THE PRODUCT OF $F$-SPACES WITH $P$-SPACES

NEIL HINDMAN
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A condition on a basically disconnected space $X$ is known which is necessary and sufficient for the product space $X \times Y$ to be basically disconnected for every P-space $Y$. This same condition, when applied to an $F'$-space $X$, guarantees that $X \times Y$ is an $F'$-space whenever $Y$ is a $P$-space and is necessary for this result. The principal result of this paper establishes that this condition is not sufficient when applied to $F$-spaces. A condition which is sufficient but not necessary is also derived.

1. Introduction. The notation and general point of view are those of the Gillman and Jerison textbook [5]. In particular, all hypothesized spaces are completely regular Hausdorff. The reader should recall from [4] the following characterizations. A space $X$ is:

- a $P$-space if and only if each cozero set is closed;
- a basically disconnected space if and only if each cozero set has open closure;
- a $U$-space if and only if disjoint cozero sets can be separated by an open-and-closed set;
- an $F$-space if and only if disjoint cozero sets can be completely separated;
- and an $F''$-space if and only if disjoint cozero sets have disjoint closures. It is clear from these characterizations that the conditions named grow progressively weaker.

In [3] Gillman asked for a necessary and sufficient condition that a product of two spaces be an $F'$-space and, parenthetically, for a necessary and sufficient condition that a product of two spaces be a basically disconnected space. Curtis had shown [2] that if $X \times Y$ is an $F'$-space then either $X$ or $Y$ must be a $P$-space. It is easily seen that if $X \times Y$ has any of the properties listed above so must both $X$ and $Y$ for $X$ and $Y$ nonempty. Observing also that the product of a space $X$ with a discrete space $Y$ has any of the above mentioned properties which $X$ has, one can rephrase the question in the form: For which spaces $X$ with property $A$ does the product $X \times Y$ have property $A$ for every $P$-space $Y$?

This question was answered for the properties $F''$ and basically disconnected in [1]. The condition was that the space be countably locally weakly Lindelöf (abbreviated CLWL). That is, for every countable collection $\{I_n\}_{n=1}^\infty$ of open covers of $X$ and each point $x$ of $X$ there must be a neighborhood $V$ of $x$ and, for each $n$, a countable subfamily $A_n$ of $I_n$ such that $V \subseteq cl \cup A_n$.

Since $F$-spaces are $F''$-spaces the condition that $X$ be CLWL is
clearly necessary for \( X \times Y \) to be an \( F \)-space for each \( P \)-space \( Y \). The obvious question, since basically disconnected spaces are \( F \)-spaces, is whether that condition is also sufficient [1, 4.7]. It is shown in §3 the answer is no. That is, there are a CLWL \( F \)-space \( X \) and a \( P \)-space \( Y \) such that \( X \times Y \) is not an \( F \)-space.

In §2 sufficient conditions for the product of two spaces to be an \( F \)-space are derived. The same conditions suffice when "\( F \)-space" is replaced by "\( U \)-space" throughout.

2. Conditions guaranteeing that a product space is an \( F \)-space. We shall have need of the following lemma from [1, 3.2].

**Lemma 2.1.** Let \( f \in C^*(X \times Y) \), where \( X \) is CLWL and \( Y \) is a \( P \)-space. If \((x_0, y_0) \in X \times Y\), then there is a neighborhood \( U \times V \) of \((x_0, y_0)\) such that \( f(x, y) = f(x, y_0) \) whenever \((x, y) \in U \times V\).

It is shown in [6] that this in fact characterizes CLWL spaces in the sense that if \( X \) is not CLWL then there is some \( P \)-space \( Y \) such that the conclusion of Lemma 2.1 fails. (The proof is a slight modification of the "necessity" proof in [1, 3.3].)

**Definition 2.2.** A point \( x \) of \( X \) is a basically disconnected point of \( X \) if whenever \( U \) is a cozero set of \( X \) and \( x \in \text{cl} \, U \) then in fact \( x \in \text{int} \, \text{cl} \, U \).

It is clear that \( X \) is basically disconnected if and only if every point of \( X \) is a basically disconnected point. The proof of the following lemma can be taken verbatim from [1, 3.4].

**Lemma 2.3.** If \( X \) is CLWL and \( x \) is a basically disconnected point of \( X \) and \( Y \) is a \( P \)-space then \((x, y)\) is a basically disconnected point of \( X \times Y \) for every \( y \) in \( Y \).

The reader should recall that a space \( X \) is weakly Lindelöf if each open cover of \( X \) has a countable subfamily whose union is dense in \( X \).

**Theorem 2.4.** If \( X \) is a CLWL \( F \)-space (respectively \( U \)-space) and there is a weakly Lindelöf subspace \( D \) of \( X \) such that every point of \( X \setminus D \) is a basically disconnected point and if \( Y \) is a \( P \)-space then \( X \times Y \) is an \( F \)-space (respectively \( U \)-space).

