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ISOMETRIC DILATIONS OF CONTRACTIONS ON BANACH SPACES

ELENA STROESCU

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This paper is concerned with the dilation, in the case of a Banach space, of operator-valued functions on a group into representations. Banach-space analogues of Sz.-Nagy's theorem and Ando's theorem are obtained.

Throughout this note Z (resp. R , resp. R^+ , resp. N , resp. C) is the set of all integer (resp. real, resp. nonnegative real, resp. nonnegative integer, resp. complex) numbers. Also G is a group, $e \in G$ its neutral element: $K: G \rightarrow R^+$ a submultiplicative function (i.e., $K(gh) \leq K(g)K(h)$ for all $g, h \in G$) with $K(e) = 1$; X a Banach space; $\mathcal{B}(X)$ the Banach algebra of all linear bounded operators on X and $I \in \mathcal{B}(X)$ the identity.

$\mathcal{E}^m(R)$ ($m \in N, m = \infty$) being the algebra of all m -times differentiable functions on R with the usual topology and $\Gamma = \{z \in C; |z| = 1\}$, $\mathcal{E}^m(\Gamma)$ is the algebra of all functions $f: \Gamma \rightarrow C$ such that $t \rightarrow f(e^{it})$ belongs to $\mathcal{E}^m(R)$, endowed with the topology induced by $\mathcal{E}^m(R)$. An operator $T \in \mathcal{B}(X)$ is called $\mathcal{E}^m(\Gamma)$ -unitary if it is $\mathcal{E}^m(\Gamma)$ -scalar ([2], [4]).

THEOREM. (See also [7] Theorem 1). *Let $\phi: G \rightarrow \mathcal{B}(X)$ be a function with the property $\|\phi_g\| \leq K(g)$ for all $g \in G$ and $\phi_e = I$.*

Then there exists a Banach space \tilde{X} containing X (by an isometric isomorphism), a norm one projection P of \tilde{X} onto X and a representation $\tilde{\phi}$ of G as a group of invertible operators on \tilde{X} such that

(0) $1/K(\gamma^{-1}) \leq \|\tilde{\phi}_\gamma\| \leq K(\gamma)$ for all $\gamma \in G$ and $\tilde{\phi}_e = \tilde{I}$.

(i) $P\tilde{\phi}_{\gamma|X} = \phi_\gamma$ for any $\gamma \in G$.

(ii) \tilde{X} is the closed vector space spanned by $\{\tilde{\phi}_\gamma x; \gamma \in G, x \in X\}$.

(iii) If ϕ takes its values from the set of contractions on X , then G is represented by $\tilde{\phi}$ as a group of invertible isometries on \tilde{X} . Moreover, if G is a topological group and for every $x \in X$, the function $g \rightarrow \phi_g x$ is left uniformly continuous, then the representation $\tilde{\phi}$ is strongly continuous.

Proof. Let Y be the vector space of all X -valued functions on G , $y(\cdot)$ with the property

$$\|y(g)\| \leq MK(g) \text{ for all } g \in G,$$

where M is a positive real constant and K the submultiplicative function from the hypothesis. (In what follows we shall denote elements of Y also by $(y_g)_{g \in G}$.) One sees easily that Y endowed with the norm

$$\|y(\cdot)\| = \sup_g \|y(g)\|K(g)^{-1}, \text{ is a Banach space.}$$

Let $X^{(G)} = \bigoplus_{g \in G} X^g$ be the direct sum with $X^g = X$ for all $g \in G$. Define a map $\Theta: X^{(G)} \rightarrow X$ by $(\Theta y)_g = \sum_h \phi_{gh} y_h$ for all $g \in G$ and $y \in X^{(G)}$. Then for every $y \in X^{(G)}$ one has $\Theta y \in Y$ and the set $\hat{X} = \{\Theta y; y \in X^{(G)}\}$ is a subspace of Y . Consider the closure of \hat{X} in Y and denote it by \tilde{X} .

