

# Pacific Journal of Mathematics

**GEOMETRIC PROPERTIES OF SOBOLEV MAPPINGS**

RONALD FRANCIS GARIEPY

## GEOMETRIC PROPERTIES OF SOBOLEV MAPPINGS

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**If  $f$  is a mapping from an open  $k$ -cube in  $R^k$  into  $R^n$ ,  $2 \leq k \leq n$ , whose coordinate functions belong to appropriate Sobolev spaces, then the area of  $f$  is the integral with respect to  $k$  dimensional Hausdorff measure over  $R^n$  of a nonnegative integer valued multiplicity function.**

1. Introduction. If  $f: Q \rightarrow R^n$ ,  $Q$  an open  $k$ -cube in  $R^k$ ,  $2 \leq k \leq n$ , is a mapping whose coordinate functions belong to appropriate Sobolev classes, it was shown in [6] that  $f$  is  $k - 1$  continuous and that the area of  $f$ , as defined in [5], is equal to the classical Jacobian integral. The purpose of this paper is to investigate, using the theory of currents as in [2], the geometric-measure theoretic properties of such a surface and to show that the area is equal to the integral with respect to  $k$  dimensional Hausdorff measure in  $R^n$  of an integer valued multiplicity function.

2. Suppose  $k$  and  $n$  are integers with  $2 \leq k \leq n$ . Let

$$Q = R^k \cap \{x: 0 < x_i < 1 \text{ for } 1 \leq i \leq k\}$$

and let  $A(k, n)$  denote the set of all  $k$ -tuples  $\lambda = (\lambda_1, \dots, \lambda_k)$  of integers such that  $1 \leq \lambda_1 < \dots < \lambda_k \leq n$ . Suppose  $f: Q \rightarrow R^n$ ,  $f = (f^1, \dots, f^n)$ ,  $f^i \in W_{p_i}^1(Q)$ ,  $p_i > k - 1$ , and  $\sum_{j=1}^k 1/p_{\lambda_j} \leq 1$  whenever  $\lambda \in A(k, n)$ . Here  $W_p^1(Q)$  denotes those functions in  $L^p(Q)$  whose distribution partial derivatives are functions in  $L^p(Q)$ .

Let  $e_1, \dots, e_n$  be the usual basis for  $R^n$  and let

$$e_\lambda = e_{\lambda_1} \wedge \dots \wedge e_{\lambda_k},$$

$\lambda \in A(k, n)$ , denote the corresponding basis for the space of  $k$ -vectors in  $R^n$ . For  $\lambda \in A(k, n)$  let  $p^\lambda$  denote the orthogonal projection of  $R^n$  onto  $R^k$  defined by letting

$$p^\lambda(y) = (y_{\lambda_1}, \dots, y_{\lambda_k}) \text{ for } y = (y_1, \dots, y_n) \in R^n.$$

For almost every (in the sense of  $k$  dimensional Lebesgue measure  $\mathcal{L}_k$ )  $x \in Q$ , let  $Jf(x) = \sum_{\lambda \in A(k, n)} Jf^\lambda(x) e_\lambda$  where  $Jf^\lambda$  denotes the determinant of the matrix of distribution partial derivatives of  $f^\lambda = p^\lambda \circ f$ . In [6] it was shown that the area of  $f$ , as defined in [5] is equal to  $\int_Q |Jf(x)| dx$  where  $|Jf(x)|$  is the Euclidean norm of the  $k$ -vector  $Jf(x)$ .

Define a current valued measure  $T$  over  $Q$  by letting

$$T(B)(\phi) = \int_B \phi(f(x)) \cdot Jf(x) dx$$

whenever  $B$  is an  $\mathcal{L}_k$  measurable subset of  $Q$  and  $\phi$  is an infinitely differentiable  $k$ -form on  $R^n$  with compact support. Let  $\sigma$  denote the finite measure defined over  $R^n$  by letting

$$\sigma(Y) = \int_{f^{-1}(Y)} |Jf(x)| dx$$

whenever  $Y$  is a Borel subset of  $R^n$ .

It will be shown in part 3 that  $T(B)$  is a locally rectifiable current whenever  $B$  is an  $\mathcal{L}_k$  measurable subset of  $Q$  and this fact will be used to define a nonnegative integer valued function  $N$  on  $R^n$  which describes the multiplicity with which  $f$  assumes its values. The main results of the paper are summarized here.

