ON THE COUNTABLE UNION OF CELLULAR DECOMPOSITIONS OF $n$-MANIFOLDS

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Suppose that $G_1, G_2 \cdots$ are cellular upper semicontinuous decompositions of an $n$-manifold with boundary $M(n \neq 4)$ such that for $i = 1, 2, \cdots, M/G_i$ is homeomorphic to $M$. Let $G$ be the decomposition of $M$ obtained from the decomposition of $G_i$ in the following manner. A set $g$ belongs to $G$ if and only if $g$ is a nondegenerate element of some $G_i$ or $g$ is a point in $M - (\bigcup_{i=1}^{\infty} H_{G_i})$. It will be shown that if the various decompositions fit together in a “continuous” manner and if $G$ is an upper semicontinuous decomposition of $M$, then $M/G$ is homeomorphic to $M$.

Our principal result thus extends previous results obtained by the author ([6], [7]) and Lamoreaux [4], by removing the 0-dimensionality restriction in [6] or, alternatively, by eliminating the finiteness condition in [7]. Furthermore, with the aid of recent work of Siebenmann [5], generalizations to $n$-manifolds ($n \neq 4$) may be made. As was observed in [7], some conditions must be imposed on the manner in which the decompositions are pieced together. The example described by Bing in [2] demonstrates that the continuity condition to be described below is a necessary one.

Notation and terminology. Suppose $G$ is an upper semicontinuous decomposition of a topological space, $X$. Then $X/G$ will denote the associated decomposition space, $P$ will denote the natural projection map from $X$ onto $X/G$, and $H_g$ will denote the collection of nondegenerate elements of $G$. If $U$ is an open subset of $X$, then $U$ is said to be saturated (with respect to $G$) in case $U = P^{-1}[P[U]]$. If $\mathcal{U}$ is a covering of a subset of $X$, then $P[\mathcal{U}] = \{P[U] : U \in \mathcal{U}\}$.

The statement that $M$ is an $n$-manifold with boundary means that $M$ is a separable metric space such that each point of $M$ has a neighborhood which is an $n$-cell. If $A$ is a subset of $M$, then $A$ is cellular in $M$ if there exists a sequence $C_1, C_2, \cdots$ of $n$-cells in $M$ such that (1) for each positive integer $i$, $C_{i+1} \subset \text{Interior } C_i$, and (2) $\bigcap_{i=1}^{\infty} C_i = A$. If $M$ is an $n$-manifold with boundary, the statement that $G$ is cellular decomposition of $M$ means that $G$ is an upper semicontinuous decomposition of $M$ and each nondegenerate element of $G$ is a cellular subset of $M$.

If $M$ is a metric space, $A$ a subset of $M$, then $S_\varepsilon(A)$ denotes the $\varepsilon$-neighborhood of $A$ and $\text{Cl} A$ denotes the closure of $A$ in $M$. If $K$
is a collection of subsets of $M$, then $K^* = \bigcup \{ k : k \in K \}$. The word map will always be used to indicate a continuous function. If $\mathcal{U}$ is a collection of subsets of $M$ and $A \subset M$, then

$$\text{St} (A, \mathcal{U}) = \bigcup \{ U \in \mathcal{U} : A \cap U \neq \emptyset \}.$$  

The main result. The principal theorem will be proved by means of repeated applications of the Lemma which appears below. We say that a cellular decomposition $G$ of a manifold $M$ satisfies condition $S$ if for each saturated open cover $\mathcal{U}$ of $H^*_0$, there exists a closed map $h$ from $M$ onto $M$ such that (1) $G = \{ h^{-1}(x) : x \in M \}$, (2) if $x \in M - \mathcal{U}^*$, then $h(x) = x$, and (3) for each $g \in G$ and $g \subset \mathcal{U}^*$, there exists a $U \in \mathcal{U}$ such that $g \cup h(g) \subset U$.

**Lemma 1.** Suppose $G$ is a cellular decomposition of an $n$-manifold with boundary $M(n \neq 4)$. Then $M/G$ is homeomorphic to $M$ if and only if $G$ satisfies condition $S$.

