NONEXPANSIVE PROJECTIONS ON SUBSETS OF BANACH SPACES

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If C is a convex subset of a Banach space E, a projection is a retraction r of C onto a subset F which for each $x \in C$ maps each point of the ray $\{r(x) + t(x - r(x)): t \geq 0\} \cap C$ onto the same point $r(x)$. A retraction r is said to be orthogonal if for each x, $x - r(x)$ is normal to F in a sense related to that of R. C. James. This paper establishes three main results. First, a nonexpansive projection is necessarily an orthogonal retraction; if E is smooth, the converse is also true. Second, if E is smooth then there can exist at most one nonexpansive projection of C onto a given subset F. Third, if E is uniformly smooth and there exists a nonexpansive retraction of C onto F, then there exists a nonexpansive projection of C onto F. The proximity mapping is a nonexpansive projection in a Hilbert space, but not in a general Banach space.

We shall adopt the following conventions throughout this paper: E always denotes a real Banach space, E^* its dual space, C a nonempty closed convex subset of E, and F a nonempty closed subset of C. We do not assume that F is convex. A mapping $f: C \to C$ is said to be *nonexpansive* if $||f(x) - f(y)|| \leq ||x - y||$ for all $x, y \in C$. F is said to be a *nonexpansive retract* of C if there exists a retraction of C onto F which is a nonexpansive mapping.

Nonexpansive retracts are of interest because they generalize two results, one linear in reflexive Banach spaces, one nonlinear in Hilbert space.

First, if E is reflexive and $L: E \to E$ is linear with $||L|| \leq 1$, then the mean ergodic theorem implies that the means $L_n = n^{-1} \sum_{j=1}^{n} L^j$ converge pointwise to a projection P of E onto $\mathrm{Ker}\,(I - L)$, and $||P|| \leq 1$. The kernel of $I - L$ is the fixed-point set of L.

Second, if E is a Hilbert space and $T: C \to C$ is nonexpansive, then $F(T)$, the fixed-point set of T, is a closed convex subset of C. If $F(T)$ is nonempty, the proximity mapping $P: C \to F(T)$ defined by $Px =$ the point of $F(T)$ which is closest to x, is known to be nonexpansive. In fact, it satisfies the stronger inequality

$$(1) \qquad ||Px - Py||^2 \leq (x - y, Px - Py)$$

for all x, y in C (Browder [3], Prop. 2).

In both of these examples the fixed-point set of a nonexpansive mapping: $C \to C$ is shown to be a nonexpansive retract of C, and a nonexpansive retraction is constructed which has a pseudolinearity

property, namely, it projects C onto the fixed-point set along rays. In Bruck [4] it was shown that if E is a reflexive, strictly convex Banach space and $T: C \to C$ is nonexpansive, then $F(T)$ is a nonexpansive retract of C. In this paper we show that under certain additional hypotheses of smoothness, this retraction can be taken to be a projection (in the sense of the abstract). Such a projection is unique and maps C onto $F(T)$ along rays which are *normal* to $F(T)$.

Although the proximity map of C onto a closed convex subset F can be defined in any strictly convex, reflexive space, and when defined is a projection, it is not, in general, nonexpansive (de Figueiredo and Karlovitz [6]), nor does it project C onto F along normals. In fact, if F is bounded, convex, and smooth, while E is finite-dimensional, it was shown in [5] that there does not exist *any* nonexpansive retraction of E onto F unless E is a Hilbert space or dim $E = 2$. Thus the class of mappings we study, the nonexpansive projections, is new and distinct from the class of proximity mappings.

Another motivation for the study of nonexpansive retracts is that they are precisely the subsets F of C for which the following extension problem can be solved for each choice of T and M:

(LEP)   Given a Lipschitzian map T of F into a metric space M, to extend T to a map $T^\sharp: C \to M$ without increasing the Lipschitz constant of T.

If r is a nonexpansive retraction of C onto F, (LEP) can be solved by letting $T^\sharp = T \circ r$; while if (LEP) can be solved on F for each T and M, then it can be solved for $M = F$ and $T = $ identity, in which case $T^\sharp$ is a nonexpansive retraction of C onto F.

In particular, if the nonexpansive retraction r can be taken to be a projection, the extension of T to $T^\sharp$ is greatly facilitated since $T^\sharp$ is constant along certain rays emanating from F and need be calculated only in some neighborhood of F.

1. **Orthogonality and the semi-inner-product.** The definition of orthogonality we use is that of Birkhoff [1] and James [8]: x is said to be orthogonal to y if

$$(2) \qquad\qquad \|x + ty\| \geq \|x\|$$

for all $t \in R^1$. We extend this definition as follows:

DEFINITION 1. x is said to be *acute* to y if (2) holds for all $t \geq 0$.

