$. Each element of $\bar{S}$ is holomorphic in some right half-plane. Let B denote the set of all sequences $(\bar{f}_n)$ of nonzero elements of $\bar{S}$ for which there exists f in L such that

$$f_n = f \text{ on } (0, n) \qquad \text{for all } n. \tag{1}$$

For a given f the set of all elements of B satisfying (1) will be

denoted $\hat{f}$. Let B^* denote the set of all elements $(\bar{g}_n)$ of B such that $(\bar{g}_n) \in \hat{g}$ where g is a nonzero element of L. Finally, let X denote the set of all sequences $(\bar{f}_n/\bar{g}_n)$ where $(\bar{f}_n) \in B$ and $(\bar{g}_n) \in B^*$. Then X consists of sequences of functions which are meromorphic in some right half-plane.

LEMMA. *Let $(\bar{f}_n) \in \hat{f}$, $(\bar{g}_n) \in \hat{g}$, $(\bar{F}_n) \in \hat{F}$ and $(\bar{G}_n) \in \hat{G}$ and suppose that g and G are nonzero elements of L. Then $f_n^* G_n = F_n^* g_n$ on $(0, n)$ for all n if and only if $\{f\{t\}\}: \{g(t)\} = \{F(t)\}: \{G(t)\}$.*

Proof. Since $f_n^* G_n = f^* G$ on $(0, n)$ and $F_n^* g_n = F^* g$ on $(0, n)$, the statements $f_n^* G_n = F_n^* g_n$ on $(0, n)$ for all n and $f^* G = F^* g$ are equivalent.

THEOREM 1. *There exists a mapping Φ of X onto Q such that if q belongs to Q, say $q = \{f(t)\}: \{g(t)\}$, and if $(\bar{f}_n)$ and $(\bar{g}_n)$ belong, respectively, to $\hat{f}$ and $\hat{g}$, then $\Phi((\bar{f}_n/\bar{g}_n)) = q$.*

Proof. Let $(\bar{f}_n/\bar{g}_n) \in X$. If $(\bar{f}_n) \in \hat{f}$ and $(\bar{g}_n) \in \hat{g}$ define

$$\Phi((\bar{f}_n/\bar{g}_n)) = \{f(t)\}: \{g(t)\} .$$

If $(\bar{f}_n/\bar{g}_n) = (\bar{F}_n/\bar{G}_n)$ then $\bar{f}_n \bar{G}_n = \bar{F}_n \bar{g}_n$ (all n), that is, $\overline{f_n^* G_n} = \overline{F_n^* g_n}$ (all n). Therefore, $f_n^* G_n = F_n^* g_n$ (all n) and hence, by the lemma,

$$\Phi((\bar{F}_n/\bar{G}_n)) = \Phi((\bar{f}_n/\bar{g}_n)) .$$

Thus, Φ is well-defined. Now, for any $q \in Q$ there exist f and g in L such that $q = \{f(t)\}: \{g(t)\}$. Let $(\bar{f}_n) \in \hat{f}$ and $(\bar{g}_n) \in \hat{g}$. Then $(\bar{f}_n/\bar{g}_n) \in X$ and $\Phi((\bar{f}_n/\bar{g}_n)) = q$. Therefore, Φ is "onto."

For each nonempty open subset Ω of the complex plane let $M(\Omega)$ denote the set of all functions which are meromorphic in Ω. We equip $M(\Omega)$ with the topology of uniform convergence on compact subsets of Ω with respect to the chordal metric. Thus $\varphi_\mu \to \varphi$ in $M(\Omega)$ if and only if

$$\lim_\mu \left[\sup_{z \in K} \frac{|\varphi_\mu(z) - \varphi(z)|}{\sqrt{1 + |\varphi_\mu(z)|^2}\sqrt{1 + |\varphi(z)|^2}} \right] = 0$$

for all compact subsets K of Ω. Let $M = \bigcup M(\Omega)$ where Ω varies over the nonempty open subsets of the complex plane and equip M with the finest topology for which all of the injections $M(\Omega) \to M$ are continuous. Let Y denote the set of all sequences in M and equip Y with the product topology. We may then endow its subset X with the relative topology. Finally, Q is given the quotient topology (relative to Φ and the topology of X). Let T denote this

topology. Thus, T is the finest topology on Q for which the function Φ: $X \to Q$ is continuous.