**Proof.** Clearly if $G$ satisfies condition $S$, then $M/G$ is homeomorphic to $M$. Suppose now that $M/G$ is homeomorphic to $M$ and that $\mathcal{U}$ is a saturated open cover of $H^*_0$. Without loss of generality we may assume that $\mathcal{U}$ is locally finite. Suppose $x \in \mathcal{U}^*$ and $U_1, \cdots, U_n$ are those sets in $\mathcal{U}$ which contain $x$. Set \[ \varepsilon_x = \max \{ d(P(x), M/G - P[U]), \cdots, d(P(x), M/G - P[U_n]) \} \] and define $f_1(x) = \varepsilon_x/2$. Then $f_1$ is a lower semicontinuous function from $\mathcal{U}^*$ into $(0, \infty)$, and, hence, there exists a continuous map $f_2$ from $\mathcal{U}^*$ into $(0, \infty)$ such that $0 < f_2 < f_1$. For $x \in \mathcal{U}^*$, define $f_3(x)$ to be $\min \{ d(P(x), M/G - P[\mathcal{U}^*]) \}$, and finally define $f(x)$ to be \[ f(x) = \min \{ f_1(x), f_2(x), f_3(x) \}. \] Siebenman's projection approximation theorem [5] may be applied to find a homeomorphism $k$ from $\mathcal{U}^*$ onto $P[\mathcal{U}^*]$ such that $d(P(x), k(x)) < f(x)$ for each $x \in \mathcal{U}^*$. Then $h = k^{-1}P$ is the desired map. To see this we need only check that for $g \in G$ and $g \subset \mathcal{U}^*$, there is a $U \in \mathcal{U}$ such that $h(g) \cup g \subset U$. Let $y = k^{-1}P(g)$. By our construction there exists a $U \in \mathcal{U}$ such that both $P(y)$ and $k(y)$ belong to $P[U]$. But $k(y) = P(g)$; therefore, $y$ and $g$ belong to $U$, which completes the proof.

Suppose $M$ is a metric space and $K$ is a collection of mutually disjoint subsets of $M$. If $g \in K$, then $K$ is said to be continuous at $g$ in case for each positive number $\varepsilon$, there exists an open subset $V$ of $M$ containing $g$ such that if $g' \in K$ and $g' \cap V \neq \emptyset$, then $g \subset S_\varepsilon(g')$ and $g' \subset S_\varepsilon(g)$.

**Theorem 1.** Suppose $G_1, G_2, \cdots$ are cellular decompositions of an $n$-manifold with boundary $M(n \neq 4)$ such that
(1) If \( g \in H_{\alpha_i} \) and \( g \cap H_{\alpha_j}^* \neq \emptyset \), then \( g \in H_{\alpha_j}^* \).
(2) For each \( k = 1, 2, \ldots \), if \( g \in H_{\alpha_k} \), then \( \{H_{\alpha_i}: i \neq k\} \cup \{g\} \) is continuous at \( g \).
(3) For \( i = 1, 2, \ldots \), \( M/G_i \) is homeomorphic to \( M \).
(4) \( G = \{g: g \in \bigcup_{i=1}^n H_{\alpha_i} \text{ or } g \text{ is a point of } M - (\bigcup_{i=1}^n H_{\alpha_i})\} \) is an upper semicontinuous decomposition of \( M \).
Then \( M/G \) is homeomorphic to \( M \).

Proof. We show that \( G \) satisfies condition \( S \). Let \( \mathcal{W} \) be a saturated open cover of \( H_\alpha^* \). The required function \( h \) will be defined as a limit of a sequence of closed, onto maps which are obtained in the following steps.

Step 1. Let \( K_1 = \{p \in M: \text{there exists a sequence of nondegenerate elements, each from a different } H_{\alpha_i}, \text{ which converges to } p\} \). Note that \( K_1 \) is a closed subset of \( M \). We construct a saturated (with respect to \( G \)) open refinement of \( \mathcal{W} \) which covers \( H_\alpha^* \) and misses \( K_1 \). For each \( g \in H_{\alpha_i} \), let \( U_g \) be a saturated open set with compact closure such that

(1) If \( \varepsilon_g = \min \{\text{diam } g, 1/2 \text{diameter } (K_1), 1\} \), then \( U_g \subset S_{\varepsilon_g}(g) \).
(2) If \( g_i \in H_{\alpha_j}^* \) and \( g_j \in H_{\alpha_j}^* \) (\( i \neq j \)) and \( g_i \) and \( g_j \) are contained in \( U_g \), then \( 1/2 \text{diameter } g_i < \text{diameter } g_j < 3/2 \text{diameter } g_i \).
(3) \( U_g \) is contained in some \( W \in \mathcal{W} \) which contains \( g \).