**THEOREM.** *Let  $H_n^k$  denote  $k$  dimensional Hausdorff measure in  $R^n$  and let  $\alpha(k)$  denote the  $\mathcal{L}_k$  measure of the unit ball in  $R^k$ .*

1. *For  $H_n^k$  almost every  $y \in R^n$*

$$N(y) = \lim_{r \rightarrow 0^+} \frac{\sigma(B(y, r))}{\alpha(k)r^k}.$$

*Here  $B(y, r)$  denotes the open ball of radius  $r$  around  $y$ .*

2. 
$$\int_{R^n} N(y) dH_n^k y = \int_Q |Jf(x)| dx.$$

3. *There exists a countable family  $F$  of  $k$  dimensional  $C^1$  submanifolds of  $R^n$  such that for  $\sigma$  almost every  $y \in R^n$  there is an  $M \in F$  with  $y \in M$ ,*

$$\lim_{r \rightarrow 0^+} \frac{\sigma(B(y, r) - M)}{\alpha(k)r^k} = 0$$

and

$$\lim_{r \rightarrow 0^+} \frac{\sigma(B(y, r) \cap M)}{\alpha(k)r^k} = N(y).$$

3. Definition of the function  $N$  and proof of the theorem. We follow a plan analogous to that of [2: 2.1]. For  $1 \leq i \leq k$  and  $r \in I = \{s: 0 < s < 1\}$  let  $P_i(r) = Q \cap \{x: x_i = r\}$ . Let  $\{f_j\}$  be a sequence of mollifiers of  $f$  as in [6] and let  $\bar{f}$  denote the pointwise limit of  $\{f_j\}$  wherever it exists. Then  $\bar{f}$  is a representative of  $f$  and according to [6], [7: Chap. 3], and [8: part 3] there exists a collection

$P$  of the sets  $P_i(r)$  such that for each  $i$ ,  $P_i(r) \in P$  for almost all (in the sense of 1 dimensional Lebesgue measure)  $r \in I$  and if  $q$  is any  $k$ -cube in  $Q$  whose  $k - 1$  faces all lie in elements of  $P$  then

- (1)  $f_j | \text{Bdry } q$  converges uniformly to  $\bar{f} | \text{Bdry } q$ ,
- (2)  $H_n^k(\bar{f} | \text{Bdry } q) = 0$
- (3)  $L_{k-1}(\bar{f} | \text{Bdry } q) = \varliminf_{j \rightarrow \infty} L_{k-1}(f_j | \text{Bdry } q)$ , where  $L_{k-1}$

denotes  $k - 1$  dimensional Lebesgue area.

Henceforth we will denote by  $f$  the pointwise limit of mollifiers  $\{f_j\}$  as described above. A  $k$ -cube  $q \subset Q$  whose  $k - 1$  faces all lie in elements of  $P$  will be called "special".

For the notation concerning currents we refer to [3].

LEMMA 1. *If  $f$  is bounded then  $T(B)$  is a rectifiable current whenever  $B$  is an  $\mathcal{L}_k$  measurable subset of  $Q$ .*

*Proof.* If  $q \subset Q$  is a special  $k$ -cube then

$$\lim_{j \rightarrow \infty} \int_q |Jf_j(x) - Jf(x)| dx = 0$$

and hence the sequence  $\{f_{j\#}(q)\}$  of currents converges weakly to  $T(q)$ . The supports of the  $f_{j\#}(q)$  and  $T(q)$  are all contained in a fixed compact set,

$$M(f_{j\#}(q)) \leq \int_q |Jf_j(x)| dx,$$

and

$$M(\partial f_{j\#}(q)) \leq L_{k-1}(f_j | \text{Bdry } q)$$

where  $M$  denotes mass in the space of currents. Thus, by [4: 8.14, 8.13],  $T(q)$  is an integral current whenever  $q$  is special.

Since it is clear that

$$M(T(A)) \leq \int_A |Jf(x)| dx$$

whenever  $A$  is an  $\mathcal{L}_k$  measurable subset of  $Q$ , the lemma follows.

Let  $\|T\|$  denote the finite measure over  $Q$  defined by letting  $\|T\|(A)$  denote the supremum of the numbers  $\sum_{j=1}^{\infty} M(T(B_j))$  taken over all countable disjoint collections of  $\mathcal{L}_k$  measurable subsets  $B_j \subset A$  whenever  $A$  is an  $\mathcal{L}_k$  measurable subset of  $Q$ . Clearly

$$\|T\|(A) \leq \int_A |Jf(x)| dx$$

whenever  $A$  is an  $\mathcal{L}_k$  measurable subset of  $Q$ .

