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The notion of terminal subcontinuum of a continuum is introduced as a generalization of the idea of terminal point and is used to study the structure of a large class $\mathscr{M}$ of hereditarily unicoherent Hausdorff continua. The class $\mathscr{M}$ contains all hereditarily unicoherent metric continua and all hereditarily decomposable, hereditarily unicoherent Hausdorff continua. The major result is that every member of $\mathscr{M}$ is irreducible about the union of its indecomposable terminal subcontinua. The known result that a hereditarily decomposable, hereditarily unicoherent Hausdorff continuum is irreducible about its terminal points is a corollary.

Introduction. That a dendrite is irreducible about its end points is a classical result. Miller generalized this by proving that a hereditarily decomposable, hereditarily unicoherent metric continuum is irreducible about its terminal points [8]. As she observed, this theorem is false for hereditarily unicoherent continua. In fact there exists a hereditarily unicoherent metric continuum containing no indecomposable subcontinuum with interior which has no terminal points (see Example 3, § 4).

The purpose of this paper is to extend Miller's definition of terminal point to that of terminal subcontinuum and to prove that every hereditarily unicoherent metric continuum is irreducible about the union of its indecomposable terminal subcontinua. Actually this result is proved for a class of hereditarily unicoherent Hausdorff continua which includes all hereditarily unicoherent metric continua and all hereditarily decomposable, hereditarily unicoherent Hausdorff continua. Miller's theorem and its generalization to the Hausdorff setting (see [5]) follow as immediate corollaries.

Fugate has given a different definition of terminal subcontinuum [2] and has used it to study chainable metric continua [2], [3]. We justify our new notion of terminal subcontinuum by proving that the two definitions are equivalent for chainable metric continua.

1. Definitions and preliminary remarks. A *continuum* is a compact, connected Hausdorff space. A continuum is *hereditarily unicoherent* if the intersection of any two of its subcontinua is connected.

NOTATION. Throughout this paper the letter $\mathscr{M}$ will denote the class of all continua M such that

(i) M is hereditarily unicoherent, and

(ii) every indecomposable subcontinuum of M is irreducible.

REMARK. Condition (ii) insures that if M is in $\mathscr{M}$, then each irreducible subcontinuum of M is contained in a maximal irreducible subcontinuum [5]. This fact is crucial in several of the proofs. It is not known if there exists an indecomposable continuum which is not irreducible, or equivalently, an indecomposable continuum with exactly one composant. Thus the class $\mathscr{M}$ may contain all hereditarily unicoherent continua. In any case, it contains all metric hereditarily unicoherent continua and all hereditarily decomposable, hereditarily unicoherent continua.

If A and B are subsets of a hereditarily unicoherent continuum H, then $\langle A, B\rangle$ will denote the unique subcontinuum which is irreducible about $A \cup B$. In particular, $\langle a, b\rangle$ denotes the unique irreducible subcontinuum from the point a to the point b.

DEFINITION. A subcontinuum H of a continuum M in $\mathscr{M}$ is said to be *terminal* if (i) H is contained in an irreducible subcontinuum of M and (ii) every irreducible subcontinuum containing H is of the form $\langle H, x\rangle$ for some x in M.

Observe that if H is a point, then the above definition is equivalent to that given by Miller [8] for a terminal point.

The reader is referred to [6] for definitions of undefined terms and for a general discussion of continua.

2. Indecomposable terminal subcontinua. In this section it is shown that the indecomposable terminal subcontinua of members of $\mathscr{M}$ possess certain properties analogous to properties of terminal points of hereditarily decomposable, hereditarily unicoherent continua. Most importantly, each member of $\mathscr{M}$ is irreducible about the union of its indecomposable terminal subcontinua.

A subset H of a continuum K is said to *cut* K if there exist points p and q of K such that each subcontinuum which contains p and q meets H.

The following theorem is analogous to Theorem 3.1 of [8]. The proof is obvious.