DEFINITION 2.   A retraction r of C onto F is *orthogonal* if for each $p \in C$ and $y \in F$, $r(p) - y$ is acute to $p - 1$ $r(p)$:

$$(3) \qquad ||(1-t)r(p) + tp - y|| \geqq ||r(p) - y|| \quad \text{for all} \quad t \geqq 0 .$$

This definition is suggested by the following observation:

PROPOSITION 1.   *Suppose r is an orthogonal retraction of E onto a convex subset F. If for some $x_0 \in E$, F has a unique supporting hyperplane $H + r(x_0)$ at its boundary point $r(x_0)$, then each $u \in H$ is orthogonal to $r(x_0) - x_0$.*

Proof.   By hypothesis, for each $y \in F$,

$$(4) \qquad ||(1-t)r(x_0) + tx_0 - y|| \geqq ||r(x_0) - y|| \quad \text{for all} \quad t \geqq 0 .$$

If K denotes the cone with vertex $r(x_0)$ generated by F,

$$K = \bigcup_{\lambda > 0} (1 - \lambda)r(x_0) + \lambda F ,$$

then (4) holds for $y \in K$ as well. For, if $\lambda > 0, y_0 \in F$, and $y = (1 - \lambda)r(x_0) + \lambda y_0$, then

$$||(1-t)r(x_0) + tx_0 - y|| = \lambda \left|\left| \left(1 - \frac{t}{\lambda}\right)r(x_0) + \frac{t}{\lambda}x_0 - y_0 \right|\right|$$

$$\geqq \lambda ||r(x_0) - y_0|| = ||r(x_0) - y||$$

since (4) holds for t/λ as well as for t.

Clearly (4) also holds for all $y \in \bar{K}$, the closure of K. Since F has a *unique* supporting hyperplane $H + r(x_0)$ at $r(x_0)$, $\bar{K}$ is the closed half-space with boundary $H + r(x_0)$, which contains F. Therefore, $H + r(x_0) \subset \bar{K}$. For any $u \in H$ we have also $-u \in H$; since $H + r(x_0) \subset \bar{K}$, we may take $y = \pm u + r(x_0)$ in (4), obtaining

$$|| u \pm t(r(x_0) - x_0) || \geqq ||u||$$

for all $t \geqq 0$. Therefore, each $u \in H$ is orthogonal to $r(x_0) - x_0$.

Let $S = \{u \in E \colon ||u|| = 1\}$.

DEFINITION 3.   The Banach space E is *smooth* if for each $(x, y) \in S \times S$ the limit

$$(5) \qquad \lim_{t \to 0} t^{-1}(||x + ty|| - ||x||)$$

exists. E is *uniformly smooth* if the limit (5) is uniform for $(x, y) \in S \times S$.

Equivalently, E is smooth iff at each point of S there is only

one supporting hyperplane to the unit ball; E is uniformly smooth iff E^* is uniformly convex (Smulian [11]).

DEFINITION 4. If E is smooth, the semi-inner-product $[\cdot, \cdot]$ on E is the Gateaux differential of $1/2 \|\cdot\|^2$ (with arguments reversed):

$$(6) \qquad [y, x] = \lim_{t \to 0} t^{-1} \left(\frac{1}{2} \|x + ty\|^2 - \frac{1}{2} \|x\|^2 \right).$$

This limit exists for each $x, y \in E$ by virtue of the existence of (5) for each $x, y \in S$.

PROPOSITION 2. *If E is smooth, the semi-inner-product satisfies the following*:
   (a) $[\cdot, x]$ *is linear for each $x \in E$*;
   (b) $[x, x] = \|x\|^2$;
   (c) $|[y, x]| \leqq \|y\| \cdot \|x\|$;
   (d) $[y, \lambda x] = \lambda[y, x]$;
   (e) $[\cdot, \cdot]$ *is continuous on $E \times E$.*
If E is uniformly smooth, the semi-inner-product is uniformly continuous on bounded subsets of $E \times E$.

REMARK. Lumer [10] defined the semi-inner-product to be any function $[\cdot, \cdot]: E \times E \to R^1$ which satisfies (a)–(c) of Proposition 2. For our later purposes the continuity condition (e) is essential. However, it follows from a result of Giles ([7], p. 439) that if $[\cdot, \cdot]$ satisfies (a)–(c) and (e), then E is smooth and $[y, x]$ is given by the limit (6). Hence there is no point in our using the more general definition.

Proof. (b), (c), and (d) follow directly from the definition of $[y, x]$ as the limit (6). (a) and (e) are true of the Gateaux differential of any continuous convex functional f for which $\lim_{t \to 0} t^{-1}(f(x + ty) - f(x))$ exists for each $x, y \in E$. Finally, if E is uniformly smooth, then the limit (6) is uniform on bounded sets of x, y and $1/2 \|\cdot\|^2$ is uniformly continuous on bounded sets. If f is any functional (convex or not!) which is uniformly continuous on bounded sets and for which $g(x, y) = \lim_{t \to 0} t^{-1}(f(x + ty) - f(x))$ exists uniformly on bounded sets, it is easy to see that g is uniformly continuous on bounded sets.

