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GEOMETRIC PROPERTIES OF SOBOLEV MAPPINGS

RONALD FRANCIS GARIEPY

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If f is a mapping from an open k -cube in R^k into R^n , $2 \leq k \leq n$, whose coordinate functions belong to appropriate Sobolev spaces, then the area of f is the integral with respect to k dimensional Hausdorff measure over R^n of a nonnegative integer valued multiplicity function.

1. Introduction. If $f: Q \rightarrow R^n$, Q an open k -cube in R^k , $2 \leq k \leq n$, is a mapping whose coordinate functions belong to appropriate Sobolev classes, it was shown in [6] that f is $k - 1$ continuous and that the area of f , as defined in [5], is equal to the classical Jacobian integral. The purpose of this paper is to investigate, using the theory of currents as in [2], the geometric-measure theoretic properties of such a surface and to show that the area is equal to the integral with respect to k dimensional Hausdorff measure in R^n of an integer valued multiplicity function.

2. Suppose k and n are integers with $2 \leq k \leq n$. Let

$$Q = R^k \cap \{x: 0 < x_i < 1 \text{ for } 1 \leq i \leq k\}$$

and let $A(k, n)$ denote the set of all k -tuples $\lambda = (\lambda_1, \dots, \lambda_k)$ of integers such that $1 \leq \lambda_1 < \dots < \lambda_k \leq n$. Suppose $f: Q \rightarrow R^n$, $f = (f^1, \dots, f^n)$, $f^i \in W_{p_i}^1(Q)$, $p_i > k - 1$, and $\sum_{j=1}^k 1/p_{\lambda_j} \leq 1$ whenever $\lambda \in A(k, n)$. Here $W_p^1(Q)$ denotes those functions in $L^p(Q)$ whose distribution partial derivatives are functions in $L^p(Q)$.

Let e_1, \dots, e_n be the usual basis for R^n and let

$$e_\lambda = e_{\lambda_1} \wedge \dots \wedge e_{\lambda_k},$$

$\lambda \in A(k, n)$, denote the corresponding basis for the space of k -vectors in R^n . For $\lambda \in A(k, n)$ let p^λ denote the orthogonal projection of R^n onto R^k defined by letting

$$p^\lambda(y) = (y_{\lambda_1}, \dots, y_{\lambda_k}) \text{ for } y = (y_1, \dots, y_n) \in R^n.$$

For almost every (in the sense of k dimensional Lebesgue measure \mathcal{L}_k) $x \in Q$, let $Jf(x) = \sum_{\lambda \in A(k, n)} Jf^\lambda(x)e_\lambda$ where Jf^λ denotes the determinant of the matrix of distribution partial derivatives of $f^\lambda = p^\lambda \circ f$. In [6] it was shown that the area of f , as defined in [5] is equal to $\int_Q |Jf(x)| dx$ where $|Jf(x)|$ is the Euclidean norm of the k -vector $Jf(x)$.

Define a current valued measure T over Q by letting

$$T(B)(\phi) = \int_B \phi(f(x)) \cdot Jf(x) dx$$

whenever B is an \mathcal{L}_k measurable subset of Q and ϕ is an infinitely differentiable k -form on R^n with compact support. Let σ denote the finite measure defined over R^n by letting

$$\sigma(Y) = \int_{f^{-1}(Y)} |Jf(x)| dx$$

whenever Y is a Borel subset of R^n .

It will be shown in part 3 that $T(B)$ is a locally rectifiable current whenever B is an \mathcal{L}_k measurable subset of Q and this fact will be used to define a nonnegative integer valued function N on R^n which describes the multiplicity with which f assumes its values. The main results of the paper are summarized here.

THEOREM. *Let H_n^k denote k dimensional Hausdorff measure in R^n and let $\alpha(k)$ denote the \mathcal{L}_k measure of the unit ball in R^k .*

1. *For H_n^k almost every $y \in R^n$*

$$N(y) = \lim_{r \rightarrow 0+} \frac{\sigma(B(y, r))}{\alpha(k)r^k}.$$

Here $B(y, r)$ denotes the open ball of radius r around y .