THEOREM 2.1 *Let M be in $\mathscr{M}$. If N is an irreducible subcontinuum of M, then either N cuts M or N is a terminal subcontinuum.*

DEFINITION. If the continuum X is irreducible from a to some other point, let $E(a)$ denote the set

$$\{x \in X;\ X \text{ is irreducible from } a \text{ to } x\} .$$

LEMMA 2.1. *Let the continuum X be irreducible from a to some other point. The following are true.*

(i) *$E(a)$ is connected.*

(ii) *If X is decomposable, then $\mathrm{cl}\,(E(a))$ is a proper subcontinuum of X.*

(iii) *If $\mathrm{cl}\,(E(a))$ is decomposable, then $E(a)$ is closed.*

Proof. (i) The set $E(a)$ is precisely the complement of the a-composant of X, which is known to be connected ([4] and [7] contain proofs for Hausdorff and metric continua respectively).

(ii) If $X = A \cup B$ is a decomposition with $a \in A$, then $\mathrm{cl}\,(E(a)) \subseteqq B$.

(iii) Suppose that $E(a)$ is not closed. Choose c in $\mathrm{cl}\,(E(a)) \backslash E(a)$ and let I be a subcontinuum irreducible from a to c. Let $\mathrm{cl}\,(E(a)) = P \cup Q$ be a decomposition with $c \in P$. Now $I \cup P$ is a proper subcontinuum of X which contains a and intersects $E(a)$. This is a contradiction.

LEMMA 2.2. *Suppose that M is in $\mathscr{M}$ and that $M = \langle K, x\rangle$ where K is an indecomposable subcontinuum. Then M is irreducible between x and some point of K.*

Proof. Assume that M is not irreducible between x and some point of K. Since M is in $\mathscr{M}$, $K = \langle p, q\rangle$ for some p and q in M. Now $\langle p, x\rangle \cap K$ is a subcontinuum of the p-composant of K and $\langle q, x\rangle \cap K$ is a subcontinuum of the q-composant of K. Since these composants are disjoint [6], it follows that $K \cap (\langle p, x\rangle \cup \langle q, x\rangle)$ is not connected. This contradicts the hereditary unicoherence of M.

COROLLARY 2.1. *Let M be in $\mathscr{M}$. If N is a proper maximal irreducible subcontinuum of M, then N is decomposable.*

Lemma 2.3 and Lemma 2.4 below are analogous to Theorem 3.2 and Theorem 3.3 respectively of [8]. They can be proved by similar methods.

LEMMA 2.3. *Suppose that $\langle a, b\rangle$ is a maximal irreducible subcontinuum of a hereditarily unicoherent continuum H, and that A is a subcontinuum of $\langle a, b\rangle$ containing a. If $\langle p, q\rangle$ is an irreducible subcontinuum of H which contains A but is not of the form $\langle A, x\rangle$ for any x in H, then $\langle p, q\rangle \subseteqq \langle a, b\rangle$.*

COROLLARY 2.2. *Let M be in $\mathscr{M}$. A subcontinuum N of M is* *terminal if and only if N is a terminal subcontinuum of a maximal irreducible subcontinuum of M.*

Proof. If N is terminal in M, then N is also terminal in any subcontinuum containing N.

Suppose that N is a terminal subcontinuum of a maximal irreducible subcontinuum $\langle a, b\rangle$. Then, either $N \cap E(a) \neq \varnothing$ or $N \cap E(b) \neq \varnothing$. Thus we may assume without loss of generality that $a \in N$. The result now follows immediately from Lemma 2.3.

LEMMA 2.4. *Suppose that H is a hereditarily unicoherent continuum, that $H = \langle a, b\rangle$, and that A is a proper subcontinuum of H containing a. If $\langle p, q\rangle$ is an irreducible subcontinuum of H which contains A but is not of the form $\langle A, x\rangle$ for any x in H, then $\langle p, q\rangle \subseteqq E(b)$.*

COROLLARY 2.3. *If H is a hereditarily unicoherent continuum which is irreducible from a to some other point, and A is a subcontinuum containing $E(a)$, then A is a terminal subcontinuum.*

THEOREM 2.2. *If M is in $\mathscr{M}$, then M contains an indecomposable terminal subcontinuum.*

Proof. Assume that M contains no indecomposable terminal subcontinua. We will construct an arbitrarily long transfinite sequence $N_1 \supset N_2 \supset \cdots \supset N_\lambda \supset \cdots$ of terminal subcontinua of M such that every inclusion is proper.