Now let X_0 be a subspace of \hat{X} of elements

$$\begin{aligned} y(\cdot) &= (\phi_g x)_{g \in G} = (\sum_h \phi_{gh} \delta_{eh} x)_{g \in G} \text{ when } x \text{ runs over } X \\ (\delta_{gh} &= 0 \text{ for } g \neq h \text{ and } \delta_{gh} = 1 \text{ for } g = h). \text{ Define a map} \\ \varphi: X_0 &\rightarrow X \text{ by } \varphi(y(\cdot)) = y(e) \text{ for all } y(\cdot) \in X_0. \end{aligned}$$

Then one has

$$\|\varphi(y(\cdot))\| = \|y(e)\| \leq \sup_g \|y(g)\|K(g)^{-1} = \|y(\cdot)\|$$

and

$$\|y(\cdot)\| = \sup_g \|\phi_g x\|K(g)^{-1} \leq \|x\| = \|y(e)\|.$$

Hence φ is an isometric isomorphism of X_0 onto X .

Let $Q: \hat{X} \rightarrow X$ be a map defined by

$$Qy(\cdot) = y(e) \text{ for all } y(\cdot) \in \hat{X}.$$

Obviously, Q is linear surjective and satisfies $\|Qy(\cdot)\| \leq \|y(\cdot)\|$ for all $y(\cdot) \in \hat{X}$. Its extension by continuity to a linear map of \tilde{X} onto X will be denoted by the same symbol. Then $\varphi^{-1}Q$ is a norm one projection of \tilde{X} onto X .

For every $\gamma \in G$, define a map $\hat{\phi}_\gamma: \hat{X} \rightarrow \hat{X}$ by

$$\hat{\phi}_\gamma \Theta y = ((\Theta y)_{g\gamma})_{g \in G} = (\sum_h \phi_{g\gamma h} y_h)_{g \in G} = (\sum_d \phi_{gd} z_d)_{g \in G} = \Theta z \in \hat{X}$$

when y runs over $X^{(G)}$. (It is made the notation $d = \gamma h$, $z_d = y_h$ for all $h \in G$; hence z with these components belongs to $X^{(G)}$.) One sees easily that $\hat{\phi}_\gamma$ is well defined and linear. Moreover, one has

$$\begin{aligned} \|\hat{\phi}_\gamma \Theta y\| &= \sup_g \|\sum_h \phi_{g\gamma h} y_h\|K(g)^{-1} \\ &= \sup_g \|\sum_h \phi_{g\gamma h} y_h\|K(g\gamma)^{-1}K(g)^{-1}K(g\gamma) \\ &\leq K(\gamma) \sup_{g\gamma} \|\sum_h \phi_{g\gamma h} y_h\|K(g\gamma)^{-1} = K(\gamma)\|\Theta y\|. \end{aligned}$$

That is

$$(1) \quad \|\hat{\phi}_\gamma \Theta y\| \leq K(\gamma) \|\Theta y\| \quad \text{for all } y \in X^{(G)}.$$

Then $\hat{\phi}_\gamma$ can be extended by continuity to an element of $\mathcal{B}(\tilde{X})$ which will be denoted by $\tilde{\phi}_\gamma$. One sees easily that $\tilde{\phi}_{\alpha\beta} = \tilde{\phi}_\alpha \tilde{\phi}_\beta$ for all $\alpha, \beta \in G$ and $\tilde{\phi}_e = \tilde{I}$. Moreover,

$$(2) \quad \|\Theta y\| \leq \|\hat{\phi}_{\gamma^{-1}} \hat{\phi}_\gamma \Theta y\| \leq K(\gamma^{-1}) \|\hat{\phi}_\gamma \Theta y\| \quad \text{for all } y \in X^{(G)}.$$

Also $\hat{\phi}_\gamma: \hat{X} \rightarrow \hat{X}$ is surjective since one has

$$\Theta y = \hat{\phi}_\gamma((\Theta y)_{g\gamma^{-1}})_{g \in G} \quad \text{for all } y \in X^{(G)} \quad \text{and } \gamma \in G.$$