2.
$$\int_{R^n} N(y) dH_n^k y = \int_Q |Jf(x)| dx.$$

3. *There exists a countable family F of k dimensional C^1 submanifolds of R^n such that for σ almost every $y \in R^n$ there is an $M \in F$ with $y \in M$,*

$$\lim_{r \rightarrow 0+} \frac{\sigma(B(y, r) - M)}{\alpha(k)r^k} = 0$$

and

$$\lim_{r \rightarrow 0+} \frac{\sigma(B(y, r) \cap M)}{\alpha(k)r^k} = N(y).$$

3. Definition of the function N and proof of the theorem. We follow a plan analogous to that of [2: 2.1]. For $1 \leq i \leq k$ and $r \in I = \{s: 0 < s < 1\}$ let $P_i(r) = Q \cap \{x: x_i = r\}$. Let $\{f_j\}$ be a sequence of mollifiers of f as in [6] and let \bar{f} denote the pointwise limit of $\{f_j\}$ wherever it exists. Then \bar{f} is a representative of f and according to [6], [7: Chap. 3], and [8: part 3] there exists a collection

P of the sets $P_i(r)$ such that for each i , $P_i(r) \in P$ for almost all (in the sense of 1 dimensional Lebesgue measure) $r \in I$ and if q is any k -cube in Q whose $k - 1$ faces all lie in elements of P then

- (1) $f_j | \text{Bdry } q$ converges uniformly to $\bar{f} | \text{Bdry } q$,
- (2) $H_n^k(\bar{f} | \text{Bdry } q) = 0$
- (3) $L_{k-1}(\bar{f} | \text{Bdry } q) = \underline{\lim}_{j \rightarrow \infty} L_{k-1}(f_j | \text{Bdry } q)$, where L_{k-1}

denotes $k - 1$ dimensional Lebesgue area.

Henceforth we will denote by f the pointwise limit of mollifiers $\{f_j\}$ as described above. A k -cube $q \subset Q$ whose $k - 1$ faces all lie in elements of P will be called "special".

For the notation concerning currents we refer to [3].

LEMMA 1. *If f is bounded then $T(B)$ is a rectifiable current whenever B is an \mathcal{L}_k measurable subset of Q .*

Proof. If $q \subset Q$ is a special k -cube then

$$\lim_{j \rightarrow \infty} \int_q |Jf_j(x) - Jf(x)| dx = 0$$

and hence the sequence $\{f_{j\#}(q)\}$ of currents converges weakly to $T(q)$. The supports of the $f_{j\#}(q)$ and $T(q)$ are all contained in a fixed compact set,

$$M(f_{j\#}(q)) \leq \int_q |Jf_j(x)| dx,$$

and

$$M(\partial f_{j\#}(q)) \leq L_{k-1}(f_j | \text{Bdry } q)$$

where M denotes mass in the space of currents. Thus, by [4: 8.14, 8.13], $T(q)$ is an integral current whenever q is special.

Since it is clear that

$$M(T(A)) \leq \int_A |Jf(x)| dx$$

whenever A is an \mathcal{L}_k measurable subset of Q , the lemma follows.

Let $\|T\|$ denote the finite measure over Q defined by letting $\|T\|(A)$ denote the supremum of the numbers $\sum_{j=1}^{\infty} M(T(B_j))$ taken over all countable disjoint collections of \mathcal{L}_k measurable subsets $B_j \subset A$ whenever A is an \mathcal{L}_k measurable subset of Q . Clearly

$$\|T\|(A) \leq \int_A |Jf(x)| dx$$

whenever A is an \mathcal{L}_k measurable subset of Q .

For \mathcal{L}_k almost every $x \in Q$ there is a k -covector ω in R^n with

$|\omega| = 1$ such that $\omega \cdot Jf(x) = |Jf(x)|$ and

$$\overline{\lim}_{r \rightarrow 0+} \frac{\|T\|(B(x, r))}{\alpha(k)r^k} \geq \lim_{r \rightarrow 0+} \frac{T(B(x, r))(\omega)}{\alpha(k)r^k} = |Jf(x)|.$$

It follows that $\|T\|(A) = \int_A |Jf(x)| dx$ whenever A is an \mathcal{L}_k measurable subset of Q .