For  $\mathcal{L}_k$  almost every  $x \in Q$  there is a  $k$ -covector  $\omega$  in  $R^n$  with

$|\omega| = 1$  such that  $\omega \cdot Jf(x) = |Jf(x)|$  and

$$\overline{\lim}_{r \rightarrow 0^+} \frac{\|T\|(B(x, r))}{\alpha(k)r^k} \geq \lim_{r \rightarrow 0^+} \frac{T(B(x, r))(\omega)}{\alpha(k)r^k} = |Jf(x)|.$$

It follows that  $\|T\|(A) = \int_A |Jf(x)| dx$  whenever  $A$  is an  $\mathcal{L}_k$  measurable subset of  $Q$ .

For each positive integer  $N$  let  $f_N = (f_N^1, \dots, f_N^n)$  where

$$f_N^i(x) = \begin{cases} N & \text{if } f^i(x) \geq N \\ f^i(x) & \text{if } |f^i(x)| < N \\ -N & \text{if } f^i(x) \leq -N. \end{cases}$$

Then  $f_N$  is bounded and  $f_N^i \in W_{p_i}^1(Q)$  for  $1 \leq i \leq n$ . For any measurable set  $B \subset Q$  let

$$T_N(B)(\phi) = \int_B \phi(f_N(x)) \cdot Jf_N(x) dx$$

whenever  $\phi$  is an infinitely differentiable  $k$ -form on  $R^n$ . Note that, if  $Y$  is a bounded Borel set in  $R^n$ , then, for sufficiently large  $N$ ,  $T_N(B) \llcorner Y = T(B) \llcorner Y$  whenever  $B$  is an  $\mathcal{L}_k$  measurable subset of  $Q$ . Consequently, if  $Y$  is a bounded open subset of  $R^n$  then  $T(B) \llcorner Y$  is rectifiable whenever  $B$  is a measurable subset of  $Q$ .

Analogous to [2: 2.1 part 3] we have

**LEMMA 2.** *There exists a countable collection  $F$  of  $k$  dimensional  $C^1$  submanifolds of  $R^n$  such that  $\sigma(R^n - \bigcup F) = 0$ .*

*Proof.* Suppose  $r$  and  $\varepsilon$  are positive real numbers and let

$$B(0, r) = R^n \cap \{y: |y| < r\}.$$

Let  $\{A_1, \dots, A_m\}$  denote a finite collection of disjoint measurable subsets of  $f^{-1}(B(0, r))$  such that  $\mathcal{L}_k(f^{-1}(B(0, r)) - \bigcup_{j=1}^m A_j) = 0$  and  $\sigma(B(0, r)) - \varepsilon < \sum_{j=1}^m M(T(A_j))$ . Note that  $T(A_j) = T(A_j) \llcorner B(0, r)$  is rectifiable for  $j = 1, \dots, m$ . Thus, by [4: 8.16], there exists for each  $j$  a countable collection  $G_j$  of  $k$ -dimensional  $C^1$  submanifolds of  $R^n$  such that  $\|T(A_j)\|(R^n - \bigcup G_j) = 0$ . Letting  $G = \bigcup_{j=1}^m G_j$ , we have

$$\begin{aligned} \varepsilon &\geq \sigma(B(0, r)) - \sum_{j=1}^m M(T(A_j)) = \sum_{j=1}^m (\|T\|(A_j) - M(T(A_j))) \\ &\geq \sum_{j=1}^m \|T\|(A_j \cap f^{-1}(R^n - \bigcup G_j)) \\ &\geq \sum_{j=1}^m \|T\|(A_j \cap f^{-1}(R^n - \bigcup G)) = \sigma(B(0, r) - \bigcup G) \end{aligned}$$

and the lemma follows.

If  $\mu$  is a measure over  $R^n$ ,  $y \in R^n$ , and  $A \subset R^n$  we let

$$\theta^k(\mu, A, y) = \lim_{r \rightarrow 0^+} \frac{\mu(A \cap B(y, r))}{\alpha(k)r^k}$$

whenever the limit exists. In case  $A = R^n$  we write  $\theta^k(\mu, y)$ .

Recall that, if  $S$  is a  $k$  dimensional rectifiable current in  $R^n$  and  $Y$  is a Borel set in  $R^n$  with  $H_n^k(Y) = 0$ , the  $S \llcorner Y = 0$ . Thus  $\sigma$  is absolutely continuous with respect to  $H_n^k$ . This fact together with Lemma 2 and the finiteness of  $\sigma$  allow us to conclude using [1: 3.1, 3.2] that:

1.  $\theta^k(\sigma, y)$  exists for  $H_n^k$  almost every  $y \in R^n$ .
2. For  $\sigma$  almost every  $y \in R^n$  there exists an  $M \in F$  such that  $y \in M$ ,  $\theta^k(\sigma, y) < \infty$ , and  $\theta^k(\sigma, R^n - M, y) = 0$ .
3.  $\int_{R^n} \theta^k(\sigma, y) dH_n^k y \leq \sigma(R^n)$ .

