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**ISOMETRIC DILATIONS OF CONTRACTIONS ON BANACH  
SPACES**

ELENA STROESCU

## ISOMETRIC DILATIONS OF CONTRACTIONS ON BANACH SPACES

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**This paper is concerned with the dilation, in the case of a Banach space, of operator-valued functions on a group into representations. Banach-space analogues of Sz.-Nagy's theorem and Ando's theorem are obtained.**

Throughout this note  $Z$  (resp.  $R$ , resp.  $R^+$ , resp.  $N$ , resp.  $C$ ) is the set of all integer (resp. real, resp. nonnegative real, resp. nonnegative integer, resp. complex) numbers. Also  $G$  is a group,  $e \in G$  its neutral element:  $K: G \rightarrow R^+$  a submultiplicative function (i.e.,  $K(gh) \leq K(g)K(h)$  for all  $g, h \in G$ ) with  $K(e) = 1$ ;  $X$  a Banach space;  $\mathcal{B}(X)$  the Banach algebra of all linear bounded operators on  $X$  and  $I \in \mathcal{B}(X)$  the identity.

$\mathcal{E}^m(R)$  ( $m \in N, m = \infty$ ) being the algebra of all  $m$ -times differentiable functions on  $R$  with the usual topology and  $\Gamma = \{z \in C; |z| = 1\}$ ,  $\mathcal{E}^m(\Gamma)$  is the algebra of all functions  $f: \Gamma \rightarrow C$  such that  $t \rightarrow f(e^{it})$  belongs to  $\mathcal{E}^m(R)$ , endowed with the topology induced by  $\mathcal{E}^m(R)$ . An operator  $T \in \mathcal{B}(X)$  is called  $\mathcal{E}^m(\Gamma)$ -unitary if it is  $\mathcal{E}^m(\Gamma)$ -scalar ([2], [4]).

**THEOREM.** (See also [7] Theorem 1). *Let  $\phi: G \rightarrow \mathcal{B}(X)$  be a function with the property  $\|\phi_g\| \leq K(g)$  for all  $g \in G$  and  $\phi_e = I$ .*

Then there exists a Banach space  $\tilde{X}$  containing  $X$  (by an isometric isomorphism), a norm one projection  $P$  of  $\tilde{X}$  onto  $X$  and a representation  $\tilde{\phi}$  of  $G$  as a group of invertible operators on  $\tilde{X}$  such that

(0)  $1/K(\gamma^{-1}) \leq \|\tilde{\phi}_\gamma\| \leq K(\gamma)$  for all  $\gamma \in G$  and  $\tilde{\phi}_e = \tilde{I}$ .

(i)  $P\tilde{\phi}_{\gamma|X} = \phi_\gamma$  for any  $\gamma \in G$ .

(ii)  $\tilde{X}$  is the closed vector space spanned by  $\{\tilde{\phi}_\gamma x; \gamma \in G, x \in X\}$ .

(iii) If  $\phi$  takes its values from the set of contractions on  $X$ , then  $G$  is represented by  $\tilde{\phi}$  as a group of invertible isometries on  $\tilde{X}$ . Moreover, if  $G$  is a topological group and for every  $x \in X$ , the function  $g \rightarrow \phi_g x$  is left uniformly continuous, then the representation  $\tilde{\phi}$  is strongly continuous.

*Proof.* Let  $Y$  be the vector space of all  $X$ -valued functions on  $G$ ,  $y(\cdot)$  with the property

$$\|y(g)\| \leq MK(g) \quad \text{for all } g \in G,$$

where  $M$  is a positive real constant and  $K$  the submultiplicative function from the hypothesis. (In what follows we shall denote elements of  $Y$  also by  $(y_g)_{g \in G}$ .) One sees easily that  $Y$  endowed with the norm

$$\|y(\cdot)\| = \sup_g \|y(g)\|K(g)^{-1}, \text{ is a Banach space.}$$

Let  $X^{(G)} = \bigoplus_{g \in G} X^g$  be the direct sum with  $X^g = X$  for all  $g \in G$ . Define a map  $\Theta: X^{(G)} \rightarrow X$  by  $(\Theta y)_g = \sum_h \phi_{gh} y_h$  for all  $g \in G$  and  $y \in X^{(G)}$ . Then for every  $y \in X^{(G)}$  one has  $\Theta y \in Y$  and the set  $\hat{X} = \{\Theta y; y \in X^{(G)}\}$  is a subspace of  $Y$ . Consider the closure of  $\hat{X}$  in  $Y$  and denote it by  $\tilde{X}$ .