Parts (1) and (2) are possible because of the continuity condition imposed on the decompositions. Define \( \mathcal{U}_1 = \{U_g: g \in H_{\alpha_i}\} \). Let \( \mathcal{U}_1 \) be a saturated open locally finite star refinement of \( \mathcal{U}_1 \) and \( \mathcal{U}_1^* = \{U \in \mathcal{U}_1: U \cap H_{\alpha_i} = \emptyset \} \). Observe that it follows from (1) that if \( p \in K_1 \), then \( p \in \mathcal{U}_1^* \). Furthermore, from (1) and (2) we have that if \( p \in K_1 \) and \( \{x_i\} \) is a sequence of points in \( \mathcal{U}_1^* \) which converge to \( p \), then the sequence \( \{\text{St}(x_i, \mathcal{U}_1)\} \) also converges to \( p \).

By Lemma 1, there exists a closed map \( h_1 \) from \( M \) onto \( M \) such that

(1) \( G_1 = \{h_1^{-1}(x): x \in M\} \).
(2) If \( x \in M - \mathcal{U}_1^* \), then \( h_1(x) = x \).
(3) If \( g \in G_1 \) and \( g \subset \mathcal{U}_1^* \), then there exists a set of \( U \in \mathcal{U}_1 \) such that \( g \cup h_1(g) \subset U \).

In addition, since \( \mathcal{U}_1 \) is saturated with respect to \( G \), part (3) holds for all \( g \in G \) which are contained in \( \mathcal{U}_1^* \).

Step 2. The decomposition \( G_2 = \{h_1(g): g \in G_2\} \) is clearly cellular and upper semicontinuous. Let \( P' \) be the projection map from \( M \) onto \( M/G_2 \) and \( P \) the projection map from \( M \) onto \( M/(G_1 \cup G_2) \). Then \( P'h_1P^{-1} \) is readily seen to be a homeomorphism from \( M/(G_1 \cup G_2) \) onto...
But it was shown in [7] that $M/(G_1 \cup G_2)$ is homeomorphic to $M$ (using Siebenman's generalization [5] of Armentrout's "projection approximation" theorem [1], the results of [7] may be extended to $n$-manifolds for $n \neq 4$).

Let $K_2 = \{ p \in M : \}$ there exists a sequence of nondegenerate elements, each from a different $H_{k_1(G_i)}$, which converges to $p$. We construct a saturated (with respect to $h_i[G]$) open refinement of $h_i[Z_1]$ which covers $H_{k_1(G)}$ and misses $K_i$. Suppose $g' = h_i(g)$ where $g \in H_{g_i} - H_{G_1}$. Choose $U_{g'}$ to be saturated (with respect to $h_i[G]$) open set such that

1. If $\varepsilon_{g'} = \min \{ \text{diam } g', 1/2 \text{d}(g', K_2), 1/2 \}$, then $U_{g'} \subset S_{g'}(g')$.
2. If $g_i \in H_{k_1(G_i)}$ and $g_j \in H_{k_1(G_j)} (i \neq j)$ and $g_i$ and $g_j$ are contained in $U_{g'}$, then $1/2 \text{diam } g_i < \text{diam } g_j < 3/2 \text{diam } g_i$.
3. $h_i^{-1}(U_{g'}) \subset S_{g_i}(g_i)$.
4. If $W = \bigcap \{ U : U \in h_i[Z_1] \text{ and } h_i(g) \subset U \}$, then $U_{g'} \subset W$.
5. If $V \in Z_1$ and $g \cup h_i(g) \subset V$, then $U_{g'} \subset V$.
6. $U_{g'} \cap \text{Cl} (h_i[H_{G_1}]) = \emptyset$.

Let $Z_2' = \{ U_{g'} : g' \in H_{k_1(G)} \}$ and let $Z_2$ be a saturated open locally finite star refinement of $Z_2'$ covering $H_{k_1(G)}$. Let

$$Z_2 = \{ U \in Z_2 : U \cap H_{k_1(G)} \neq \emptyset \}.$$ 

Note that $h_i^{-1}(Z_1') \subset S_{h_i(H_{G_1})}$ and $h_i^{-1}(Z_2') \subset S_{h_i(H_{G_1})}$.