We begin by defining N_1. Let $I_1 = \langle a_1, b_1\rangle$ be a maximal irreducible subcontinuum of M. According to Corollary 2.1, I_1 is decomposable. By Lemma 2.1, $\mathrm{cl}\,(E(b_1))$ is proper in I_1 and, by Corollary 2.3, $\mathrm{cl}\,(E(b_1))$ is a terminal subcontinuum of I_1. Hence, by Corollary 2.2, $\mathrm{cl}\,(E(b_1))$ is a terminal subcontinuum of M. By assumption $\mathrm{cl}\,(E(b_1))$ is decomposable. It follows from Lemma 2.1 that $\mathrm{cl}\,(E(b_1)) = E(b_1)$. Define N_1 to be the terminal subcontinuum $E(b_1)$.

Next we define N_β for an arbitrary ordinal β. Suppose that N_α is defined for each $\alpha < \beta$. If β is a nonlimit ordinal, let $I_\beta = \langle a_\beta, b_\beta\rangle$ be a maximal irreducible subcontinuum of $N_{\beta-1}$ and define N_β to be $E(b_\beta)$. If β is a limit ordinal, let $N_\beta = \bigcap \{N_\alpha; \alpha < \beta\}$.

It must be verified inductively that for a fixed ordinal β, $\{N_\alpha; \alpha < \beta\}$ is a strictly decreasing transfinite sequence of terminal subcontinua of M. We will indicate the procedures involved by proving that N_2 and N_ω (ω denotes the first infinite ordinal) have the desired properties.

The set N_2 is by definition $E(b_2)$, where $I_2 = \langle a_2, b_2\rangle$ is some maximal irreducible subcontinuum of $N_1 = E(b_1)$. Since N_1 is decomposable, it follows from Corollary 2.1 that I_2 is decomposable. An argument like that used for N_1 shows that $N_2 = E(b_2) = \mathrm{cl}\,(E(b_2))$ is a terminal subcontinuum of I_2 and hence of N_1 (Corollary 2.2). Suppose that N_2 is not terminal in M. We will assume that $a_1 \in N_2$ (since $N_2 \subsetneqq E(b_1)$). By definition, there exists an irreducible subcontinuum $\langle p, q\rangle$ containing N_2 such that $\langle p, q\rangle \neq \langle N_2, x\rangle$ for any $x \in M$. According to Lemma 2.3, $\langle p, q\rangle \subsetneqq I_1$. Thus, by Lemma 2.4, $\langle p, q\rangle \subsetneqq E(b_1) = N_1$. Consequently N_2 is not terminal in N_1 which is a contradiction.

A straight-forward induction argument (analogous to the preceding proof for $i = 2$) shows that $\{N_i; i < \omega\}$ is a strictly decreasing sequence of terminal subcontinua of M.

Since $N_\omega = \bigcap \{N_i; i < \omega\}$, it follows that N_ω is a proper subcontinuum of each N_i. Suppose that N_ω is not terminal in M. We will assume that $a_i \in N_\omega$ for each $i < \omega$ (since $N_\omega \subsetneqq E(b_i)$). By definition, there exists an irreducible subcontinuum $\langle p, q\rangle$ such that $N_\omega \subsetneqq \langle p, q\rangle$ and $\langle p, q\rangle \neq \langle N_\omega, x\rangle$ for any $x \in M$. As in the discussion about N_2, it follows that $\langle p, q\rangle \subsetneqq N_1$. Applying this argument inductively, we find that $\langle p, q\rangle \subsetneqq N_i$ for all $i < \omega$. Consequently $\langle p, q\rangle \subsetneqq \bigcap N_i = N_\omega$, which is contradiction. Hence N_ω is a terminal subcontinuum of M.

