A VERY WEAK TOPOLOGY FOR THE MIKUSINSKI FIELD OF OPERATORS

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Using a generalized Laplace transformation the Mikusinski field is given a topology $T$ such that sequences which converge in the sense of Mikusinski converge with respect to $T$, such that the mapping $q \rightarrow q^{-1}$ is continuous and such that the series $\sum (-\lambda)^n s^n/n!$ converges to the translation operator $e^{-\lambda s}$.

In [3] it is shown that the notion of convergence defined in [8] for the Mikusinski field of operators is not topological. Topologies for the Mikusinski field are given in [1], [3], and [9]. In the present paper we endow this field with a topology $T$ such that sequences which converge in the sense of Mikusinski converge with respect to $T$, such that the identity

\[ e^{-zs} = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} s^k \quad (\lambda > 0) \]

holds and such that the mapping $q \rightarrow q^{-1}$ is continuous. The author wishes to acknowledge that this paper constitutes proofs of assertions proposed by Gregers Krabbe [7].

Let $L$ denote the family of complex-valued functions which are locally integrable on $[0, \infty)$. Under addition and convolution $L$ is an integral domain. If $Q$ denotes the quotient field of $L$ then $Q$ is the Mikusinski field of operators. Elements of $Q$ will be denoted $\{f(t)\}$; $\{g(t)\}$ and the injection of $L$ into $Q$ will be denoted $f \rightarrow \{f(t)\}$. We define $S$ to be the set of all $f$ in $L$ for which the integral

\[ \int_0^{\infty} e^{-zt} f(t) dt \]

converges for some $z$. For $f$ in $S$ let

\[ \bar{f}(z) = \int_0^{\infty} e^{-zt} f(t) dt \]

and $\bar{S} = \{\bar{f} : f \in S\}$. Each element of $\bar{S}$ is holomorphic in some right half-plane. Let $B$ denote the set of all sequences $(f_n)$ of nonzero elements of $\bar{S}$ for which there exists $f$ in $L$ such that

(1) \quad $f_n = f$ on $(0, n)$ \quad for all $n$.

For a given $f$ the set of all elements of $B$ satisfying (1) will be
denoted $\hat{f}$. Let $B^*$ denote the set of all elements $(\bar{g}_n)$ of $B$ such that $(\bar{g}_n) \in \hat{g}$ where $g$ is a nonzero element of $L$. Finally, let $X$ denote the set of all sequences $(\bar{f}_n/\bar{g}_n)$ where $(\bar{f}_n) \in \hat{B}$ and $(\bar{g}_n) \in B^*$. Then $X$ consists of sequences of functions which are meromorphic in some right half-plane.

**Lemma.** Let $(\bar{f}_n) \in \hat{f}$, $(\bar{g}_n) \in \hat{g}$, $(\bar{F}_n) \in \hat{F}$ and $(\bar{G}_n) \in \hat{G}$ and suppose that $g$ and $G$ are nonzero elements of $L$. Then $f_n^*G_n = F_n^*g_n$ on $(0, n)$ for all $n$ if and only if $\{f(t)\}: \{g(t)\} = \{F(t)\}: \{G(t)\}$.

**Proof.** Since $f_n^*G_n = f^*G$ on $(0, n)$ and $F_n^*g_n = F^*g$ on $(0, n)$, the statements $f_n^*G_n = F_n^*g_n$ on $(0, n)$ for all $n$ and $f^*G = F^*g$ are equivalent.

**Theorem 1.** There exists a mapping $\Phi$ of $X$ onto $Q$ such that if $q$ belongs to $Q$, say $q = \{f(t)\}: \{g(t)\}$, and if $(\bar{f}_n)$ and $(\bar{g}_n)$ belong, respectively, to $\hat{f}$ and $\hat{g}$, then $\Phi((\bar{f}_n/\bar{g}_n)) = q$.