By Lemma 1, there is a closed map $h_2$ from $M$ onto $M$ such that

1. $G_2' = \{ h_i^{-1}(x) : x \in M \}$.
2. If $x \in M - Z_2^*$, then $h_2(x) = x$.
3. For each $g' \in G_2'$ contained in $Z_2^*$, there exists a $U \in Z_2$ such that $h_2(g') \cup g' \subset U$.

Claim. For each $g \in G$ contained in $Z_1^*$, there exists a $W \in Z_1$ such that $g \cup h_2 h_i(g) \subset W$.

Proof of Claim. Suppose $g \in G$ and $g \subset Z_1^*$. Then there exists $U \in Z_1$ such that $h_i(g) \cup g \subset U$. If $g \in H_{G_i}$ or if $h_i(g)$ is not contained in $Z_2^*$, then $h_2 h_i(g) = h_i(g)$, and we are done. Suppose then that $g \in H_{G_1}$ and $h_i(g) \cap Z_2^* \neq \emptyset$. Since $Z_2'$ is a refinement of $h_i[Z_1]$ and $Z_2$ is a locally finite star refinement of $Z_2'$, we may find $U_2 \in Z_2$ and $U_{g'} \in Z_2'$, where $h_i(g) = g'$, such that $h_i(g) \cup U_2 \subset \text{St} (U_{g'}, Z_2) \subset U_{g'}$.

We first show that there exists a $V \in Z_1$ such that $U_{g'} \subset V$. Of course, $h_i(g) = g' \subset U_{g'}$. Let $V_1, V_2, \ldots, V_n$ be those members of $Z_1$ which contain $g$. Then by our construction of $Z_2'$,

$$U_{g'} \subset h_i(V_1) \cap \cdots \cap h_i(V_n).$$

Since $h_i(g) \subset U_{g'}$, it follows that $g \subset V_1 \cap \cdots \cap V_n$. But for at least
one \( i = 1, 2, \ldots, \) or \( n, \) \( h_i(g) \cap g \cup V_i. \) Therefore, by (5) in our construction of \( \mathcal{U}'_i, \) it must be the case that \( U_{i^*} \) is contained in \( V_i. \)

We need only observe now that if \( Z \in \mathcal{U}'_z \) and \( h_i(g) \subseteq Z, \) then \( Z \subseteq V_i. \) This is clear since \( Z \subseteq \text{St}(U_{i+1}, \mathcal{U}_i) \subseteq U_{i^*} \subseteq V_i. \) Hence, we have that \( \text{St}(h_i(g), \mathcal{U}_i) \) is contained in \( V_i \) and since

\[
h_i h_i(g) \subseteq \text{St}(h_i(g), \mathcal{U}_i),
\]

the proof of the claim is complete.

We continue inductively. Assume now that covers \( \mathcal{U}'_1, \cdots, \mathcal{U}'_i, \mathcal{U}_i, \cdots, \mathcal{U}_n, \mathcal{V}_1, \cdots, \mathcal{V}_n \) have been defined so that the conditions listed below are satisfied. We denote \( h_k h_{k-1} \cdots h_1 \) by \( h_k \) and \( h_0 = \text{id}. \) For \( i = 1, 2, \ldots, n, \) let \( K_i = \{ p \in M: \) there exists a sequence of nondegenerate elements converging to \( p \) where each element is a member of a different \( H_{h_k} \} \).

(1) \( \mathcal{U}'_i = \{ U_g: g' \in H_{h_k} \} \) is a collection of saturated (with respect to \( h_k \}) \) open sets which refines \( \mathcal{U}'_{i-1} \) and misses \( K_i. \) For each \( g', U_{g'} \) is chosen to contain \( g' \) such that

(a) \( \varepsilon_{g'} = \min \{ \text{diam } g', 1/2 \text{d}(g', K_i) \} \), then \( U_{g'} \subseteq S_{1/2}(g'). \)

(b) \( \text{If } g_j \in H_{h_k} \text{ and } g_k \in H_{h_k} \text{ (} j \neq k \), and } g_j \text{ and } g_k \text{ are contained in } U_{g'}, \text{ then } 1/2 \text{diam } g_j < \text{diam } g_k < 3/2 \text{diam } g_j. \)

(2) \( \mathcal{V}_i \) is a saturated open locally finite star refinement of \( \mathcal{U}'_i \) and \( \mathcal{V}_i^* = \{ U \in \mathcal{V}_i: U \cap H_{h_k}^* \neq \emptyset \}. \)