A proof of the following statement concerning rectifiable currents can be found in [2: 2.1 part 7]: If  $S$  is a rectifiable  $k$  dimensional current in  $R^n$ ,  $M$  is a  $k$  dimensional  $C^1$  submanifold of  $R^n$ ,

$$y \in M - \text{spt } \partial S,$$

$\theta^k(\|S\|, y) < \infty$ ,  $\theta^k(\|S\|, R^n - M, y) = 0$ , and  $P$  is an oriented  $k$  plane tangent to  $M$  at  $y$ , then there exists a unique integer  $m$  such that

$$\lim_{r \rightarrow 0^+} \frac{1}{\alpha(k)r^k} F[S \llcorner B(y, r) - m(P \cap B(y, r))] = 0$$

where  $F$  denotes the flat norm [4: 3.2].

Now suppose  $q$  is a special  $k$ -cube in  $Q$  and  $y \in R^n$ . If there is an  $M \in F$  with  $y \in M - f(\text{Bdry } q)$ ,  $\theta^k(\sigma, y) < \infty$ , and

$$\theta^k(\sigma, R^n - M, y) = 0,$$

let  $P$  denote an oriented  $k$ -plane tangent to  $M$  at  $y$ , let  $m(q, y)$  denote the integer such that

$$\lim_{r \rightarrow 0^+} \frac{1}{\alpha(k)r^k} F[T(q) \llcorner B(y, r) - m(q, y)(P \cap B(y, r))] = 0$$

and set  $\alpha(q, y) = m(q, y) \zeta(y)$  where  $\zeta(y)$  is the simple unit  $k$ -vector orienting  $P$ . Otherwise set  $\alpha(q, y) = 0$ .

Then, for  $H_n^k$  almost every  $y \in R^n$ ,

$$\theta^k(T(q) \llcorner \phi, y) = \lim_{r \rightarrow 0} \frac{[T(q) \llcorner B(y, r)](\phi)}{\alpha(k)r^k} = \phi(y) \cdot \alpha(q, y)$$

whenever  $\phi$  is an infinitely differentiable  $k$ -form in  $R^n$ . Consequently  $T(q)(\phi) = \int_{R^n} \phi(y) \cdot \alpha(q, y) dH_n^k y$  whenever  $\phi$  is an infinitely differentiable  $k$ -form and hence

$$M(T(q)) \leq \int_{R^n} |\alpha(q, y)| dH_n^k y$$

whenever  $q$  is a special  $k$ -cube.

For  $y \in R^n$  let  $N(y)$  denote the supremum of the numbers  $\sum_{q \in G} |\alpha(q, y)|$  taken over all finite collections  $G$  of nonoverlapping special  $k$ -cubes in  $Q$ .

Suppose  $N(y) \neq 0$  and  $G$  is a finite collection of nonoverlapping special  $k$ -cubes such that  $\alpha(q, y) \neq 0$  for  $q \in G$ . Let  $\omega$  denote a  $k$ -covector with  $|\omega| = 1$  and  $\omega \cdot \zeta(y) = 1$ . Then

$$\begin{aligned} \sum_{q \in G} |\alpha(q, y)| &= \sum_{q \in G} |\theta^k(T(q) \lrcorner \omega, y)| \\ &= \lim_{r \rightarrow 0} \sum_{q \in G} \frac{|[T(q) \lrcorner B(y, r)](\omega)|}{\alpha(k)r^k} \\ &\leq \theta^k(\sigma, y). \end{aligned}$$

Thus  $N(y) \leq \theta^k(\sigma, y)$  for  $H_n^k$  almost every  $y \in R^n$ .

On the other hand, if  $G$  is any finite collection of nonoverlapping special  $k$ -cubes,

$$\sum_{q \in G} M(T(q)) \leq \int_{R^n} \sum_{q \in G} |\alpha(q, y)| dH_n^k y.$$

The supremum of the numbers  $\sum_{q \in G} M(T(q))$  taken over all finite collections  $G$  of nonoverlapping special  $k$ -cubes is readily seen to be  $\sigma(R^n)$  and hence

$$\sigma(R^n) \leq \int_{R^n} N(y) dH_n^k y \leq \int_{R^n} \theta^k(\sigma, y) dH_n^k y \leq \sigma(R^n).$$

Thus  $N(y) = \theta^k(\sigma, y)$   $H_n^k$  almost everywhere and

$$\int_{R^n} N(y) dH_n^k y = \int_Q |Jf(x)| dx.$$

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