PROPOSITION 3. *Suppose E is smooth. Then x is orthogonal to y iff $[y, x] = 0$; x is acute to y iff $[y, x] \geqq 0$.*

Proof. We will prove the equivalence for x acute to y, the other part being similar.

Let $f(t) = 1/2\|x + ty\|^2$. Then f is differentiable and $f'(t) = [y, x + ty]$.

By definition, x is acute to y iff $f(t) \geq f(0)$ for all $t \geq 0$. Since f is convex, this is so iff $f'(0) \geq 0$, i.e., $[y, x] \geq 0$.

2. Approximating the nonexpansive retraction. We will prove our main results with the aid of an approximation scheme of F. E. Browder [2].

DEFINITION 5. If $p \in E$ and $F \neq \varnothing$, the approximation region between p and F is the set

$$A(p; F) = \{u \in E: u - y \text{ is acute to } p - u \text{ for all } y \in F\} .$$

Equivalently, $A(p; F)$ is the set of u which satisfy

$$\|(1 - t)u + tp - y\| \geq \|u - y\|$$

for all $t \geq 0$ and $y \in F$. If E is smooth, by Prop. 3 and Prop. 2(d)

$$(7) \qquad A(p; F) = \{u \in E: [u - p, u - y] \leq 0 \quad \text{for all} \quad y \in F\} .$$

Referring to Def. 2, we also have the following characterization of orthogonal retractions:

(8) r is an orthogonal retraction of C onto F iff $r(p) \in F \cap A(p; F)$ for each $p \in C$.

LEMMA 1. *The approximation region $A(p; F)$ is always closed and bounded. If E is smooth then $F \cap A(p; F)$ can contain at most one point.*

Proof. Suppose $\{u_n\}$ is a sequence in $A(p; F)$ and $u_n \to u$. Then for each $t \geq 0$ and $y \in F$,

$$\|(1 - t)u_n + tp - y\| \geq \|u_n - y\| .$$

Letting $n \to \infty$, we find

$$(9) \qquad \|(1 - t)u + tp - y\| \geq \|u - y\| ,$$

so $u \in A(p; F)$. Therefore, the approximation region is closed.

Next, suppose $u \in A(p; F)$. Then (9) holds for $t = 1$ in particular, so $\|p - y\| \geq \|u - y\|$. Thus $A(p; F)$ is a subset of the ball of radius $\|p - y\|$, centered at y. The approximation region is therefore bounded. (Recall that F is assumed to be nonempty.)

Finally, suppose E is smooth and $u_1, u_2 \in F \cap A(p; F)$. By the equivalent formulation (7) of $A(p; F)$ in the smooth case,

(10) $[u_i - p, u_i - y] \leqq 0$

for $i = 1, 2$ and $y \in F$. Since $u_1, u_2 \in F$ we may take $i = 1, y = u_2$ in (10), obtaining

(11) $[u_1 - p, u_1 - u_2] \leqq 0 \,,$

then $i = 2, y = u_1$, obtaining

$$[u_2 - p, u_2 - u_1] \leqq 0 \,.$$

By Prop. 2, $[u_2 - p, u_2 - u_1] = [p - u_2, u_1 - u_2]$; thus

(12) $[p - u_2, u_1 - u_2] \leqq 0 \,.$

Adding (11) and (12),

$$[u_1 - u_2, u_1 - u_2] = [u_1 - p + p - u_2, u_1 - u_2] \leqq 0 \,.$$

But $[u_1 - u_2, u_1 - u_2] = \|u_1 - u_2\|^2$, therefore $u_1 = u_2$.

LEMMA 2. *Suppose E is uniformly smooth and $\{x_n\}$ is a sequence in $A(p; F)$ such that $\lim_n \operatorname{dis}(x_n, F) = 0$. Then $F \cap A(p; F) \neq \varnothing$ and $\{x_n\}$ converges to the unique point in $F \cap A(p; F)$.*

Proof. Choose a sequence $\{y_n\}$ in F so that $x_n - y_n \to 0$. By Lemma 1, $A(p; F)$ is bounded, so $\{x_n\}$ and $\{y_n\}$ are bounded.

By the uniform continuity of the semi-inner-product on bounded sets (Prop. 2),

$$\lim_n ([y_n - p, y_n - y_m] - [x_n - p, x_n - y_m]) = 0$$

uniformly in m. But $[x_n - p, x_n - y_m] \leqq 0$ by (7), hence there exists a sequence $t_n \to 0 +$ such that

(13) $[y_n - p, y_n - y_m] \leqq t_n$ for all m, n .