(3) For \( i = 1, 2, \ldots, n \) and \( 1 \leq j \leq i - 1, \)

\[
h_j^{-1} \cdots h_{i-1}^{-1} \subseteq S_{1/2}(h_j, H_{h_k}^*)
\]

and

\[
h_j^{-1} \cdots h_{i-1}^{-1} \subseteq S_{1/2}(h_j, H_{h_k}^*)
\]

(4) For \( i = 1, 2, \ldots, n, \) \( h_i \) is a closed map from \( M \) onto \( M \) such that if \( G_i = \{ \hat{h}_i(g): g \in G_i \} \) then

(1) \( G_i = \{ h_i(x): x \in M \}. \)

(2) \( \text{If } x \in M - \mathcal{V}_i^*, \text{ then } h_i(x) = x. \)

(3) \( \text{For each } g' \in G_i \) which is contained in \( \mathcal{V}_i^*, \text{ there exists } \)

\( \hat{h}_i(g') \cup g' \subseteq U. \)

(5) \( \text{For } i = 1, 2, \ldots, n \) and \( 0 \leq j \leq i - 1 \), if \( g \in G \) and \( \hat{h}_i(g) \)

is contained in \( \mathcal{V}_i^*, \text{ then there exists } U \in \mathcal{V}_i^* \) such that \( \hat{h}_i(g) \cup \hat{h}_i(g) \subseteq U. \)

(6) \( \mathcal{V}_i^* \cap \text{Cl}(h_{i-1}(H_{\hat{h}_i}^* \cup \cdots \cup H_{\hat{h}_i}^*)) = \emptyset. \)

Step \( n + 1. \) Let \( G_n = \{ \hat{h}_n(g): g \in G_n \}. \) A proof similar to that employed in Step 2 shows that \( M/G_n \) is homeomorphic to \( M. \) Let \( K_n = \{ p \in M: \) there exists a sequence of nondegenerate elements converging to \( p \) where each element is a member of a different \( H_{h_k} \}. \)
We construct a saturated (with respect to $\hat{h}_n[G]$) open refinement of $h_n[\mathcal{U}_n]$ which covers $H_{\hat{h}_n[G]}$ and misses $K_{n+1}$. Let $g' = \hat{h}_n(g)$ where $g \in H_g = (H_{\hat{h}_1} \cup \cdots \cup H_{\hat{h}_n})$. Choose $U_{g'}$ to be a saturated open set containing $g'$ such that

1. If $\varepsilon_{g'} = \min \{\text{diam } g', 1/2 \text{diam } (g', K_{n+1}), 1/n + 1\}$, then $U_{g'} \subset S_{g'}(g')$.
2. If $g_i \in H_{\hat{h}[G]}$ and $g_j \in H_{\hat{h}[G]} (i \neq j)$ and $g_i$ and $g_j$ are contained in $U_{g'}$, then $1/2 \text{diam } g_i < \text{diam } g_j < 3/2 \text{diam } g_i$.
3. For $i = 1, 2, \ldots, n$, $(h_{i}h_{i+1} \cdots h_{n})^{-1}(U_{g'}) \subset S_{i/n}(\hat{h}_{i-1}(g'))$.
4. For $i = 1, 2, \ldots, n$, if $U^i$ is the intersection of those sets in $\mathcal{U}_i$ which contain $\hat{h}_{i-1}(g)$, then

$$U_{g'} \subset \hat{h}_n(U^1) \cap \hat{h}_n h_{n-1} \cdots h_2(U^2) \cap \cdots \cap h_n(U^n).$$

5. For $0 \leq i < n$, if $\hat{h}_i(g) \cup \hat{h}_n(g) \subset U \in \mathcal{U}_n$, then $U_{g'} \subset U$.
6. $U_{g'} \cap \text{Cl } \hat{h}_n[H_{\hat{h}_1}^* \cup \cdots \cup H_{\hat{h}_n}^*] = \emptyset$.

Let $\mathcal{U}_{n+1} = \{g' \in H_{\hat{h}_n[G]}\}$, let $\mathcal{U}_{n+1}$ be a saturated open locally finite star refinement of $\mathcal{U}_{n+1}$ and let $\mathcal{V}_{n+1} = \{U \in \mathcal{U}_{n+1}: U \cap \hat{h}_n[H_{\hat{h}_n[G]}^*] = \emptyset\}$. By Lemma 1 there exists a closed map $h_{n+1}$ from $M$ onto $M$ such that

1. $G_{n+1}^* = \{h_{n+1}^{-1}(x): x \in M\}$.
2. If $x \in M - \mathcal{V}_{n+1}$, then $h_{n+1}(x) = x$.
3. For each $g \in G_{n+1}^*$ contained in $\mathcal{V}_{n+1}^*$, there exists $U \in \mathcal{U}_{n+1}$ such that $g \cup h_{n+1}(g) \subset U$.

