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**A VERY WEAK TOPOLOGY FOR THE MIKUSINSKI FIELD OF
OPERATORS**

HARRIS SAMUEL SHULTZ

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Using a generalized Laplace transformation the Mikusinski field is given a topology T such that sequences which converge in the sense of Mikusinski converge with respect to T , such that the mapping $q \rightarrow q^{-1}$ is continuous and such that the series $\sum (-\lambda)^n s^n / n!$ converges to the translation operator $e^{-\lambda s}$.

In [3] it is shown that the notion of convergence defined in [8] for the Mikusinski field of operators is not topological. Topologies for the Mikusinski field are given in [1], [3], and [9]. In the present paper we endow this field with a topology T such that sequences which converge in the sense of Mikusinski converge with respect to T , such that the identity

$$e^{-\lambda s} = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} s^k \quad (\lambda > 0)$$

holds and such that the mapping $q \rightarrow q^{-1}$ is continuous. The author wishes to acknowledge that this paper constitutes proofs of assertions proposed by Gregers Krabbe [7].

Let L denote the family of complex-valued functions which are locally integrable on $[0, \infty)$. Under addition and convolution L is an integral domain. If Q denotes the quotient field of L then Q is the Mikusinski field of operators. Elements of Q will be denoted $\{f(t)\}$; $\{g(t)\}$ and the injection of L into Q will be denoted $f \rightarrow \{f(t)\}$. We define S to be the set of all f in L for which the integral

$$\int_0^{\infty} e^{-zt} f(t) dt$$

converges for some z . For f in S let

$$\bar{f}(z) = \int_0^{\infty} e^{-zt} f(t) dt$$

and $\bar{S} = \{\bar{f} : f \in S\}$. Each element of \bar{S} is holomorphic in some right half-plane. Let B denote the set of all sequences (\bar{f}_n) of nonzero elements of \bar{S} for which there exists f in L such that

$$(1) \quad \bar{f}_n = f \text{ on } (0, n) \quad \text{for all } n.$$

For a given f the set of all elements of B satisfying (1) will be

denoted \hat{f} . Let B^* denote the set of all elements (\bar{g}_n) of B such that $(\bar{g}_n) \in \hat{g}$ where g is a nonzero element of L . Finally, let X denote the set of all sequences (\bar{f}_n/\bar{g}_n) where $(\bar{f}_n) \in \hat{B}$ and $(\bar{g}_n) \in B^*$. Then X consists of sequences of functions which are meromorphic in some right half-plane.

LEMMA. Let $(\bar{f}_n) \in \hat{f}$, $(\bar{g}_n) \in \hat{g}$, $(\bar{F}_n) \in \hat{F}$ and $(\bar{G}_n) \in \hat{G}$ and suppose that g and G are nonzero elements of L . Then $f_n^* G_n = F_n^* g_n$ on $(0, n)$ for all n if and only if $\{f(t)\}: \{g(t)\} = \{F(t)\}: \{G(t)\}$.

Proof. Since $f_n^* G_n = f^* G$ on $(0, n)$ and $F_n^* g_n = F^* g$ on $(0, n)$, the statements $f_n^* G_n = F_n^* g_n$ on $(0, n)$ for all n and $f^* G = F^* g$ are equivalent.

THEOREM 1. There exists a mapping Φ of X onto Q such that if q belongs to Q , say $q = \{f(t)\}: \{g(t)\}$, and if (\bar{f}_n) and (\bar{g}_n) belong, respectively, to \hat{f} and \hat{g} , then $\Phi((\bar{f}_n/\bar{g}_n)) = q$.

Proof. Let $(\bar{f}_n/\bar{g}_n) \in X$. If $(\bar{f}_n) \in \hat{f}$ and $(\bar{g}_n) \in \hat{g}$ define

$$\Phi((\bar{f}_n/\bar{g}_n)) = \{f(t)\}: \{g(t)\}.$$

If $(\bar{f}_n/\bar{g}_n) = (\bar{F}_n/\bar{G}_n)$ then $\bar{f}_n \bar{G}_n = \bar{F}_n \bar{g}_n$ (all n), that is, $\bar{f}_n^* \bar{G}_n = \bar{F}_n^* \bar{g}_n$ (all n). Therefore, $f_n^* G_n = F_n^* g_n$ (all n) and hence, by the lemma,

$$\Phi((\bar{F}_n/\bar{G}_n)) = \Phi((\bar{f}_n/\bar{g}_n)).$$

Thus, Φ is well-defined. Now, for any $q \in Q$ there exist f and g in L such that $q = \{f(t)\}: \{g(t)\}$. Let $(\bar{f}_n) \in \hat{f}$ and $(\bar{g}_n) \in \hat{g}$. Then $(\bar{f}_n/\bar{g}_n) \in X$ and $\Phi((\bar{f}_n/\bar{g}_n)) = q$. Therefore, Φ is "onto."