Now let  $X_0$  be a subspace of  $\hat{X}$  of elements

$$\begin{aligned} y(\cdot) &= (\phi_g x)_{g \in G} = (\sum_h \phi_{gh} \delta_{eh} x)_{g \in G} \text{ when } x \text{ runs over } X \\ (\delta_{gh} &= 0 \text{ for } g \neq h \text{ and } \delta_{gh} = 1 \text{ for } g = h). \text{ Define a map} \\ \varphi: X_0 &\rightarrow X \text{ by } \varphi(y(\cdot)) = y(e) \text{ for all } y(\cdot) \in X_0. \end{aligned}$$

Then one has

$$\|\varphi(y(\cdot))\| = \|y(e)\| \leq \sup_g \|y(g)\|K(g)^{-1} = \|y(\cdot)\|$$

and

$$\|y(\cdot)\| = \sup_g \|\phi_g x\|K(g)^{-1} \leq \|x\| = \|y(e)\|.$$

Hence  $\varphi$  is an isometric isomorphism of  $X_0$  onto  $X$ .

Let  $Q: \hat{X} \rightarrow X$  be a map defined by

$$Qy(\cdot) = y(e) \text{ for all } y(\cdot) \in \hat{X}.$$

Obviously,  $Q$  is linear surjective and satisfies  $\|Qy(\cdot)\| \leq \|y(\cdot)\|$  for all  $y(\cdot) \in \hat{X}$ . Its extension by continuity to a linear map of  $\tilde{X}$  onto  $X$  will be denoted by the same symbol. Then  $\varphi^{-1}Q$  is a norm one projection of  $\tilde{X}$  onto  $X$ .

For every  $\gamma \in G$ , define a map  $\hat{\phi}_\gamma: \hat{X} \rightarrow \hat{X}$  by

$$\hat{\phi}_\gamma \Theta y = ((\Theta y)_{g\gamma})_{g \in G} = (\sum_h \phi_{g\gamma h} y_h)_{g \in G} = (\sum_d \phi_{gd} z_d)_{g \in G} = \Theta z \in \hat{X}$$

when  $y$  runs over  $X^{(G)}$ . (It is made the notation  $d = \gamma h$ ,  $z_d = y_h$  for all  $h \in G$ ; hence  $z$  with these components belongs to  $X^{(G)}$ .) One sees easily that  $\hat{\phi}_\gamma$  is well defined and linear. Moreover, one has

$$\begin{aligned} \|\hat{\phi}_\gamma \Theta y\| &= \sup_g \|\sum_h \phi_{g\gamma h} y_h\|K(g)^{-1} \\ &= \sup_g \|\sum_h \phi_{g\gamma h} y_h\|K(g\gamma)^{-1}K(g)^{-1}K(g\gamma) \\ &\leq K(\gamma) \sup_{g\gamma} \|\sum_h \phi_{g\gamma h} y_h\|K(g\gamma)^{-1} = K(\gamma)\|\Theta y\|. \end{aligned}$$

That is

$$(1) \quad \|\hat{\phi}_\gamma \theta y\| \leq K(\gamma) \|\theta y\| \quad \text{for all } y \in X^{(G)}.$$

Then  $\hat{\phi}_\gamma$  can be extended by continuity to an element of  $\mathcal{B}(\tilde{X})$  which will be denoted by  $\tilde{\phi}_\gamma$ . One sees easily that  $\tilde{\phi}_{\alpha\beta} = \tilde{\phi}_\alpha \tilde{\phi}_\beta$  for all  $\alpha, \beta \in G$  and  $\tilde{\phi}_e = \tilde{I}$ . Moreover,

$$(2) \quad \|\theta y\| \leq \|\hat{\phi}_{\gamma^{-1}} \hat{\phi}_\gamma \theta y\| \leq K(\gamma^{-1}) \|\hat{\phi}_\gamma \theta y\| \quad \text{for all } y \in X^{(G)}.$$

Also  $\hat{\phi}_\gamma: \hat{X} \rightarrow \hat{X}$  is surjective since one has

$$\theta y = \hat{\phi}_\gamma((\theta y)_{g\gamma^{-1}})_{g \in G} \quad \text{for all } y \in X^{(G)} \quad \text{and } \gamma \in G.$$

Thus the property (0) is proved. To show (i) we see that

$$((\varphi^{-1}Q)\tilde{\phi}_\gamma)\varphi^{-1}(x) = \varphi^{-1}(\phi_\gamma x) \quad \text{for all } x \in X \quad \text{and } \gamma \in G.$$

Identifying  $X_0$  and  $X$  via  $\varphi$  and writing  $P$  instead of  $\varphi^{-1}Q$ , this equality reads more naturally as  $P\tilde{\phi}_{\gamma^{-1}x} = \phi_\gamma$ . The property (ii) is immediate noting that every  $\theta y \in \hat{X}$  can be written  $\theta y = \Sigma_h \tilde{\phi}_h \varphi^{-1}(y_h)$ . The first assertion of (iii) is immediate because taking  $K(g) = 1$  for all  $g \in G$ , the above inequalities (1) and (2) become

$$(3) \quad \|\hat{\phi}_\gamma \theta y\| = \|\theta y\| \quad \text{for all } y \in X^{(G)} \quad \text{and } \gamma \in G.$$