Thus the property (0) is proved. To show (i) we see that

$$((\varphi^{-1}Q)\tilde{\phi}_\gamma)\varphi^{-1}(x) = \varphi^{-1}(\phi_\gamma x) \quad \text{for all } x \in X \quad \text{and } \gamma \in G.$$

Identifying X_0 and X via φ and writing P instead of $\varphi^{-1}Q$, this equality reads more naturally as $P\tilde{\phi}_{\gamma|X} = \phi_\gamma$. The property (ii) is immediate noting that every $\Theta y \in \hat{X}$ can be written $\Theta y = \sum_h \tilde{\phi}_h \varphi^{-1}(y_h)$. The first assertion of (iii) is immediate because taking $K(g) = 1$ for all $g \in G$, the above inequalities (1) and (2) become

$$(3) \quad \|\hat{\phi}_\gamma \Theta y\| = \|\Theta y\| \quad \text{for all } y \in X^{(G)} \quad \text{and } \gamma \in G.$$

To prove the second assertion of (iii) we assume still that G is a topological group and $g \rightarrow \phi_g x$ is left uniformly continuous for each $x \in X$. Taking into account of (ii) it is enough to show that for any fixed $\gamma \in G$ and $y(\cdot) \in X_0$, the map $\alpha \rightarrow \tilde{\phi}_\alpha(\tilde{\phi}_\gamma y)(\cdot) = (\tilde{\phi}_{\alpha\gamma} y)(\cdot)$ is continuous. As this map is the composition of $\alpha \rightarrow \alpha\gamma$ and $\alpha\gamma \rightarrow (\tilde{\phi}_{\alpha\gamma} y)(\cdot)$, we need only show that for each $y(\cdot) \in X_0$, the map $\alpha \rightarrow (\tilde{\phi}_\alpha y)(\cdot)$ is continuous. For this it is sufficient to show the continuity at $\alpha = e$. But this fact is immediate from the left uniform continuity of $g \rightarrow \phi_g x$ for every $x \in X$, because $\|(\tilde{\phi}_\alpha y)(\cdot) - y(\cdot)\| = \sup_g \|\phi_{g\alpha} y(e) - \phi_g y(e)\|$.

COROLLARY 1. *Let $\{T_t\}_{t \in \mathbb{R}^+} \subset \mathcal{B}(X)$ be a semigroup of contractions. Then there exists a Banach space \tilde{X} containing X , a norm one projection P of \tilde{X} onto X and a group $\{U_t\}_{t \in \mathbb{R}}$ of invertible isometries on \tilde{X} such that:*

- (i) $PU_t x = T_{|t|} x$, for all $x \in X$, $t \in \mathbb{R}$.
- (ii) \tilde{X} is the closed vector space spanned by

$$\{U_t x; t \in \mathbb{R}, x \in X\}.$$

(iii) *If $\{T_t\}_{t \in \mathbb{R}^+}$ is strongly continuous, then $\{U_t\}_{t \in \mathbb{R}}$ is also strongly continuous.*

Proof. Taking $G = R$, the additive group of real numbers defining ϕ by $\phi_t = T_{|t|}$, and K by $K(t) = 1$, for any $t \in R$, we are in assumptions of the previous theorem.

REMARK 1. An invertible isometry is a $\mathcal{C}^m(I)$ -unitary operator with $m > 1$, ([2], Proposition 5.1.4). Hence Corollary 1 can be understood as a Banach space analogue of Sz.-Nagy’s theorem ([9]) about of the dilation of a semigroup of contractions into a group of unitary operators.

COROLLARY 2. (See [9], Theorem IV). *Let $T \in \mathcal{B}(X)$ be a contraction. Then there exists a Banach space \tilde{X} containing X , a norm one projection P of \tilde{X} onto X and an invertible isometry U on \tilde{X} such that:*

- (i) $PU^n x = T^{|n|}x$, for all $x \in X$, $n \in Z$.
- (ii) \tilde{X} is the closed vector space spanned by

$$\{U^n x; n \in Z, x \in X\} .$$

Proof. Obviously, for this case one takes $G = Z$ the additive group of integer numbers, ϕ defined by $\phi_n = T^{|n|}$ and K by $K(n) = 1$, for all $n \in Z$.