For each nonempty open subset Ω of the complex plane let $M(\Omega)$ denote the set of all functions which are meromorphic in Ω . We equip $M(\Omega)$ with the topology of uniform convergence on compact subsets of Ω with respect to the chordal metric. Thus $\varphi_\mu \rightarrow \varphi$ in $M(\Omega)$ if and only if

$$\lim_{\mu} \left[\sup_{z \in K} \frac{|\varphi_\mu(z) - \varphi(z)|}{\sqrt{1 + |\varphi_\mu(z)|^2} \sqrt{1 + |\varphi(z)|^2}} \right] = 0$$

for all compact subsets K of Ω . Let $M = \bigcup M(\Omega)$ where Ω varies over the nonempty open subsets of the complex plane and equip M with the finest topology for which all of the injections $M(\Omega) \rightarrow M$ are continuous. Let Y denote the set of all sequences in M and equip Y with the product topology. We may then endow its subset X with the relative topology. Finally, Q is given the quotient topology (relative to Φ and the topology of X). Let T denote this

topology. Thus, T is the finest topology on Q for which the function $\Phi: X \rightarrow Q$ is continuous.

THEOREM 2. *If q_k converges to q in the sense of Mikusinski then q_k converges to q with respect to the topology T .*

Proof. Suppose q_k converges to q in the sense of Mikusinski. Then there exists g, f and $f_k (k = 1, 2, \dots)$ in L such that $\{g(t)\}q = \{f_k(t)\}$ and $\{g(t)\}q = \{f(t)\}$ and such that f_k converges to f uniformly on compact subsets of $[0, \infty)$. Define

$$\bar{f}_{k,n}(z) = \int_0^n e^{-zt} f_k(t) dt$$

and

$$\bar{f}_n(z) = \int_0^n e^{-zt} f(t) dt .$$

Then $(\bar{f}_{k,n}) \in \hat{f}_k$ and $(\bar{f}_n) \in \hat{f}$. Moreover, $\bar{f}_{k,n}$ and \bar{f}_n are entire functions and

$$\lim_{k \rightarrow \infty} [\sup_{z \in K} |\bar{f}_{k,n}(z) - \bar{f}_n(z)|] = 0 \quad (n = 1, 2, \dots)$$

for any compact set K . Let $(\bar{g}_n) \in \hat{g}$ and, for each n , choose a non-empty open set Ω_n such that \bar{g}_n is holomorphic and nonvanishing in Ω_n . Then $\bar{f}_{k,n}/\bar{g}_n$ is holomorphic in Ω_n and

$$\lim_{k \rightarrow \infty} \bar{f}_{k,n}/\bar{g}_n = \bar{f}_n/\bar{g}_n$$

in $M(\Omega_n)$ and therefore in M . Thus,

$$\lim_{k \rightarrow \infty} (\bar{f}_{k,n}/\bar{g}_n) = (\bar{f}_n/\bar{g}_n) \text{ in } X .$$

But $\Phi((\bar{f}_{k,n}/\bar{g}_n)) = q_k$ and $\Phi((\bar{f}_n/\bar{g}_n)) = q$ by Theorem 1. Therefore, since Φ is continuous, it follows that

$$\lim_{k \rightarrow \infty} q_k = q .$$

Let us define

$$h_\beta(t) = \frac{t^{\beta-1}}{(\beta-1)!} \quad (\beta = 1, 2, \dots)$$

$s^0 =$ the identity element of Q

$$s^\beta = \{h_\beta(t)\}^{-1} \quad (\beta = 1, 2, \dots) .$$

We also define $e^{-\lambda s} = s\{f(t)\}$, where

$$f(t) = \begin{cases} 0 & 0 \leq t < \lambda \\ 1 & 0 < \lambda \leq t. \end{cases}$$

Then s is the *differential operator* and $e^{-\lambda s}$ is the *translation operator*.

THEOREM 3. $e^{-\lambda s} = \sum_{k=0}^{\infty} (-\lambda)^k / k! s^k$.