For each positive integer N let $f_N = (f_N^1, \dots, f_N^n)$ where

$$f_N^i(x) = \begin{cases} N & \text{if } f^i(x) \geq N \\ f^i(x) & \text{if } |f^i(x)| < N \\ -N & \text{if } f^i(x) \leq -N. \end{cases}$$

Then f_N is bounded and $f_N^i \in W_{p_i}^1(Q)$ for $1 \leq i \leq n$. For any measurable set $B \subset Q$ let

$$T_N(B)(\phi) = \int_B \phi(f_N(x)) \cdot Jf_N(x) dx$$

whenever ϕ is an infinitely differentiable k -form on R^n . Note that, if Y is a bounded Borel set in R^n , then, for sufficiently large N , $T_N(B) \lfloor Y = T(B) \lfloor Y$ whenever B is an \mathcal{L}_k measurable subset of Q . Consequently, if Y is a bounded open subset of R^n then $T(B) \lfloor Y$ is rectifiable whenever B is a measurable subset of Q .

Analogous to [2: 2.1 part 3] we have

LEMMA 2. *There exists a countable collection F of k dimensional C^1 submanifolds of R^n such that $\sigma(R^n - \bigcup F) = 0$.*

Proof. Suppose r and ε are positive real numbers and let

$$B(0, r) = R^n \cap \{y: |y| < r\}.$$

Let $\{A_1, \dots, A_m\}$ denote a finite collection of disjoint measurable subsets of $f^{-1}(B(0, r))$ such that $\mathcal{L}_k(f^{-1}(B(0, r)) - \bigcup_{j=1}^m A_j) = 0$ and $\sigma(B(0, r)) - \varepsilon < \sum_{j=1}^m M(T(A_j))$. Note that $T(A_j) = T(A_j) \lfloor B(0, r)$ is rectifiable for $j = 1, \dots, m$. Thus, by [4: 8.16], there exists for each j a countable collection G_j of k -dimensional C^1 submanifolds of R^n such that $\|T(A_j)\|(R^n - \bigcup G_j) = 0$. Letting $G = \bigcup_{j=1}^m G_j$, we have

$$\begin{aligned} \varepsilon &\geq \sigma(B(0, r)) - \sum_{j=1}^m M(T(A_j)) = \sum_{j=1}^m (\|T\|(A_j) - M(T(A_j))) \\ &\geq \sum_{j=1}^m \|T\|(A_j \cap f^{-1}(R^n - \bigcup G_j)) \\ &\geq \sum_{j=1}^m \|T\|(A_j \cap f^{-1}(R^n - \bigcup G)) = \sigma(B(0, r) - \bigcup G) \end{aligned}$$

and the lemma follows.

If μ is a measure over R^n , $y \in R^n$, and $A \subset R^n$ we let

$$\theta^k(\mu, A, y) = \lim_{r \rightarrow 0^+} \frac{\mu(A \cap B(y, r))}{\alpha(k)r^k}$$

whenever the limit exists. In case $A = R^n$ we write $\theta^k(\mu, y)$.

Recall that, if S is a k dimensional rectifiable current in R^n and Y is a Borel set in R^n with $H_n^k(Y) = 0$, the $S \llcorner Y = 0$. Thus σ is absolutely continuous with respect to H_n^k . This fact together with Lemma 2 and the finiteness of σ allow us to conclude using [1: 3.1, 3.2] that:

1. $\theta^k(\sigma, y)$ exists for H_n^k almost every $y \in R^n$.
2. For σ almost every $y \in R^n$ there exists an $M \in F$ such that $y \in M$, $\theta^k(\sigma, y) < \infty$, and $\theta^k(\sigma, R^n - M, y) = 0$.
3. $\int_{R^n} \theta^k(\sigma, y) dH_n^k y \leq \sigma(R^n)$.