THEOREM 2. *If q_k converges to q in the sense of Mikusinski then q_k converges to q with respect to the topology T.*

Proof. Suppose q_k converges to q in the sense of Mikusinski. Then there exists g, f and f_k $(k = 1, 2, \cdots)$ in L such that $\{g(t)\}q = \{f_k(t)\}$ and $\{g(t)\}q = \{f(t)\}$ and such that f_k converges to f uniformly on compact subsets of $[0, \infty)$. Define

$$\bar{f}_{k,n}(z) = \int_0^n e^{-zt} f_k(t)dt$$

and

$$\bar{f}_n(z) = \int_0^n e^{-zt} f(t)dt \,.$$

Then $(\bar{f}_{k,n}) \in \hat{f}_k$ and $(\bar{f}_n) \in \hat{f}$. Moreover, $\bar{f}_{k,n}$ and $\bar{f}_n$ are entire functions and

$$\lim_{k\to\infty} [\sup_{z\in K} |\bar{f}_{k,n}(z) - \bar{f}_n(z)|] = 0 \qquad (n = 1, 2, \cdots)$$

for any compact set K. Let $(\bar{g}_n) \in \hat{g}$ and, for each n, choose a nonempty open set Ω_n such that $\bar{g}_n$ is holomorphic and nonvanishing in Ω_n. Then $\bar{f}_{k,n}/\bar{g}_n$ is holomorphic in Ω_n and

$$\lim_{k\to\infty} \bar{f}_{k,n}/\bar{g}_n = \bar{f}_n/\bar{g}_n$$

in $M(\Omega_n)$ and therefore in M. Thus,

$$\lim_{k\to\infty} (\bar{f}_{k,n}/\bar{g}_n) = (\bar{f}_n/\bar{g}_n) \quad \text{in } X \,.$$

But $\Phi((\bar{f}_{k,n}/\bar{g}_n)) = q_k$ and $\Phi((\bar{f}_n/\bar{g}_n)) = q$ by Theorem 1. Therefore, since Φ is continuous, it follows that

$$\lim_{k\to\infty} q_k = q \,.$$

Let us define

$$h_\beta(t) = \frac{t^{\beta-1}}{(\beta-1)!} \qquad (\beta = 1, 2, \cdots)$$

$$s^0 = \text{the identity element of } Q$$

$$s^\beta = \{h_\beta(t)\}^{-1} \qquad (\beta = 1, 2, \cdots) \,.$$

We also define $e^{-\lambda s} = s\{f(t)\}$, where

$$f(t) = \begin{cases} 0 & 0 \leqq t < \lambda \\ 1 & 0 < \lambda \leqq t \,. \end{cases}$$

Then s is the *differential* operator and $e^{-\lambda s}$ is the *translation* operator.

THEOREM 3. $e^{-\lambda s} = \sum_{k=0}^{\infty} (-\lambda)^k / k!\, s^k$.

Proof. If f and h_β are defined as above then $\bar{f}(z) = e^{-\lambda z}/z$ and $\bar{h}_\beta(z) = z^{-\beta}$ $(\beta = 1, 2, \cdots)$. Let

$$\varphi_k(z) = \frac{(-\lambda)^k}{k!} z^k \qquad (k = 0, 1, 2, \cdots) \,.$$

Then

$$\frac{\bar{f}(z)}{\bar{h}_1(z)} = e^{-\lambda z} = \sum_{k=0}^{\infty} \varphi_k(z)$$

where the convergence is uniform on compact sets. Therefore,

$$\bar{f}/\bar{h}_1 = \sum_{k=0}^{\infty} \varphi_k \qquad \text{(convergence in } M) \,.$$

That is,

$$\bar{f}/\bar{h}_1 = \lim_{N \to \infty} \sum_{k=0}^{N} \varphi_k \qquad \text{(convergence in } M) \,.$$