To prove the second assertion of (iii) we assume still that  $G$  is a topological group and  $g \rightarrow \phi_g x$  is left uniformly continuous for each  $x \in X$ . Taking into account of (ii) it is enough to show that for any fixed  $\gamma \in G$  and  $y(\cdot) \in X_0$ , the map  $a \rightarrow \tilde{\phi}_a(\tilde{\phi}_\gamma y)(\cdot) = (\tilde{\phi}_{a\gamma} y)(\cdot)$  is continuous. As this map is the composition of  $a \rightarrow a\gamma$  and  $a\gamma \rightarrow (\tilde{\phi}_{a\gamma} y)(\cdot)$ , we need only show that for each  $y(\cdot) \in X_0$ , the map  $a \rightarrow (\tilde{\phi}_a y)(\cdot)$  is continuous. For this it is sufficient to show the continuity at  $a = e$ . But this fact is immediate from the left uniform continuity of  $g \rightarrow \phi_g x$  for every  $x \in X$ , because  $\|(\tilde{\phi}_a y)(\cdot) - y(\cdot)\| = \sup_g \|\phi_{ga} y(e) - \phi_g y(e)\|$ .

**COROLLARY 1.** *Let  $\{T_t\}_{t \in R^+} \subset \mathcal{B}(X)$  be a semigroup of contractions. Then there exists a Banach space  $\tilde{X}$  containing  $X$ , a norm one projection  $P$  of  $\tilde{X}$  onto  $X$  and a group  $\{U_t\}_{t \in R}$  of invertible isometries on  $\tilde{X}$  such that:*

- (i)  $PU_t x = T_{|t|x}$ , for all  $x \in X$ ,  $t \in R$ .
- (ii)  $\tilde{X}$  is the closed vector space spanned by

$$\{U_t x; t \in R, x \in X\}.$$

(iii) *If  $\{T_t\}_{t \in R^+}$  is strongly continuous, then  $\{U_t\}_{t \in R}$  is also strongly continuous.*

*Proof.* Taking  $G = R$ , the additive group of real numbers defining  $\phi$  by  $\phi_t = T_{|t|}$ , and  $K$  by  $K(t) = 1$ , for any  $t \in R$ , we are in assumptions of the previous theorem.

REMARK 1. An invertible isometry is a  $\mathcal{E}^m(\Gamma)$ -unitary operator with  $m > 1$ , ([2], Proposition 5.1.4). Hence Corollary 1 can be understood as a Banach space analogue of Sz.-Nagy's theorem ([9]) about the dilation of a semigroup of contractions into a group of unitary operators.

COROLLARY 2. (See [9], Theorem IV). *Let  $T \in \mathcal{B}(X)$  be a contraction. Then there exists a Banach space  $\tilde{X}$  containing  $X$ , a norm one projection  $P$  of  $\tilde{X}$  onto  $X$  and an invertible isometry  $U$  on  $\tilde{X}$  such that:*

- (i)  $PU^n x = T^{|n|} x$ , for all  $x \in X$ ,  $n \in \mathbb{Z}$ .
- (ii)  $\tilde{X}$  is the closed vector space spanned by

$$\{U^n x; n \in \mathbb{Z}, x \in X\}.$$

*Proof.* Obviously, for this case one takes  $G = \mathbb{Z}$  the additive group of integer numbers,  $\phi$  defined by  $\phi_n = T^{|n|}$  and  $K$  by  $K(n) = 1$ , for all  $n \in \mathbb{Z}$ .

COROLLARY 3. *Let  $\{T_1, T_2, \dots, T_p\} \subset \mathcal{B}(X)$  be a finite system of not necessarily commuting contractions. Then there exists a Banach space  $\tilde{X}$  containing  $X$ , a norm one projection  $P$  of  $\tilde{X}$  onto  $X$  and a finite system of commutative invertible isometries  $\{U_1, U_2, \dots, U_p\}$  on  $\tilde{X}$  such that:*

$$(i) \quad PU_1^{n_1} U_2^{n_2} \dots U_p^{n_p} x = T_1^{|n_1|} T_2^{|n_2|} \dots T_p^{|n_p|} x,$$

for any

$$n_1, n_2, \dots, n_p \in \mathbb{Z}, x \in X.$$

- (ii)  $\tilde{X}$  is the closed vector space spanned by

$$\{U_1^{n_1} U_2^{n_2} \dots U_p^{n_p} x; n_1, n_2, \dots, n_p \in \mathbb{Z}, x \in X\}.$$

*Proof.* We take  $G = \mathbb{Z}_1 \times \mathbb{Z}_2 \times \dots \times \mathbb{Z}_p$  with  $\mathbb{Z}_i = \mathbb{Z}$  for  $i = 1, 2, \dots, p$ ; define  $\phi$  by  $\phi(n_1, n_2, \dots, n_p) = T_1^{|n_1|} T_2^{|n_2|} \dots T_p^{|n_p|}$  and  $K$  by  $K(n_1, n_2, \dots, n_p) = 1$  for any  $n_1, n_2, \dots, n_p \in \mathbb{Z}$ , then apply the above theorem.