By symmetry,

(14) $[y_m - p, y_m - y_n] \leqq t_m$ for all m, n .

Adding (13) and (14),

$$\|y_n - y_m\|^2 = [y_n - p + p - y_m, y_n - y_m] \leqq t_n + t_m \,.$$

$\{y_n\}$ is therefore a Cauchy sequence in F, $\{x_n\}$ is a Cauchy sequence in $A(p; F)$, and both sequences converge to the same limit z since $x_n - y_n \to 0$. But since F and $A(p; F)$ are both closed, the limit z belongs to $F \cap A(p; F)$.

DEFINITION 6. A mapping $T: C \to E$ is *firmly nonexpansive* if for all $x, y \in C$ and $t \geqq 0$ there holds the inequality

$$\| (1 - t)(Tx - Ty) + t(x - y) \| \geqq \| Tx - Ty \| .$$

By taking $t = 1$ we see that a firmly nonexpansive mapping is nonexpansive.

REMARK. Definition 6 requires that $Tx - Ty$ be acute to $(x - y) - (Tx - Ty)$ for all $x, y \in C$. If E is smooth, this implies

$$[x - y - (Tx - Ty), Tx - Ty] \geqq 0$$

by Prop. 3, so

$$\| Tx - Ty \|^2 \leqq [x - y, Tx - Ty] .$$

Our terminology is therefore consistent with that of Browder [2], where, if E is a Hilbert space, $T: C \to E$ is called *firmly contractive* if $\| Tx - Ty \|^2 \leqq (x - y, Tx - Ty)$ for all $x, y \in C$ (see also (1)).

Our basic approximation result assures the existence of a large family of firmly nonexpansive mappings:

LEMMA 3. *Suppose $T: C \to C$ is nonexpansive and $F = \{y: Ty = y\} \neq \varnothing$. Then for each $\lambda \in (0, 1)$ there is exactly one mapping $x_\lambda: C \to C$ with the property that $x_\lambda = \lambda T \circ x_\lambda + (1 - \lambda)I$, where $I =$ identity. x_λ and T have the same fixed-point set, x_λ is firmly nonexpansive, and $x_\lambda(p) \in A(p; F)$ for each $p \in C$.*

Proof. Since $0 < \lambda < 1$ and T maps the convex set C into itself, for fixed $p \in C$ the mapping $\lambda T + (1 - \lambda)p$ is a λ-contraction of C into C. By the Banach Contraction Principle, it has a unique fixed point. If $x_\lambda(p)$ denotes this fixed point, then $x_\lambda: C \to C$ has been defined so as to satisfy $x_\lambda = \lambda T \circ x_\lambda + (1 - \lambda)I$; and by the *uniqueness* of the fixed point, there is only one such mapping x_λ.

If $x_\lambda(p) = p$, then $p = \lambda T(p) + (1 - \lambda)p$, so $T(p) = p$. Conversely, if $T(p) = p$, then p is a solution z of $z = \lambda Tz + (1 - \lambda)p$; since $x_\lambda(p)$ is the only such solution, $x_\lambda(p) = p$. Therefore, x_λ and T have the same fixed-point set.

We see that x_λ is firmly nonexpansive as follows: let $p, q \in C$, $u = x_\lambda(p), v = x_\lambda(q)$, so

(15) $$u - v = \lambda(Tu - Tv) + (1 - \lambda)(p - q) .$$

Let $w(s) = (1 - s)(Tu - Tv) + s(p - q)$. Then $w(0) = Tu - Tv, w(1) = p - q$, and, from (15), $w(1 - \lambda) = u - v$. Since $0 < 1 - \lambda < 1, u - v$ lies on the line segment joining $Tu - Tv$ and $p - q$. Therefore, for

any $t \geqq 0$ the three points $Tu - Tv$, $u - v$, and $(1 - t)(u - v) + t(p - q)$ (in that order) are collinear. Since $\| Tu - Tv \| \leqq \| u - v \|$ it follows that

$$\| (1 - t)(u - v) + t(p - q) \| \geqq \| u - v \| \quad \text{for all} \quad t \geqq 0 \, .$$

Substituting $u = x_\lambda(p)$ and $v = x_\lambda(q)$,

$$\| (1 - t)(x_\lambda(p) - x_\lambda(q)) + t(p - q) \| \geqq \| x_\lambda(p) - x_\lambda(q) \|$$
(16)
$$\text{for all} \quad t \geqq 0 \, ,$$

i.e., x_λ is firmly nonexpansive.