Claim. Suppose $g' = \hat{h}_{n+1}(g)$ is contained in $\mathcal{U}_{n+1}^*$ ($g'$ is an element of $G$). Suppose $0 \leq i < n + 1$. Then there exists $U \in \mathcal{U}_{i+1}$ such that $g' \cup \hat{h}_i(g) \subset U$.

A proof patterned after the proof of the Claim in Step 2 may be used to establish this Claim.

Define $h = \text{Lim } \hat{h}_n$. To see that $h$ is well defined, we observe that for each $x \in M$, there exists an integer $N$ such that for $n > N$,

$$\hat{h}_n(x) = \hat{h}_N(x) = h(x).$$

This is clearly the case if $x \in H_{\hat{h}_n}^*$, since if $N$ is the first integer such that $x \in H_{\hat{h}_N}^*$ then $h_{n}(x)$ does not belong to the succeeding $\mathcal{U}_n^*$, and, hence, is left fixed. If $x \in \text{Cl } H_{\hat{h}_n}^*$ then choose $N$ such that

$$d(x, \text{Cl } H_{\hat{h}_n}^*) > \frac{1}{N}.$$
hand, if no such $U$ exists, then there is a sequence $\{g_n\}$ of nondegenerate elements from distinct decompositions $G_{i_n}$ which converges to $x$. But it was noted in Step 1 that in this case $x \in \mathcal{U}_i^*$ and thus $h(x) = x$.

We next show that $h$ is continuous. Suppose $\{x_i\}$ is a sequence of points in $M$ converging to a point $x$. If there exists an open set $U$ containing $x$ such that $U \cap H_{a_i}^* = \emptyset$ for all but at most a finite number of $i$, then it follows again from (3) of the induction Step $n + 1$ that $\{h(x_i)\}$ converges to $h(x)$. If no such $U$ exists, then there are two cases to consider.

**Case 1.** $x \in (\text{Cl } H_{a_i}^* - H_{a_i}^*)$. Suppose for each $i$, $x_i \in g_{n_i} \in G_{i_n}$. We may assume that the $x_i$ lie in $\mathcal{U}_i^*$ since if not $h(x_i) = x_i$. But as it was observed in Step 1, since the sequence $\{g_n\}$ converges to $x$, we have that the corresponding sequence $\{\text{St}(g_{n_i}, \mathcal{U}_i)\}$ also converges to $x$. It follows from the Claim in Step $n + 1$, that $h(x_i) \in \text{St}(g_{n_i}, \mathcal{U}_i)$, and, therefore, $\{h(x_i)\}$ converges to $h(x)$.

**Case 2.** $x \in H_{a_i}^*$. Let $n$ be the first integer such that $x \in g_n \in H_{a_n}$. But then $h_n(g_n)$ is a point and our construction in the inductive steps reduces this case to Case 1.

That $h$ is onto may be seen by the following argument. Suppose $p$ is a point in $M$. We assume that $p \in g' \subset G$ where $g' \subset \mathcal{U}_i^*$ (if not, $h(p) = p$). For each positive integer $i$, there exists a point $x_i$ in $\mathcal{U}_i^*$ such that $h_i(x_i) = p$. It follows from the Claim in Step $n + 1$ that for each $i$, $x_i \in \text{St}(g', \mathcal{U}_i)$. Since $\text{St}(g', \mathcal{U}_i)$ has compact closure (see Step 1), there exists an accumulation point $x$ of the sequence $\{x_i\}$. For simplicity of notation let us assume that $\{x_i\}$ converges to $x$. We show that $h(x) = p$.

Let $g \in G$ be the member of the decomposition which contains $x$. Choose $N$ large enough so that $h_n(g) = h(g)$ for each $n \geq N$. First we suppose that there exists a positive integer $K \geq N$ such that for $n \geq K$, $S_{iK}(g) \cap H_{a_n}^* = \emptyset$. Of course, the sequence $\{h_K(x_i)\}$ converges to $\hat{h}_K(x)$. But it follows from (3) of Step $n + 1$, that for $i$ sufficiently large, we will have $\hat{h}_k(x_i) = \hat{h}_i(x_i) = h(x_i)$. Thus $h(x) = p$, since $\hat{h}_i(x_i) = p$ for all $i$.