REMARK 2. Corollary 3 is a Banach space analogue of Ando's theorem ([1]). We remark that it is not necessarily to assume any

property of commutativity also we can take a number of more than two contractions, (in a Hilbert space this is not true, see [5]).

REMARK 3. The above theorem also asserts that for any sequence  $\{T_n\}_{n \in Z} \subset \mathcal{B}(X)$  of contractions with  $T_0 = 1$ , there exists a Banach space  $\tilde{X} \supset X$ , a norm one projection  $P$  of  $\tilde{X}$  onto  $X$  and a invertible isometry  $U$  on  $\tilde{X}$  such that  $T_n = PU^n|_X$  for any  $n \in Z$ . Also  $\tilde{X}$  is the closed vector space spanned by  $\{U^n x; n \in Z, x \in X\}$ . (This fact is true in a Hilbert space if and only if  $T_n$  is a positive definite sequence.)

COROLLARY 4. Let  $\{T_t\}_{t \in R^+} \subset \mathcal{B}(X)$  be a semigroup of operators such that  $\|T_t\| \leq Me^{at}$  (resp.  $\|T_t\| \leq t^\alpha + 1$ , with  $0 \leq \alpha \leq 1$ ) for all  $t \in R^+$ , where  $a$  and  $M$  are real positive constants. Then there exists a Banach space  $\tilde{X} \supset X$ , a norm one projection  $P$  of  $\tilde{X}$  onto  $X$  and a group of invertible (resp.  $\mathcal{E}^m(\Gamma)$ -unitary with  $m > \alpha + 1$ ) operators on  $\tilde{X}$ ,  $\{U_t\}_{t \in R}$  such that:

(0)  $M^{-1}e^{-a|t|} \leq \|U_t\| \leq Me^{a|t|}$  for all  $t \in R$ , if  $M > 1$ , or  $e^{-a|t|} \leq \|U_t\| \leq e^{a|t|}$  for all  $t \in R$ , if  $M \leq 1$ , (resp.  $(|t|^\alpha + 1)^{-1} \leq \|U_t\| \leq |t|^\alpha + 1$  for all  $t \in R$ ).

- (i)  $PU_t x = T_{|t|} x$  for all  $t \in R, x \in X$ ,
- (ii)  $\tilde{X}$  is the closed vector space spanned by  $\{U_t x; t \in R, x \in X\}$ .

Proof. Taking  $G = R$  the additive group of real numbers, defining  $\phi$  by  $\phi_t = T_{|t|}$  for all  $t \in R$  and  $K$  thus: if  $M > 1, K(t) = Me^{a|t|}$  for  $t \neq 0$ , and  $K(0) = 1$ ; or if  $M \leq 1, K(t) = e^{a|t|}$  for  $t \neq 0$  and  $K(0) = 1$ , (resp.  $K(t) = |t|^\alpha + 1$  for any  $t \in R$ ), we have the hypothesis of the theorem.

Moreover, for the second case we obtain

$$\|U_{nt}\| = \|(U_t)^n\| \leq |n|^\alpha (|t|^\alpha + 1)$$

for all  $|n| > 1, t \in R$ . Then applying Proposition 5.1.4 from [2], it follows that  $U_t$  is a  $\mathcal{E}^m(\Gamma)$ -unitary operator with  $m > \alpha + 1$ , for each  $t \in R$ .

COROLLARY 5. Let  $T \in \mathcal{B}(X)$ , satisfying  $\|T^n\| \leq n^\alpha + 1$  for all  $n \in N$ , with  $0 \leq \alpha \leq 1$ . Then there exists a Banach space  $\tilde{X} \supset X$ , a norm one projection  $P$  of  $\tilde{X}$  onto  $X$  and a  $\mathcal{E}^m(\Gamma)$ -unitary operator, with  $m > \alpha + 1, U$  on  $\tilde{X}$  such that:

- (0)  $(|n|^\alpha + 1)^{-1} \leq \|U^n\| \leq |n|^\alpha + 1$  for all  $n \in Z$ .
- (i)  $PU^n x = T^{|n|} x$  for all  $n \in Z, x \in X$ .
- (ii)  $\tilde{X}$  is the closed vector space spanned by

$$\{U^n x; n \in Z, x \in X\}.$$

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