Thus,

$$(\bar{f}/\bar{h}_1, \bar{f}/\bar{h}_1, \cdots) = \lim_{N \to \infty} \left(\sum_{k=0}^{N} \varphi_k, \sum_{k=0}^{N} \varphi_k, \cdots \right)$$

where the convergence is in X. But $\Phi((\bar{f}/\bar{h}_1, \bar{f}/\bar{h}_1, \cdots)) = e^{-\lambda s}$ and

$$\Phi\left(\left(\sum_{k=0}^{N} \varphi_k, \sum_{k=0}^{N} \varphi_k, \cdots \right) \right) = \sum_{k=0}^{N} \frac{(-\lambda)^k}{k!} s^k \,.$$

Since Φ is continuous it follows that

$$e^{-\lambda s} = \lim_{N \to \infty} \sum_{k=0}^{N} \frac{(-\lambda)^k}{k!} s^k \,.$$

Let Q^* denote the set of nonzero elements of Q and define Γ: $Q^* \to Q^*$ by the equation $\Gamma(q) = q^{-1}$ (all q in Q^*).

THEOREM 4. *The function Γ is continuous.*

Proof. Let $X^* = \{x \in X: \Phi(x) \in Q^*\}$. Since Q^* has the quotient topology (relative to Φ and the topology of X^*) it suffices to show

that the composition $\Gamma^{\circ}\Phi$ is continuous [5, p. 95, Theorem 9]. Suppose x_μ is a net in X^* which converges to x in X^*. Let $x_\mu = (\bar{f}_{\mu,n}/\bar{g}_{\mu,n})$ and $x = (\bar{f}_n/\bar{g}_n)$. If $(\bar{f}_{\mu,n}) \in \hat{f}$ then $f_\mu \neq 0$ (since $x_\mu \in X^*$) and therefore $(\bar{f}_{\mu,n}) \in B^*$. Similarly, $(\bar{f}_n) \in B^*$. Therefore, $(\bar{g}_{\mu,n}/\bar{f}_{\mu,n})$ and $(\bar{g}_n/\bar{f}_n)$ belong to X^*. Since $x_\mu \to x$ it follows that $\bar{f}_{\mu,n}/\bar{g}_{\mu,n} \to \bar{f}_n/\bar{g}_n$ in M for each n. Therefore, for each n there exists Ω_n such that

$$\bar{f}_{\mu,n}/\bar{g}_{\mu,n} \longrightarrow \bar{f}_n/\bar{g}_n$$

in $M(\Omega_n)$. Since the reciprocals $\bar{g}_{\mu,n}/\bar{f}_{\mu,n}$ and $\bar{g}_n/\bar{f}_n$ are also meromorphic in Ω_n, the identity

$$\frac{\left|\frac{1}{z} - \frac{1}{w}\right|}{\sqrt{1 + \left|\frac{1}{z}\right|^2}\sqrt{1 + \left|\frac{1}{w}\right|^2}} = \frac{|z - w|}{\sqrt{1 + |z|^2}\ \sqrt{1 + |w|^2}}$$

implies that $\bar{g}_{\mu,n}/\bar{f}_{\mu,n} \to \bar{g}_n/\bar{f}_n$ in $M(\Omega_n)$ and therefore in M. Since this is true for each n it follows that $(\bar{g}_{\mu,n}/\bar{f}_{\mu,n}) \to (\bar{g}_n/\bar{f}_n)$ in X^*. Therefore, $\Phi((\bar{g}_{\mu,n}/\bar{f}_{\mu,n})) \to \Phi((\bar{g}_n/\bar{f}_n))$ in Q^*. But, by Theorem 1, $\Phi((\bar{g}_{\mu,n}/\bar{f}_{\mu,n})) = \Gamma(\Phi(x_\mu))$ and $\Phi((\bar{g}_n/\bar{f}_n)) = \Gamma(\Phi(x))$. Therefore,

$$\Gamma(\Phi(x_\mu)) \longrightarrow \Gamma(\Phi(x)) ,$$

from which we may conclude that the function $\Gamma^{\circ}\Phi$ is continuous.