COROLLARY 3. *Let $\{T_1, T_2, \dots, T_p\} \subset \mathcal{B}(X)$ be a finite system of not necessarily commuting contractions. Then there exists a Banach space \tilde{X} containing X , a norm one projection P of \tilde{X} onto X and a finite system of commutative invertible isometries $\{U_1, U_2, \dots, U_p\}$ on \tilde{X} such that:*

- (i) $PU_1^{n_1}U_2^{n_2} \dots U_p^{n_p}x = T_1^{|n_1|}T_2^{|n_2|} \dots T_p^{n_p}x$,

for any

$$n_1, n_2, \dots, n_p \in Z, x \in X .$$

- (ii) \tilde{X} is the closed vector space spanned by

$$\{U_1^{n_1}U_2^{n_2} \dots U_p^{n_p}x; n_1, n_2, \dots, n_p \in Z, x \in X\} .$$

Proof. We take $G = Z_1 \times Z_2 \times \dots \times Z_p$ with $Z_i = Z$ for $i = 1, 2, \dots, p$; define ϕ by $\phi(n_1, n_2, \dots, n_p) = T_1^{|n_1|}T_2^{|n_2|} \dots T_p^{|n_p|}$ and K by $K(n_1, n_2, \dots, n_p) = 1$ for any $n_1, n_2, \dots, n_p \in Z$, then apply the above theorem.

REMARK 2. Corollary 3 is a Banach space analogue of Ando’s theorem ([1]). We remark that it is not necessarily to assume any

property of commutativity also we can take a number of more than two contractions, (in a Hilbert space this is not true, see [5]).

REMARK 3. The above theorem also asserts that for any sequence $\{T_n\}_{n \in Z} \subset \mathcal{B}(X)$ of contractions with $T_0 = 1$, there exists a Banach space $\tilde{X} \supset X$, a norm one projection P of \tilde{X} onto X and a invertible isometry U on \tilde{X} such that $T_n = PU^n|_X$ for any $n \in Z$. Also \tilde{X} is the closed vector space spanned by $\{U^n x; n \in Z, x \in X\}$. (This fact is true in a Hilbert space if and only if T_n is a positive definite sequence.)

COROLLARY 4. Let $\{T_t\}_{t \in R^+} \subset \mathcal{B}(X)$ be a semigroup of operators such that $\|T_t\| \leq Me^{at}$ (resp. $\|T_t\| \leq t^\alpha + 1$, with $0 \leq \alpha \leq 1$) for all $t \in R^+$, where a and M are real positive constants. Then there exists a Banach space $\tilde{X} \supset X$, a norm one projection P of \tilde{X} onto X and a group of invertible (resp. $\mathcal{C}^m(\Gamma)$ -unitary with $m > \alpha + 1$) operators on \tilde{X} , $\{U_t\}_{t \in R}$ such that:

(0) $M^{-1}e^{-a|t|} \leq \|U_t\| \leq Me^{a|t|}$ for all $t \in R$, if $M > 1$, or $e^{-a|t|} \leq \|U_t\| \leq e^{a|t|}$ for all $t \in R$, if $M \leq 1$, (resp. $(|t|^\alpha + 1)^{-1} \leq \|U_t\| \leq |t|^\alpha + 1$ for all $t \in R$).

(i) $PU_t x = T_{|t|} x$ for all $t \in R, x \in X$,

(ii) \tilde{X} is the closed vector space spanned by $\{U_t x; t \in R, x \in X\}$.

Proof. Taking $G = R$ the additive group of real numbers, defining ϕ by $\phi_t = T_{|t|}$ for all $t \in R$ and K thus: if $M > 1, K(t) = Me^{a|t|}$ for $t \neq 0$, and $K(0) = 1$; or if $M \leq 1, K(t) = e^{a|t|}$ for $t \neq 0$ and $K(0) = 1$, (resp. $K(t) = |t|^\alpha + 1$ for any $t \in R$), we have the hypothesis of the theorem.