Now suppose that each open set containing $x$ intersects an infinite number of the $H_{a_i}^*$, and, hence, each open set containing $\hat{h}_n(x)$ will also intersect infinitely many of the sets $H_{a_i}^*$. Thus, $\hat{h}_n(x)$ belongs to $K_{n+1}$ (see Step $n + 1$). Since $\{\hat{h}_n(x_i)\}$ converges to $\hat{h}_n(x)$, it follows from conditions (1) and (3) of Step $n + 1$ that the sequence

$$\{\text{St}(\hat{h}_n(x_i), \mathcal{U}_n)\}$$

also converges to $\hat{h}_n(x)$.

But the Claim in this step ensures that for $j > N$, $\hat{h}_j(x_i) \cup \hat{h}_N(x_i)$ belongs to $\text{St}(\hat{h}_N(x_i), \mathcal{U}_N)$. In particular then for $i > N$,
\[ \mathring{h}_i(x_i) \cup \mathring{h}_N(x_i) \subseteq \text{St} (\mathring{h}_N(x_i), \mathcal{U}_N), \]
and since, \( \mathring{h}_i(x_i) = p \), it again follows that \( h(x) = p \). Thus \( h \) is an onto map.

It is easily seen from our construction of \( h \) that \( G = \{ h^{-1}(x) : x \in M \} \).

Finally, we must show that \( h \) is closed. It suffices to show that if \( K \) is a compact subset of \( M \), then \( h^{-1}(K) \) is also compact. Since \( h \) is onto, for each \( x \in K \), there exists a unique element \( g_x \in G \) such that \( h(g_x) = x \). If \( g_x \in \mathcal{U}^*_1 \), let \( U_{g_x} \) be a member of \( \mathcal{U}_1 \) which contains \( g_x \). If \( g_x \) is not contained in \( \mathcal{U}^*_1 \) let \( U_{g_x} \) be an open set containing \( g_x \) with compact closure. Note that it follows from Step 1 that if \( g_x \) is contained in \( \mathcal{U}^*_1 \), then \( \text{St} (U_{g_x}, \mathcal{U}_1) \) has compact closure. Since if \( g_x \in \mathcal{U}^*_1 \), then \( g_x \cup h(g_x) \subseteq \text{St} (U_{g_x}, \mathcal{U}_1) \), and if \( g_x \) is not contained in \( \mathcal{U}^*_1 \), then \( h(g_x) = g_x \), the collection \( \{ U_{g_x} : x \in K \} \) is an open cover of \( K \). Let \( U_{g_{\pi_1}}, \ldots, U_{g_{\pi_n}} \) be a finite subcover of \( K \), where the first \( i \) terms are members of \( \mathcal{U}_1 \). To finish the proof we need only observe that

\[ h^{-1}(K) \subseteq \text{St} (g_{\pi_1}, \mathcal{U}_1) \cup \cdots \cup \text{St} (g_{\pi_i}, \mathcal{U}_1) \cup U_{g_{\pi_{i+1}}} \cup \cdots \cup U_{g_{\pi_n}} \]

and that the right hand set has compact closure. Thus, the conditions of property \( S \) have been satisfied, and, hence, \( M/G \) is homeomorphic to \( M \).

A decomposition of a metric space is said to be nondegenerately continuous if for each \( g \in G, H_g \cup \{ g \} \) is continuous at \( g \).

**Corollary 1.** Suppose \( G \) is a cellular nondegenerately continuous upper semicontinuous decomposition of \( E^3 \). Suppose there exists a countable number of planes in \( E^3, Q_1, Q_2, \ldots \) such that for each \( g \in H_g, g \) is contained in at least one of these planes. Then \( E^3/G \) is homeomorphic to \( E^3 \).

**Proof.** For \( i = 1, 2, \ldots \), let \( G_i \) be the decomposition of \( E^3 \) such that \( H_{G_i} = \{ g \in H_G : g \subseteq Q_i \} \). Then \( E^3/G_i \) is homeomorphic to \( E^3 \) [3], and since it is readily verified that \( G_i, G_2, \ldots \) satisfy the conditions of Theorem 1, \( E^3/G \) is homeomorphic to \( E^3 \).

**References**


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