It is now clear how to verify inductively that for an arbitrary ordinal β, $\{N_\alpha; \alpha < \beta\}$ is a strictly decreasing transfinite sequence of terminal subcontinua of M. Choosing an ordinal β with cardinal number strictly larger than the cardinal number of M, we obtain a contradiction.

COROLLARY 2.4. *If M is in $\mathscr{M}$ and M is irreducible, then M contains indecomposable terminal subcontinua A and B such that $\langle A, B\rangle = M$.*

Proof. Let $M = \langle a, b\rangle$ and assume that M is decomposable. Then $\mathrm{cl}\,(E(a))$ and $\mathrm{cl}\,(E(b))$ are proper subcontinua. If $\mathrm{cl}\,(E(a))$ (respectively $\mathrm{cl}\,(E(b))$ is indecomposable, then let $A = \mathrm{cl}\,(E(a))$ (respectively $B = \mathrm{cl}\,(E(b))$. Otherwise, by the proof of Theorem 2.2, $E(a)$ (respectively $E(b)$) contains an indecomposable terminal subcontinuum A (respectively B). Clearly $\langle A, B\rangle = M$.

COROLLARY 2.5. *If M is in $\mathscr{M}$, then M is irreducible about the union of its indecomposable terminal subcontinua.*

Proof. Suppose that N is a proper subcontinuum of M which contains all of the indecomposable terminal subcontinua. Choose $p \in M\backslash N$ and let I be a maximal irreducible subcontinuum of M containing p. By Corollary 2.4, $I = \langle A, B \rangle$ where A and B are indecomposable terminal subcontinua of I. Consequently A and B are indecomposable terminal subcontinua of M and $A \cup B \subseteqq N$. Since M is hereditarily unicoherent, we have $I \subseteqq N$, which is a contradiction.

COROLLARY 2.6. (*Miller*) *A hereditarily decomposable, hereditarily unicoherent continuum is irreducible about its terminal points.*

REMARKS. It is tempting to conjecture that a terminal subcontinuum of a continuum in $\mathscr{M}$ must contain a minimal terminal subcontinuum, or, at least, an indecomposable terminal subcontinuum. Example 1, § 4 shows that both conjectures are false.

It is known that a hereditarily decomposable, hereditarily unicoherent continuum is not separated by any subset of its terminal points [5], [8]. There seems to be no satisfactory analogue to this theorem for continua belonging to $\mathscr{M}$. There is a hereditarily unicoherent metric continuum which is separated by a single indecomposable terminal subcontinuum (Example 3, § 4), and another which is separated by a subset of its terminal points (Example 2, § 4). It is easy to verify that no continuum is separated by a single terminal point.

3. Terminal subcontinua of atriodic continua.

In this section we characterize the terminal subcontinua of the atriodic members of $\mathscr{M}$. As a corollary of the characterization, it follows that our definition of terminal subcontinuum is equivalent to Fugate's [2] for chainable metric continua.

A continuum K is a *triod* if there exists a proper subcontinuum H of K such that $K \backslash H$ is the union of three mutually disjoint open sets. A continuum which contains no triods is said to be *atriodic*. A continuum is said to be a *type* 1 *triod* [9] if it is the union of three subcontinua which have a point in common and such that no one of them is a subset of the union of the other two.

The following lemma, proved for metric continua in [9], is easily seen to be valid for arbitrary continua.

LEMMA 3.1. *Every unicoherent type* 1 *triod is a triod.*

The next lemma is a partial generalization of Theorem 2.1 of [8] to the Hausdorff setting.