**Proof.** Let $(\bar{f}_n/\bar{g}_n) \in X$. If $(\bar{f}_n) \in \hat{f}$ and $(\bar{g}_n) \in \hat{g}$ define

$$\Phi((\bar{f}_n/\bar{g}_n)) = \{f(t)\}: \{g(t)\}.$$ 

If $(\bar{f}_n/\bar{g}_n) = (\bar{F}_n/\bar{G}_n)$ then $\bar{f}_n\bar{G}_n = \bar{F}_n\bar{g}_n$ (all $n$), that is, $\bar{f}_n^*\bar{G}_n = \bar{F}_n^*\bar{g}_n$ (all $n$). Therefore, $f_n^*G_n = F_n^*g_n$ (all $n$) and hence, by the lemma,

$$\Phi((\bar{F}_n/\bar{G}_n)) = \Phi((\bar{f}_n/\bar{g}_n)).$$

Thus, $\Phi$ is well-defined. Now, for any $q \in Q$ there exist $f$ and $g$ in $L$ such that $q = \{f(t)\}: \{g(t)\}$. Let $(\bar{f}_n) \in \hat{f}$ and $(\bar{g}_n) \in \hat{g}$. Then $(\bar{f}_n/\bar{g}_n) \in X$ and $\Phi((\bar{f}_n/\bar{g}_n)) = q$. Therefore, $\Phi$ is "onto."

For each nonempty open subset $\Omega$ of the complex plane let $M(\Omega)$ denote the set of all functions which are meromorphic in $\Omega$. We equip $M(\Omega)$ with the topology of uniform convergence on compact subsets of $\Omega$ with respect to the chordal metric. Thus $\varphi_\mu \to \varphi$ in $M(\Omega)$ if and only if

$$\lim_\mu \left[ \sup_{z \in K} \frac{|\varphi_\mu(z) - \varphi(z)|}{\sqrt{1 + |\varphi_\mu(z)|^2} \sqrt{1 + |\varphi(z)|^2}} \right] = 0$$

for all compact subsets $K$ of $\Omega$. Let $M = \bigcup M(\Omega)$ where $\Omega$ varies over the nonempty open subsets of the complex plane and equip $M$ with the finest topology for which all of the injections $M(\Omega) \to M$ are continuous. Let $Y$ denote the set of all sequences in $M$ and equip $Y$ with the product topology. We may then endow its subset $X$ with the relative topology. Finally, $Q$ is given the quotient topology (relative to $\Phi$ and the topology of $X$). Let $T$ denote this
topology. Thus, $T$ is the finest topology on $Q$ for which the function $\Phi: X \to Q$ is continuous.

**Theorem 2.** If $q_k$ converges to $q$ in the sense of Mikusinski then $q_k$ converges to $q$ with respect to the topology $T$.

**Proof.** Suppose $q_k$ converges to $q$ in the sense of Mikusinski. Then there exists $g, f$ and $f_k (k = 1, 2, \cdots)$ in $L$ such that $\{g(t)\}q = \{f_k(t)\}$ and $\{g(t)\}q = \{f(t)\}$ and such that $f_k$ converges to $f$ uniformly on compact subsets of $[0, \infty)$. Define

$$\bar{f}_{k,n}(z) = \int_0^1 e^{-zt}f_k(t)\,dt$$

and

$$\bar{f}_n(z) = \int_0^1 e^{-zt}f(t)\,dt.$$  

Then $(\bar{f}_{k,n}) \in \tilde{f}_k$ and $(\bar{f}_n) \in \tilde{f}$. Moreover, $\bar{f}_{k,n}$ and $\bar{f}_n$ are entire functions and

$$\lim\sup_{n \to \infty} |\bar{f}_{k,n}(z) - \bar{f}_n(z)| = 0 \quad (n = 1, 2, \cdots)$$

for any compact set $K$. Let $(\bar{g}_n) \in \tilde{g}$ and, for each $n$, choose a non-empty open set $\Omega_n$ such that $\bar{g}_n$ is holomorphic and nonvanishing in $\Omega_n$. Then $\bar{f}_{k,n}/\bar{g}_n$ is holomorphic in $\Omega_n$ and

$$\lim_{k \to \infty} \frac{s_{k,n}}{\bar{g}_n} = \frac{\bar{f}_n}{\bar{g}_n}$$

in $M(\Omega_n)$ and therefore in $M$. Thus,

$$\lim_{k \to \infty} \left( \frac{\bar{f}_{k,n} / \bar{g}_n}{\bar{f}_n / \bar{g}_n} \right) = (\bar{f}_n / \bar{g}_n) \quad \text{in } X.$$  

But $\Phi((\bar{f}_{k,n} / \bar{g}_n)) = q_k$ and $\Phi((\bar{f}_n / \bar{g}_n)) = q$ by Theorem 1. Therefore, since $\Phi$ is continuous, it follows that

$$\lim_{k \to \infty} q_k = q .$$

Let us define

$$h_\beta(t) = \frac{t^{\beta-1}}{(\beta - 1)!} \quad (\beta = 1, 2, \cdots)$$

$s^0 = \text{the identity element of } Q$

$s^\beta = \{h_\beta(t)\}^{-1} \quad (\beta = 1, 2, \cdots) .$

We also define $e^{-\lambda s} = s\{f(t)\}$, where
Then \( s \) is the differential operator and \( e^{-\lambda s} \) is the translation operator.