Moreover, for the second case we obtain

$$\|U_{nt}\| = \|(U_t)^n\| \leq |n|^\alpha(|t|^\alpha + 1)$$

for all $|n| > 1, t \in R$. Then applying Proposition 5.1.4 from [2], it follows that U_t is a $\mathcal{C}^m(\Gamma)$ -unitary operator with $m > \alpha + 1$, for each $t \in R$.

COROLLARY 5. Let $T \in \mathcal{B}(X)$, satisfying $\|T^n\| \leq n^\alpha + 1$ for all $n \in N$, with $0 \leq \alpha \leq 1$. Then there exists a Banach space $\tilde{X} \supset X$, a norm one projection P of \tilde{X} onto X and a $\mathcal{C}^m(\Gamma)$ -unitary operator, with $m > \alpha + 1, U$ on \tilde{X} such that:

(0) $(|n|^\alpha + 1)^{-1} \leq \|U^n\| \leq |n|^\alpha + 1$ for all $n \in Z$.

(i) $PU^n x = T^{|n|} x$ for all $n \in Z, x \in X$.

(ii) \tilde{X} is the closed vector space spanned by

$$\{U^n x; n \in Z, x \in X\} .$$

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K. Adachi, Masuo Suzuki and M. Yoshida, <i>Continuation of holomorphic mappings, with values in a complex Lie group</i>	1
Michael Aschbacher, <i>A characterization of the unitary and symplectic groups over finite fields of characteristic at least 5</i>	5
Larry Eugene Bobisud and James Calvert, <i>Energy bounds and virial theorems for abstract wave equations</i>	27
Christer Borell, <i>A note on an inequality for rearrangements</i>	39
Peter Southcott Bullen and S. N. Mukhopadhyay, <i>Peano derivatives and general integrals</i>	43
Wendell Dan Curtis, Yu-Lee Lee and Forrest Miller, <i>A class of infinite dimensional subgroups of $\text{Diff}^r(X)$ which are Banach Lie groups</i>	59
Paul C. Eklof, <i>The structure of ultraproducts of abelian groups</i>	67
William Alan Feldman, <i>Axioms of countability and the algebra $C(X)$</i>	81
Jack Tilden Goodykoontz, Jr., <i>Aposyndetic properties of hyperspaces</i>	91
George Grätzer and J. Płonka, <i>On the number of polynomials of an idempotent algebra. II</i>	99
Alan Trinler Huckleberry, <i>The weak envelope of holomorphy for algebras of holomorphic functions</i>	115
John Joseph Hutchinson and Julius Martin Zelmanowitz, <i>Subdirect sum decompositions of endomorphism rings</i>	129
Gary Douglas Jones, <i>An asymptotic property of solutions of $y''' + py' + qy = 0$</i>	135
Howard E. Lacey, <i>On the classification of Lindenstrauss spaces</i>	139
Charles Dwight Lahr, <i>Approximate identities for convolution measure algebras</i>	147
George William Luna, <i>Subdifferentials of convex functions on Banach spaces</i>	161
Nelson Groh Markley, <i>Locally circular minimal sets</i>	177
Robert Wilmer Miller, <i>Endomorphism rings of finitely generated projective modules</i>	199
Donald Steven Passman, <i>On the semisimplicity of group rings of linear groups</i>	221
Bennie Jake Pearson, <i>Dendritic compactifications of certain dendritic spaces</i>	229
Ryōtarō Satō, <i>Abel-ergodic theorems for subsequences</i>	233
Henry S. Sharp, Jr., <i>Locally complete graphs</i>	243
Harris Samuel Shultz, <i>A very weak topology for the Mikusinski field of operators</i>	251
Elena Stroescu, <i>Isometric dilations of contractions on Banach spaces</i>	257
Charles W. Trigg, <i>Versum sequences in the binary system</i>	263
William L. Voxman, <i>On the countable union of cellular decompositions of n-manifolds</i>	277
Robert Francis Wheeler, <i>The strict topology, separable measures, and paracompactness</i>	287