LEMMA 3.2. *Let M be in $\mathscr{M}$. Then M is atriodic if and only if each nondegenerate subcontinuum of M is irreducible.*

Proof. If each nondegenerate subcontinuum of M is irreducible, then M is clearly atriodic. Suppose that M is atriodic and let N be a nondegenerate subcontinuum. According to [5], N contains a maximal irreducible subcontinuum, say $\langle a, b\rangle$. If $c \in N\backslash\langle a, b\rangle$, then $a \notin \langle c, b\rangle$ and $b \notin \langle c, a\rangle$. Thus no point of $\{a, b, c\}$ cuts between the other two in N. This contradicts Lemma 5.1 of [5].

THEOREM 3.1. *Let M be an atriodic member of $\mathscr{M}$. Then the subcontinuum N of M is terminal if and only if* (*) *for each pair K and L of subcontinua of M which intersect N, either $K \subseteqq N \cup L$ or $L \subseteqq N \cup K$.*

Proof. Suppose that N is a terminal subcontinuum of M and that condition (*) fails. There exist subcontinua K and L of M, intersecting N, such that $K \not\subseteq N \cup L$ and $L \not\subseteq N \cup K$. Choose $k \in K\backslash(N \cup L)$, $l \in L\backslash(N \cup K)$, and $n \in N$. According to Lemma 5.1 of [5], some point of $\{k, l, n\}$ cuts between the other two. Consequently $n \in \langle l, k\rangle$. Now N is not contained in $\langle l, k\rangle$ since N is terminal. Thus $\langle n, k\rangle \cup \langle n, l\rangle \cup N$ is a type 1 triod. This contradicts Lemma 3.1 and the fact that M is atriodic.

Suppose that N is not terminal. By Lemma 3.2, N is contained in an irreducible subcontinuum, hence there exists an irreducible subcontinuum $\langle p, q\rangle$ containing N which is not of the form $\langle N, x\rangle$ for any $x \in M$. The subcontinua $K = \langle N, p\rangle$ and $L = \langle N, q\rangle$ do not satisfy condition (*).

Fugate [2] takes condition (*) of Theorem 3.1 as the definition of terminal subcontinuum. Since every chainable metric continuum is atriodic and a member of $\mathscr{M}$, it follows that the two definitions of terminal subcontinuum are equivalent for chainable metric continua. It is perhaps worth noting that the two definitions need not agree for irreducible continua in $\mathscr{M}$ (see Example 1, § 4).

4. Examples.

EXAMPLE 1. Let T denote a simple triod and let M be any metric compactification of a half-ray having T as remainder. Then M is clearly an irreducible, hereditarily unicoherent, hereditarily decomposable metric continuum. Let T' denote a subtriod of T which misses the end points of T. Then T' is a terminal subcontinuum of M, since every irreducible subcontinuum containing T' also contains T. Clearly T' does not satisfy the condition (*) of Theorem 3.1. Consequently T' is not a terminal subcontinuum according to Fugate's definition [2]. Also T' contains no minimal terminal subcontinuum of M and no terminal point of M. Thus T' contains no indecomposable terminal subcontinuum of M.

EXAMPLE 2. Let M denote the pseudo-arc. Then, by [1], M is a hereditarily unicoherent metric continuum, each of whose points is a terminal point. It follows that M contains a subset of terminal points which separates.

EXAMPLE 3. Let S denote the dyadic solenoid. Then S is an indecomposable, hereditarily unicoherent metric continuum, each proper subcontinuum of which is an arc. Clearly S contains no proper terminal subcontinua (hence no terminal points). Now let C denote the Cantor set, and define $M = S \times C/\{s\} \times C$ for some fixed $s \in S$. The continuum M is a hereditarily unicoherent metric continuum which contains no indecomposable subcontinua with interior and no terminal points. Furthermore, each copy of S in M is an indecomposable terminal subcontinuum of M which separates M.

To obtain a simple example containing an indecomposable terminal subcontinuum which separates, let $A = [0, 1]$, fix s in the dyadic solenoid S, and define M to be the continuum obtained by identifying 1/2 with s. Then M is a hereditarily unicoherent metric continuum with precisely three indecomposable terminal subcontinua, namely S, $\{0\}$, and $\{1\}$. Notice that M is not irreducible about any two of these terminal subcontinua and that S separates M.

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