**Theorem 3.** \( e^{-\lambda s} = \sum_{k=0}^{\infty} (-\lambda)^k / k! \cdot s^k \).

*Proof.* If \( f \) and \( h_\beta \) are defined as above then \( \tilde{f}(z) = e^{-\lambda s} / z \) and \( \tilde{h}_\beta(z) = z^{-\beta} \) (\( \beta = 1, 2, \ldots \)). Let

\[
\Phi_k(z) = \frac{(-\lambda)^k}{k!} z^k \quad (k = 0, 1, 2, \ldots).
\]

Then

\[
\frac{\tilde{f}(z)}{\tilde{h}_1(z)} = e^{-\lambda s} = \sum_{k=0}^{\infty} \Phi_k(z)
\]

where the convergence is uniform on compact sets. Therefore,

\[
\frac{\tilde{f}}{\tilde{h}_1} = \sum_{k=0}^{\infty} \Phi_k \quad \text{(convergence in } M).\]

That is,

\[
\frac{\tilde{f}}{\tilde{h}_1} = \lim_{N \to \infty} \sum_{k=0}^{N} \Phi_k \quad \text{(convergence in } M).\]

Thus,

\[
(\frac{\tilde{f}}{\tilde{h}_1}, \frac{\tilde{f}}{\tilde{h}_1}, \cdots) = \lim_{N \to \infty} \left( \sum_{k=0}^{N} \Phi_k, \sum_{k=0}^{N} \Phi_k, \cdots \right)
\]

where the convergence is in \( X \). But \( \Phi((\frac{\tilde{f}}{\tilde{h}_1}, \frac{\tilde{f}}{\tilde{h}_1}, \cdots)) = e^{-\lambda s} \) and

\[
\Phi \left( \left( \sum_{k=0}^{N} \Phi_k, \sum_{k=0}^{N} \Phi_k, \cdots \right) \right) = \sum_{k=0}^{N} \frac{(-\lambda)^k}{k!} \cdot s^k.
\]

Since \( \Phi \) is continuous it follows that

\[
e^{-\lambda s} = \lim_{N \to \infty} \sum_{k=0}^{N} \frac{(-\lambda)^k}{k!} \cdot s^k.
\]

Let \( Q^* \) denote the set of nonzero elements of \( Q \) and define \( \Gamma : Q^* \to Q^* \) by the equation \( \Gamma(q) = q^{-1} \) (all \( q \) in \( Q^* \)).

**Theorem 4.** The function \( \Gamma \) is continuous.

*Proof.* Let \( X^* = \{ x \in X : \Phi(x) \in Q^* \} \). Since \( Q^* \) has the quotient topology (relative to \( \Phi \) and the topology of \( X^* \)) it suffices to show
that the composition $I^n \Phi$ is continuous [5, p. 95, Theorem 9]. Suppose $x_\mu$ is a net in $X^*$ which converges to $x$ in $X^*$. Let $x_\mu = (\tilde{f}_{\mu,n}/\tilde{g}_{\mu,n})$ and $x = (\tilde{f}_n/\tilde{g}_n)$. If $(\tilde{f}_{\mu,n}) \in \tilde{F}$ then $f_\mu \neq 0$ (since $x_\mu \in X^*$) and therefore $(\tilde{f}_{\mu,n}) \in B^*$. Similarly, $(\tilde{f}_n) \in B^*$. Therefore, $(\tilde{g}_{\mu,n}/\tilde{f}_{\mu,n})$ and $(\tilde{g}_n/\tilde{f}_n)$ belong to $X^*$. Since $x_\mu \to x$ it follows that $\tilde{f}_{\mu,n}/\tilde{g}_{\mu,n} \to \tilde{f}_n/\tilde{g}_n$ in $M$ for each $n$. Therefore, for each $n$ there exists $\Omega_n$ such that