Finally, $x_\lambda(p) \in A(p;F)$. For any $y \in F$ we have $x_\lambda(y) = y$ since x_λ and T have the same fixed points. But x_λ is firmly nonexpansive, so taking $q = y$ and noting that $x_\lambda(y) = y$, (16) reduces to

$$\| (1 - t)x_\lambda(p) + tp - y \| \geqq \| x_\lambda(p) - y \|$$

for all $t \geqq 0$ and y in F, i.e., $x_\lambda(p) \in A(p; F)$.

3. Projections and retractions. We are now in a position to prove our principal results.

DEFINITION 7. A *projection* is a mapping f which satisfies the following: whenever $p \in \mathrm{Dom} f$, $t \geqq 0$, and $p^* = f(p) + t(p - f(p))$ is in $\mathrm{Dom} f$, then $f(p^*) = f(p)$. Thus f retracts its domain onto its range along rays. (In [4] such a mapping was called a ray retraction.)

THEOREM 1. *Suppose r is a retraction of C onto F. Then each of the following conditions implies the next:*
  (a) *r is a nonexpansive projection;*
  (b) *r is firmly nonexpansive;*
  (c) *r is an orthogonal retraction.*
If E is smooth these conditions are equivalent and there can exist at most one nonexpansive projection of C onto F.

Proof. (a) $\Rightarrow$ (b). Suppose r is a nonexpansive projection of C onto F, and let the mappings x_λ be defined as in Lemma 3 for $T = r$, i.e., $x_\lambda = \lambda r \circ x_\lambda + (1 - \lambda)I$. Then (16) holds for all $t \geqq 0$.

Now $x_\lambda(p) = \lambda r(x_\lambda(p)) + (1 - \lambda)p$, so p lies on the ray from $r(x_\lambda(p))$ which extends through $x_\lambda(p)$. Since r is a projection, $r(p) = r(x_\lambda(p))$, and hence $x_\lambda(p) = \lambda r(p) + (1 - \lambda)p$. Therefore, $x_\lambda(p) \to r(p)$ and $x_\lambda(q) \to r(q)$ strongly as $\lambda \to 1-$. Letting $\lambda \to 1-$ in (16), we obtain

$$\| (1 - t)(r(p) - r(q)) + t(p - q) \| \geqq \| r(p) - r(q) \|$$
(17)

for all $t \geqq 0$. That is, r is firmly nonexpansive.

(b) $\Rightarrow$ (c). If r is a firmly nonexpansive retraction of C onto F, then for p in C, q in F, (17) holds for all $t \geq 0$, and $r(q) = q$. This reduces to (3) with $y = q$, so r is orthogonal.

(c) $\Rightarrow$ (a) when E is smooth. Whether or not E is smooth, whenever r is an orthogonal retraction of C onto F, $p \in C, \lambda \geq 0$, and $p^* = r(p) + \lambda(p - r(p))$, it follows that $r(p) \in A(p^*; F) \cap F$. In fact, since r is orthogonal $r(p) - y$ is acute to $p - r(p)$ for any y in F, so for any $t \geq 0$,

$$\| r(p) - y + t(p^* - r(p)) \| = \| r(p) - y + \lambda t(p - r(p)) \| \geq \| r(p) - y \|$$

since $p^* - r(p) = \lambda(p - r(p))$ and $\lambda t \geq 0$. This shows that $r(p) - y$ is acute to $p^* - r(p)$, so $r(p) \in A(p^*; F)$. Since $r(p) \in F$ also, we have $r(p) \in F \cap A(p^*; F)$.

In particular, if E is smooth and $p^* \in C$ then (8) and Lemma 1 imply that $F \cap A(p^*; F)$ consists of the single point $r(p^*)$. Since $r(p)$ also belongs to this set, we must have $r(p^*) = r(p)$.

We have proven that if E is smooth and r is an orthogonal retraction, then r is a projection; it remains to show that r is non-expansive. If $p, q \in C$ then $r(q) \in F$ so $r(p) - p$ must be acute to $r(q) - r(p)$, i.e.,

(18) $[r(p) - p, r(p) - r(q)] \leq 0 .$

By symmetry,

(19) $[r(q) - q, r(q) - r(p)] \leq 0 .$

Adding (18) and (19),

$$[r(p) - r(q) - (p - q), r(p) - r(q)] \leq 0 ,$$

from which we conclude

(20) $\| r(p) - r(q) \|^2 \leq [p - q, r(p) - r(q)] .$

But $[p - q, r(p) - r(q)] \leq \| p - q \| \cdot \| r(p) - r(q) \|$, hence (20) implies that r is nonexpansive.

Finally, if E is smooth it follows from (8) and Lemma 1 that there can exist at most one orthogonal retraction of C onto F.