REFERENCES

1. J. Anthony, *Topologies for quotient fields of commutative integral domains*, Pacific J. Math., **36** (1971), 585-601.

2. L. Berg, *Asymptotische Auffassung der Operatorenrechung*, Studia Math., **21** (1962), 215-229.

3. T. K. Boehme, *On Mikusinski operators*, Studia Math., **33** (1969), 127-140.

4. W. A. Ditkin, *On the theory of operational calculus*, Dokl. Akad. Nauk SSSR, **123** (1958), 395-396.

5. J. Kelley, *General Topology*, Van Nostrand, Princeton, New Jersey, 1955.

6. G. Krabbe, *Ratios of Laplace transforms, Mikusinski operational calculus*, Math. Annalen, **162** (1966), 237-245.

7. ———, *A very weak topology for the Mikusinski field*, Notices Amer. Math. Soc., **19** (1972), 358.

8. J. Mikusinski, *Operational Calculus*, Pergamon Press, New York, 1959.

9. R. Struble, *A genuine topology for the field of Mikusinski operators*, Canad. Math. Bull., **11** (1968), 297-299.

Received April 26, 1972.

CALIFORNIA STATE UNIVERSITY, FULLERTON

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

D. GILBARG AND J. MILGRAM
Stanford University
Stanford, California 94305

J. DUGUNDJI*
Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT
University of Washington
Seattle, Washington 98105

RICHARD ARENS
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH B. H. NEUMANN F. WOLF K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

* C. DePrima will replace J. Dugundji until August 1974.

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics
Vol. 47, No. 1 January, 1973

K. Adachi, Masuo Suzuki and M. Yoshida, *Continuation of holomorphic mappings, with values in a complex Lie group* 1
Michael Aschbacher, *A characterization of the unitary and symplectic groups over finite fields of characteristic at least* 5 5
Larry Eugene Bobisud and James Calvert, *Energy bounds and virial theorems for abstract wave equations* 27
Christer Borell, *A note on an inequality for rearrangements* 39
Peter Southcott Bullen and S. N. Mukhopadhyay, *Peano derivatives and general integrals* 43
Wendell Dan Curtis, Yu-Lee Lee and Forrest Miller, *A class of infinite dimensional subgroups of* $\mathrm{Diff}^r(X)$ *which are Banach Lie groups* 59
Paul C. Eklof, *The structure of ultraproducts of abelian groups* 67
William Alan Feldman, *Axioms of countability and the algebra* $C(X)$ 81
Jack Tilden Goodykoontz, Jr., *Aposyndetic properties of hyperspaces* 91
George Grätzer and J. Płonka, *On the number of polynomials of an idempotent algebra. II* 99
Alan Trinler Huckleberry, *The weak envelope of holomorphy for algebras of holomorphic functions* 115
John Joseph Hutchinson and Julius Martin Zelmanowitz, *Subdirect sum decompositions of endomorphism rings* 129
Gary Douglas Jones, *An asymptotic property of solutions of* $y''' + py' + qy = 0$ 135
Howard E. Lacey, *On the classification of Lindenstrauss spaces* 139
Charles Dwight Lahr, *Approximate identities for convolution measure algebras* 147
George William Luna, *Subdifferentials of convex functions on Banach spaces* 161
Nelson Groh Markley, *Locally circular minimal sets* 177
Robert Wilmer Miller, *Endomorphism rings of finitely generated projective modules* 199
Donald Steven Passman, *On the semisimplicity of group rings of linear groups* 221
Bennie Jake Pearson, *Dendritic compactifications of certain dendritic spaces* 229
Ryōtarō Satō, *Abel-ergodic theorems for subsequences* 233
Henry S. Sharp, Jr., *Locally complete graphs* 243
Harris Samuel Shultz, *A very weak topology for the Mikusinski field of operators* 251
Elena Stroescu, *Isometric dilations of contractions on Banach spaces* 257
Charles W. Trigg, *Versum sequences in the binary system* 263
William L. Voxman, *On the countable union of cellular decompositions of n-manifolds* 277
Robert Francis Wheeler, *The strict topology, separable measures, and paracompactness* 287