Proof. If f and h_β are defined as above then $\bar{f}(z) = e^{-\lambda z}/z$ and $\bar{h}_\beta(z) = z^{-\beta}$ ($\beta = 1, 2, \dots$). Let

$$\varphi_k(z) = \frac{(-\lambda)^k}{k!} z^k \quad (k = 0, 1, 2, \dots).$$

Then

$$\frac{\bar{f}(z)}{\bar{h}_1(z)} = e^{-\lambda z} = \sum_{k=0}^{\infty} \varphi_k(z)$$

where the convergence is uniform on compact sets. Therefore,

$$\bar{f}/\bar{h}_1 = \sum_{k=0}^{\infty} \varphi_k \quad (\text{convergence in } M).$$

That is,

$$\bar{f}/\bar{h}_1 = \lim_{N \rightarrow \infty} \sum_{k=0}^N \varphi_k \quad (\text{convergence in } M).$$

Thus,

$$(\bar{f}/\bar{h}_1, \bar{f}/\bar{h}_1, \dots) = \lim_{N \rightarrow \infty} \left(\sum_{k=0}^N \varphi_k, \sum_{k=0}^N \varphi_k, \dots \right)$$

where the convergence is in X . But $\Phi((\bar{f}/\bar{h}_1, \bar{f}/\bar{h}_1, \dots)) = e^{-\lambda s}$ and

$$\Phi\left(\left(\sum_{k=0}^N \varphi_k, \sum_{k=0}^N \varphi_k, \dots\right)\right) = \sum_{k=0}^N \frac{(-\lambda)^k}{k!} s^k.$$

Since Φ is continuous it follows that

$$e^{-\lambda s} = \lim_{N \rightarrow \infty} \sum_{k=0}^N \frac{(-\lambda)^k}{k!} s^k.$$

Let Q^* denote the set of nonzero elements of Q and define $\Gamma: Q^* \rightarrow Q^*$ by the equation $\Gamma(q) = q^{-1}$ (all q in Q^*).

THEOREM 4. *The function Γ is continuous.*

Proof. Let $X^* = \{x \in X: \Phi(x) \in Q^*\}$. Since Q^* has the quotient topology (relative to Φ and the topology of X^*) it suffices to show

that the composition $\Gamma^\circ \Phi$ is continuous [5, p. 95, Theorem 9]. Suppose x_μ is a net in X^* which converges to x in X^* . Let $x_\mu = (\bar{f}_{\mu,n}/\bar{g}_{\mu,n})$ and $x = (\bar{f}_n/\bar{g}_n)$. If $(\bar{f}_{\mu,n}) \in \hat{f}$ then $f_\mu \neq 0$ (since $x_\mu \in X^*$) and therefore $(\bar{f}_{\mu,n}) \in B^*$. Similarly, $(\bar{f}_n) \in B^*$. Therefore, $(\bar{g}_{\mu,n}/\bar{f}_{\mu,n})$ and (\bar{g}_n/\bar{f}_n) belong to X^* . Since $x_\mu \rightarrow x$ it follows that $\bar{f}_{\mu,n}/\bar{g}_{\mu,n} \rightarrow \bar{f}_n/\bar{g}_n$ in M for each n . Therefore, for each n there exists Ω_n such that

$$\bar{f}_{\mu,n}/\bar{g}_{\mu,n} \longrightarrow \bar{f}_n/\bar{g}_n$$

in $M(\Omega_n)$. Since the reciprocals $\bar{g}_{\mu,n}/\bar{f}_{\mu,n}$ and \bar{g}_n/\bar{f}_n are also meromorphic in Ω_n , the identity

$$\frac{\left| \frac{1}{z} - \frac{1}{w} \right|}{\sqrt{1 + \left| \frac{1}{z} \right|^2} \sqrt{1 + \left| \frac{1}{w} \right|^2}} = \frac{|z - w|}{\sqrt{1 + |z|^2} \sqrt{1 + |w|^2}}$$

implies that $\bar{g}_{\mu,n}/\bar{f}_{\mu,n} \rightarrow \bar{g}_n/\bar{f}_n$ in $M(\Omega_n)$ and therefore in M . Since this is true for each n it follows that $(\bar{g}_{\mu,n}/\bar{f}_{\mu,n}) \rightarrow (\bar{g}_n/\bar{f}_n)$ in X^* . Therefore, $\Phi((\bar{g}_{\mu,n}/\bar{f}_{\mu,n})) \rightarrow \Phi((\bar{g}_n/\bar{f}_n))$ in Q^* . But, by Theorem 1, $\Phi((\bar{g}_{\mu,n}/\bar{f}_{\mu,n})) = \Gamma(\Phi(x_\mu))$ and $\Phi((\bar{g}_n/\bar{f}_n)) = \Gamma(\Phi(x))$. Therefore,

$$\Gamma(\Phi(x_\mu)) \longrightarrow \Gamma(\Phi(x)) ,$$

from which we may conclude that the function $\Gamma^\circ \Phi$ is continuous.

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