EXAMPLE 1. In general, if E is not smooth the nonexpansive projection need not be unique. Let $E = R^2$ normed by $\|(x, y)\| = |x| + |y|$. If $r_1(x, y) = (0, y)$, then r_1 is a nonexpansive projection onto $L = \{(0, y): y \in R^1\}$. But $r_2(x, y) = (0, x + y)$ is also a nonexpansive projection of E onto L, and $r_1 \neq r_2$.

If E is not smooth, the implications (b) $\Rightarrow$ (a) and (c) $\Rightarrow$ (a) are false in general. Let $f: R^1 \to R^1$ be any nonexpansive nonlinear function for which $f(0) = 0$ (for example, $f = \arctan$). The retraction $r: E \to L$ defined by $r(x, y) = (0, f(x) + y)$ is easily calculated to be firmly nonexpansive, but it is not a projection.

THEOREM 2. *Suppose E is smooth, F is a nonexpansive retract of C, and either*
  (a) *E is uniformly smooth, or*
  (b) *F is locally weakly compact and whenever $x_n \to 0$ weakly, $[y, x_n] \to 0$ for each $y \in E$.*
Then there exists a nonexpansive projection of C onto F.

Proof. By virtue of (8) and Theorem 1 it suffices to show that $F \cap A(p; F) \neq \varnothing$ for each $p \in C$.

Let f be any nonexpansive retraction of C onto F, let $\lambda_n \to 1-$, and define $\{x_n\}$ by $x_n = \lambda_n f(x_n) + (1 - \lambda_n)p$. By Lemma 3, $\{x_n\}$ is bounded, and since

$$x_n - f(x_n) = (1 - \lambda_n^{-1})(x_n - p)$$

while $\lambda_n \to 1$, we have $x_n - f(x_n) \to 0$ strongly. But $f(x_n) \in F$, so $\lim_n \operatorname{dis}(x_n, F) = 0$.

If hypothesis (a) holds, then by Lemma 2, $F \cap A(p; F) \neq \varnothing$ and we are finished.

If hypothesis (b) holds, set $y_n = f(x_n)$, so $y_n \in F$ and $x_n - y_n \to 0$. The sequence $\{y_n\}$ is therefore bounded, and since F is locally weakly compact, there exists $y \in F$ and a subsequence of $\{y_n\}$ which converges weakly to y. Without loss of generality we may suppose $y_n \to y$ weakly. Then also $x_n \to y$ weakly. Since $x_n \in A(p; F)$ and $y \in F$,

$$[x_n - p, x_n - y] \leqq 0 \,,$$

which implies

(21) $$\| x_n - y \|^2 \leqq [p - y, x_n - y] \,.$$

But $x_n - y \to 0$ weakly. By hypothesis (b), the right-hand-side (and hence the left-hand-side) of (21) goes to 0. Thus $x_n \to y$ strongly.

Since $x_n - y_n \to 0$ strongly, y is the strong limit of two sequences, $\{x_n\}$ in $A(p; F)$ and $\{y_n\}$ in F. Since F and $A(p; F)$ are closed, $y \in F \cap A(p; F)$.

COROLLARY 1. *If E is strictly convex and uniformly smooth, $T: C \to C$ is nonexpansive, and $F = \{x: Tx = x\}$ is nonempty, then there exists a nonexpansive projection of C onto F.*

Proof. E is reflexive since E^* is uniformly convex. By Theorem 1 of Bruck [4], F is a nonexpansive retract of C; by Theorem 2(a) above, there exists a nonexpansive projection of C onto F.

THEOREM 3. *Suppose F is a nonexpansive retract of C which is locally compact, i.e., such that each bounded sequence in F has a strongly convergent subsequence. Then there exists a firmly nonexpansive retraction of C onto F.*

Proof. Let T be a nonexpansive retraction of C onto F, and define the mappings x_λ as in Lemma 3. By Lemma 3, $T \circ x_\lambda$ is seen to be a nonexpansive retraction of C onto F. Set $f_\lambda = Tx_\lambda$ and fix $y_0 \in F$.

If we let $F_p = \{u \in F: ||u - y_0|| \leq ||p - y_0||\}$, then F_p is compact and by Tychonoff's theorem the product $\prod_{p \in C} F_p$ is compact. Each f_λ belongs to this product space since f_λ is nonexpansive and $f_\lambda(y_0) = y_0$:

$$||f_\lambda(p) - y_0|| = ||f_\lambda(p) - f_\lambda(y_0)|| \leq ||p - y_0|| .$$

If $0 < t < 1$ define H_t to be the closure in $\prod F_p$ of $\{f_\lambda: t \leq \lambda < 1\}$. The family $\{H_t: 0 < t < 1\}$ is seen to have the finite intersection property, and since each H_t is compact, there exists f in $\cap \{H_t: 0 < t < 1\}$. We claim that f is firmly nonexpansive and retracts C onto F.