A proof of the following statement concerning rectifiable currents can be found in [2: 2.1 part 7]: If S is a rectifiable k dimensional current in R^n , M is a k dimensional C^1 submanifold of R^n ,

$$y \in M - \text{spt } \partial S,$$

$\theta^k(\|S\|, y) < \infty$, $\theta^k(\|S\|, R^n - M, y) = 0$, and P is an oriented k plane tangent to M at y , then there exists a unique integer m such that

$$\lim_{r \rightarrow 0^+} \frac{1}{\alpha(k)r^k} F[S \llcorner B(y, r) - m(P \cap B(y, r))] = 0$$

where F denotes the flat norm [4: 3.2].

Now suppose q is a special k -cube in Q and $y \in R^n$. If there is an $M \in F$ with $y \in M - f(\text{Bdry } q)$, $\theta^k(\sigma, y) < \infty$, and

$$\theta^k(\sigma, R^n - M, y) = 0,$$

let P denote an oriented k -plane tangent to M at y , let $m(q, y)$ denote the integer such that

$$\lim_{r \rightarrow 0^+} \frac{1}{\alpha(k)r^k} F[T(q) \llcorner B(y, r) - m(q, y) (P \cap B(y, r))] = 0$$

and set $\alpha(q, y) = m(q, y) \zeta(y)$ where $\zeta(y)$ is the simple unit k -vector orienting P . Otherwise set $\alpha(q, y) = 0$.

Then, for H_n^k almost every $y \in R^n$,

$$\theta^k(T(q) \llcorner \phi, y) = \lim_{r \rightarrow 0} \frac{[T(q) \llcorner B(y, r)](\phi)}{\alpha(k)r^k} = \phi(y) \cdot \alpha(q, y)$$

whenever ϕ is an infinitely differentiable k -form in R^n . Consequently $T(q)(\phi) = \int_{R^n} \phi(y) \cdot \alpha(q, y) dH_n^k y$ whenever ϕ is an infinitely differentiable k -form and hence

$$M(T(q)) \leq \int_{R^n} |\alpha(q, y)| dH_n^k y$$

whenever q is a special k -cube.

For $y \in R^n$ let $N(y)$ denote the supremum of the numbers $\sum_{q \in G} |\alpha(q, y)|$ taken over all finite collections G of nonoverlapping special k -cubes in Q .

Suppose $N(y) \neq 0$ and G is a finite collection of nonoverlapping special k -cubes such that $\alpha(q, y) \neq 0$ for $q \in G$. Let ω denote a k -covector with $|\omega| = 1$ and $\omega \cdot \zeta(y) = 1$. Then

$$\begin{aligned} \sum_{q \in G} |\alpha(q, y)| &= \sum_{q \in G} |\theta^k(T(q) \lrcorner \omega, y)| \\ &= \lim_{r \rightarrow 0} \sum_{q \in G} \frac{|[T(q) \lrcorner B(y, r)](\omega)|}{\alpha(k)r^k} \\ &\leq \theta^k(\sigma, y). \end{aligned}$$

Thus $N(y) \leq \theta^k(\sigma, y)$ for H_n^k almost every $y \in R^n$.

On the other hand, if G is any finite collection of nonoverlapping special k -cubes,

$$\sum_{q \in G} M(T(q)) \leq \int_{R^n} \sum_{q \in G} |\alpha(q, y)| dH_n^k y.$$

The supremum of the numbers $\sum_{q \in G} M(T(q))$ taken over all finite collections G of nonoverlapping special k -cubes is readily seen to be $\sigma(R^n)$ and hence

$$\sigma(R^n) \leq \int_{R^n} N(y) dH_n^k y \leq \int_{R^n} \theta^k(\sigma, y) dH_n^k y \leq \sigma(R^n).$$

Thus $N(y) = \theta^k(\sigma, y)$ H_n^k almost everywhere and

$$\int_{R^n} N(y) dH_n^k y = \int_Q |Jf(x)| dx.$$

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