$\tilde{f}_{\mu,n}/\tilde{g}_{\mu,n} \to \tilde{f}_n/\tilde{g}_n$

in $M(\Omega_n)$. Since the reciprocals $\tilde{g}_{\mu,n}/\tilde{f}_{\mu,n}$ and $\tilde{g}_n/\tilde{f}_n$ are also meromorphic in $\Omega_n$, the identity

$$\left| \frac{1}{z} - \frac{1}{w} \right| \sqrt{1 + \left| \frac{1}{z} \right|^2} \frac{1}{\sqrt{1 + \left| \frac{1}{w} \right|^2}} = \frac{|z - w|}{\sqrt{1 + |z|^2} \sqrt{1 + |w|^2}}$$

implies that $\tilde{g}_{\mu,n}/\tilde{f}_{\mu,n} \to \tilde{g}_n/\tilde{f}_n$ in $M(\Omega_n)$ and therefore in $M$. Since this is true for each $n$ it follows that $(\tilde{g}_{\mu,n}/\tilde{f}_{\mu,n}) \to (\tilde{g}_n/\tilde{f}_n)$ in $X^*$. Therefore, $\Phi((\tilde{g}_{\mu,n}/\tilde{f}_{\mu,n})) \to \Phi((\tilde{g}_n/\tilde{f}_n))$ in $Q^*$. But, by Theorem 1, $\Phi((\tilde{g}_{\mu,n}/\tilde{f}_{\mu,n})) = \Gamma(\Phi(x_\mu))$ and $\Phi((\tilde{g}_n/\tilde{f}_n)) = \Gamma(\Phi(x))$. Therefore,

$$\Gamma(\Phi(x_\mu)) \to \Gamma(\Phi(x)),$$

from which we may conclude that the function $I^n \Phi$ is continuous.

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Received April 26, 1972.

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Larry Eugene Bobisud and James Calvert, *Energy bounds and virial theorems for abstract wave equations* ................................................................. 27
Christer Borell, *A note on an inequality for rearrangements* ......................... 39
Peter Southcott Bullen and S. N. Mukhopadhyay, *Peano derivatives and general integrals* ................................................................. 43
Wendell Dan Curtis, Yu-Lee Lee and Forrest Miller, *A class of infinite dimensional subgroups of Diff^r (X) which are Banach Lie groups* ............. 59
Paul C. Eklof, *The structure of ultraproducts of abelian groups* ...................... 67
William Alan Feldman, *Axioms of countability and the algebra C(X)* .......... 81
Jack Tilden Goodykoontz, Jr., *Aposyndetic properties of hyperspaces* .......... 91
George Grätzer and J. Plonka, *On the number of polynomials of an idempotent algebra. II* ................................................................. 99
Alan Trinler Huckleberry, *The weak envelope of holomorphy for algebras of holomorphic functions* ................................................................. 115
John Joseph Hutchinson and Julius Martin Zelmanowitz, *Subdirect sum decompositions of endomorphism rings* ............................................. 129
Gary Douglas Jones, *An asymptotic property of solutions of y'''' + py' + qy = 0* ................................................................. 135
Howard E. Lacey, *On the classification of Lindenstrauss spaces* .................... 139
Charles Dwight Lahr, *Approximate identities for convolution measure algebras* ................................................................. 147
George William Luna, *Subdifferentials of convex functions on Banach spaces* ................................................................. 161
Nelson Groh Markley, *Locally circular minimal sets* ..................................... 177
Robert Wilmer Miller, *Endomorphism rings of finitely generated projective modules* ................................................................. 199
Donald Steven Passman, *On the semisimplicity of group rings of linear groups* ................................................................. 221
Bennie Jake Pearson, *Dendritic compactifications of certain dendritic spaces* ................................................................. 229
Ryōtarō Satō, *Abel-ergodic theorems for subsequences* ................................ 233
Henry S. Sharp, Jr., *Locally complete graphs* ............................................. 243
Harris Samuel Shultz, *A very weak topology for the Mikusinski field of operators* ................................................................. 251
Elena Stroescu, *Isometric dilations of contractions on Banach spaces* ............. 257
Charles W. Trigg, *Versum sequences in the binary system* .............................. 263
William L. Voxman, *On the countable union of cellular decompositions of n-manifolds* ................................................................. 277
Robert Francis Wheeler, *The strict topology, separable measures, and paracompactness* ................................................................. 287