If $\lambda_n \in (0, 1)$ we abbreviate f_{λ_n} and x_{λ_n} to f_n and x_n respectively. It follows from the definition of the product topology and the fact that $f \in H_t$ for all $t \in (0, 1)$ that for each $p, q \in C$ there exists a sequence $\lambda_n \to 1-$ such that $f_n(p) \to f(p)$ and $f_n(q) \to f(q)$.

But $x_n(p) - f_n(p) \to 0$ and $x_n(q) - f_n(q) \to 0$ since

$$x_n - f_n = (1 - \lambda_n^{-1})(x_n - I) .$$

Therefore, $x_n(p) \to f(p)$ and $x_n(q) \to f(q)$. Since x_n is firmly nonexpansive (Lemma 3),

$$||(1 - t)(x_n(p) - x_n(q)) + t(p - q)|| \geq ||x_n(p) - x_n(q)||$$

for all $t \geq 0$, so in the limit

$$||(1 - t)(fp - fq) + t(p - q)|| \geq ||fp - fq|| .$$

Therefore, f is firmly nonexpansive.

That f retracts C onto F is trivial. First, the range of f is a subset of F since $f \in \prod F_p$. Next, if $p \in F$ we can choose $\lambda_n \to 1-$ such that $f_n(p) \to f(p)$. But $f_n(p) \equiv p$ since $p \in F$, therefore $f(p) = p$.

We have shown that f is a firmly nonexpansive retraction of C on F.

4. **Other results.** Corollary 1 suggests the following question: can we find more general hypotheses than those of Corollary 1 which guarantee the existence of a nonexpansive projection onto the fixed-point set of T? We can relax the nonexpansive condition if we strengthen the compactness assumption:

THEOREM 4. *Suppose E is smooth and $T: C \to C$ is continuous and satisfies*

$$(22) \qquad [Tx - x, x - y] \leqq 0 \quad \text{for all} \quad x \in C, y \in F,$$

where $F = \{x: Tx = x\}$. If the image of T is relatively compact then there exists a nonexpansive projection of C onto F.

Proof. For $p \in C$ and $0 < \lambda < 1$, the map $\lambda T + (1 - \lambda)p$ sends C into C and has a fixed point by Schauder's theorem. Let x_λ denote such a fixed point. (Curiously, although there may exist many choices of x_λ for a given λ, the strong $\lim_{\lambda \to 1-} x_\lambda$ exists independently of the choice of x_λ.)

Thus $\lambda(Tx_\lambda - x_\lambda) = (1 - \lambda)(x_\lambda - p)$. By Schauder's theorem, $F \neq \varnothing$. For any $y \in F$, we therefore have

$$(1 - \lambda)[x_\lambda - p, x_\lambda - y] = \lambda[Tx_\lambda - x_\lambda, x_\lambda - y] \leqq 0$$

by virtue of (22). Thus $x_\lambda \in A(p; F)$.

Let $\lambda_n \to 1-$ and set $x_n = x_{\lambda_n}$. Since $T(C)$ is relatively compact, some subsequence of $\{Tx_n\}$ converges strongly. Without loss of generality we may suppose $Tx_n \to y$ in C. Since $x_n = \lambda_n Tx_n + (1 - \lambda_n)p$, it follows that also $x_n \to y$ and hence $Ty = y$.

That is, $\{x_n\}$ is a sequence in $A(p; F)$ which converges to a point $y \in F$. Since $A(p; F)$ is closed, $y \in F \cap A(p; F)$. Theorem 1 and (8) now imply the existence of a nonexpansive projection of C onto F.

A concrete condition which implies (22) is that T be quasi-nonexpansive, i.e., that whenever $x \in C$ and $Ty = y$, there hold the inequality $\| Tx - y \| \leqq \| x - y \|$. This is because

$$[Tx - y, x - y] \leqq \| Tx - y \| \cdot \| x - y \| \leqq \| x - y \|^2 = [x - y, x - y]$$

from which (22) easily follows. We have proven:

COROLLARY 2. *If E is smooth and $T: C \to C$ is compact, continuous, and quasi-nonexpansive, then there exists a nonexpansive projection of C onto the fixed-point set of T.*

REMARK. It is not known whether smoothness of E or compactness of T are essential in this Corollary; except that any mapping

without fixed points is vacuously quasi-nonexpansive, so that any such extension of the corollary would have to consider the empty set to be the range of a nonexpansive projection by convention.

Recently Karlovitz [9] has proven that if $\dim E = 2$ then *every* closed convex subset of E is the range of a nonexpansive projection. This follows easily from a result of [4] and our present results if we merely note that a line in E is always a nonexpansive retract of E.

THEOREM 5.   *If* $\dim E = 2$ *and* F *is a nonempty closed convex subset of* E, *then there exists a nonexpansive projection of* E *onto* F.

Proof.   Let L be a line in E. Then there exist x_0, y_0 with $\|x_0\| = 1$ such that $L = \{y_0 + tx_0 : t \in R^1\}$. By a corollary of the Hahn-Banach theorem, there exists $w \in E^*$ with $\|w\| = 1$ and $w(x_0) = 1$. The mapping $f : E \to L$ defined by $f(x) = y_0 + w(x - y_0) \cdot x_0$ is easily seen to be a nonexpansive projection of E onto L. Thus every line in E is the range of a nonexpansive projection.

It follows that any closed half-space H in E is a nonexpansive retract of E. Indeed, let f be a nonexpansive projection of E onto the boundary line ∂H of H; define $g : E \to H$ by $g(x) = x$ for $x \in H$, $g(x) = f(x)$ for $x \notin H$. Obviously g is a retraction of E onto H. We claim that g is nonexpansive.

We must prove that $\|gx - gy\| \leq \|x - y\|$ for all $x, y \in E$. This is obvious if x, y are both in H or both in $E \backslash H$. Therefore, it is enough to show that $\|g(x) - y\| \leq \|x - y\|$ whenever $x \notin H$ and $y \in H$. Let z be the point of ∂H which lies on the segment joining $y \in H$ to $x \in E \backslash H$. Then $g(x) - g(z) = f(x) - f(z)$ and $g(z) - y = z - y$, so

$$\|g(x) - y\| \leq \|g(x) - g(z)\| + \|g(z) - y\| = \|f(x) - f(z)\| + \|z - y\|$$
$$\leq \|x - z\| + \|z - y\| = \|x - y\|$$

since x, z, y are collinear. Therefore g is nonexpansive and we have shown that H is a nonexpansive retract of E.

Now suppose that E is both smooth and strictly convex. (The general case will be proven later.) By Theorem 2 of Bruck [4], the intersection of nonexpansive retracts of a reflexive, strictly convex Banach space E is also a nonexpansive retract of E. Since F is the intersection of closed half-spaces, which are nonexpansive retracts of a two-dimensional space E, it follows that F is a nonexpansive retract of E. A smooth finite-dimensional space is uniformly smooth, so by Theorem 2 (a) there exists a nonexpansive projection of E onto F.

The case of a general norm $\|\cdot\|$ is treated as a limiting case. Let $\varepsilon_n \to 0+$ and for each positive integer n choose a norm $\|\cdot\|_n$ on

E with respect to which E is both smooth and strictly convex, and such that

$$\|x\| \leqq \|x\|_n \leqq (1 + \varepsilon_n)\|x\| \,.$$

By our previous case, for each n there exists a projection f_n of E onto F such that $\|f_n x - f_n y\|_n \leqq \|x - y\|_n$ for each $x, y \in E$. Thus

$$(23) \qquad \|f_n x - f_n y\| \leqq \|f_n x - f_n y\|_n \leqq (1 + \varepsilon_n)\|x - y\| \,.$$

If $y_0 \in F$ is fixed, we also have

$$\|f_n x - y_0\| = \|f_n x - f_n y_0\| \leqq (1 + \varepsilon_n)\|x - y_0\| \,.$$

Thus on bounded sets the family $\{f_n\}$ is equibounded and equicontinuous. By a variant of the Ascoli-Arzèla theorem (in which the mappings are into E, not R^1), there exists a subsequence of $\{f_n\}$ which converges uniformly on bounded sets to some $f: E \to E$. Without loss of generality we can suppose this subsequence is $\{f_n\}$ itself. It follows from (23) that f is nonexpansive with respect to $\|\cdot\|$. Obviously $fx = x$ for all $x \in F$, while $f_n x \in F$ for all n implies $fx \in F$. We have shown that f is a nonexpansive retraction of $(E, \|\cdot\|)$ onto F.

To prove that f is a projection, fix $t \geqq 0$ and $x \in E$ and set $x_n = f_n(x)$, $x_n^* = x_n + t(x - x_n)$, and $x^* = f(x) + t(x - f(x))$. Thus $x_n \to f(x)$, $x_n^* \to x^*$. But each f_n is a projection, so

$$(24) \qquad\qquad f_n(x_n^*) = x_n \,.$$

The right-hand-side of (24) converges to $f(x)$, the left-hand-side to $f(x^*)$ since $f_n \to f$ uniformly on bounded sets and $x_n^* \to x^*$. Therefore, $f(x^*) = f(x)$, i.e., f is a projection.

EXAMPLE 2. It may happen that there exists a nonexpansive projection of C onto F even if F is not convex. For example, let $E = R^2$ normed by $\|(x, y)\| = \max(|x|, |y|)$, and define $f: E \to E$ by $f(x, y) = (x, |x|)$. Then f is a nonexpansive projection of E onto $F = \{(x, y): y = |x|\}$, but F is not convex.

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