**Proof.** Let \( f \in C^*(X \times Y) \). By Theorem 3.3 of [1] \( X \times Y \) is an \( F' \)-space so \( \text{cl} \, \text{pos} \, f \cap \text{cl} \, \text{neg} \, f = \emptyset \). (Here \( \text{pos} \, f = \{(x, y) : f(x, y) > 0\} \) and \( \text{neg} \, f = \{(x, y) : f(x, y) < 0\} \).) To show that \( X \times Y \) is an \( F \)-space it suffices to show that \( \text{pos} \, f \) and \( \text{neg} \, f \) can be completely separated.
Define an equivalence relation on \( Y \) by agreeing that \( y_1 \sim y_2 \) if and only if the following three conditions hold for every \( x \in D \): (1) \( f(x, y_1) = f(x, y_2) \); (2) \( (x, y_i) \in \text{cl pos } f \) if and only if \( (x, y_2) \in \text{cl pos } f \); and (3) \( (x, y_i) \in \text{cl neg } f \) if and only if \( (x, y_2) \in \text{cl neg } f \). It is clear that \( \sim \) is an equivalence relation. Let \( \Gamma \) be the set of \( \sim \) equivalence classes.

We claim that each element \( V \) of \( \Gamma \) is open. To see this let \( V \in \Gamma \) and \( y_0 \in V \). For each \( x \) in \( D \) there is a neighborhood \( U_x \times V \) of \( (x, y_0) \) such that \( f(x', y') = f(x', y_0) \) whenever \( (x', y') \in U_x \times V \), by Lemma 2.1. Further, since \( X \) and \( Y \) are completely regular, \( U_x \) and \( V \) may be chosen to be cozero sets in \( X \) and \( Y \). Now \( \{ U_x : x \in D \} \) is an open cover of \( D \) so there exists a countable subset \( \{ x(n) \}^\infty_{n=1} \) of \( D \) such that \( D \subseteq \text{cl } \bigcup_{n=1}^\infty U_{x(n)} \). Let \( V_0 = \bigcap_{n=1}^\infty V_{x(n)} \). Since \( Y \) is a P-space \( V_0 \) is a neighborhood of \( y_0 \). We claim that \( V_0 \subseteq V \), and hence that \( V \) is open as desired. To see this, let \( y_1 \in V \). We will show that \( y_1 \sim y_0 \). To see that condition (1) holds suppose instead that \( f(x, y_1) \neq f(x, y_0) \) for some \( x \in D \). Without loss of generality we may assume that \( f(x, y_1) < f(x, y_0) \) so that there exist neighborhoods \( U' \times V' \) of \( (x, y_0) \) and \( U'' \times V'' \) of \( (x, y_1) \) such that \( f(x', y') < f(x'', y'') \) whenever \( (x', y') \in U' \times V' \) and \( (x'', y'') \in U'' \times V'' \). Now \( U' \cap U'' \) is a neighborhood of \( x \), a point of \( D \), so there is some \( n \) and some \( \bar{x} \) such that \( \bar{x} \in U_{x(n)} \cap U' \cap U'' \). Now \( (\bar{x}, y_1) \in U_{x(n)} \times V_{x(n)} \) and \( (\bar{x}, y_0) \in U_{x(n)} \times V_{x(n)} \) so \( f(\bar{x}, y_1) = f(\bar{x}, y_0) \). But \( (\bar{x}, y_0) \in U' \times V' \) and \( (\bar{x}, y_1) \in U'' \times V'' \) so \( f(\bar{x}, y_0) < f(\bar{x}, y_1) \), a contradiction.

To see that condition (2) holds suppose instead that there is some \( x \) in \( D \) such that either \( (x, y_1) \in \text{cl pos } f \) and \( (x, y_0) \not\in \text{cl pos } f \) or \( (x, y_1) \not\in \text{cl pos } f \) and \( (x, y_0) \in \text{cl pos } f \). Suppose that the former case holds. Then there is a neighborhood \( U' \times V' \) of \( (x, y_0) \), where \( U' \) and \( V' \) are cozero sets, such that \( (U' \times V') \cap \text{pos } f = \varnothing \). For each \( n \) in \( N \) let \( U_n = U_{x(n)} \cap U' \). Then \( U_n \) is a cozero set in \( X \) so \( \bigcup_{n=1}^\infty U_n \) is a cozero set of \( X \). Also, since \( Y \) is a P-space, \( \bigcap_{n=1}^\infty V_{x(n)} = V_0 \) is a cozero set. Therefore, \( \bigcup_{n=1}^\infty U_n \times V_0 \) is the cozero set of some continuous function on \( X \times Y \), say \( \bigcup_{n=1}^\infty U_n \times V_0 = \text{coz } g \). Further, if \( (x', y') \in \bigcup_{n=1}^\infty U_n \times V_0 \) then \( f(x', y') = f(x', y_0) \), since \( (x', y') \in U_{x(n)} \times V_{x(n)} \) for some \( n \), and \( f(x', y_0) \not\leq 0 \) since \( (x', y_0) \in U' \times V' \). Thus \( f(x', y') \not\leq 0 \) and \( \text{coz } g \) and \( \text{pos } f \) are disjoint cozero sets. Consequently \( \text{cl } \text{coz } g \cap \text{cl pos } f = \varnothing \) and so there is a neighborhood \( U'' \times V'' \) of \( (x, y_0) \) which misses \( \text{coz } g \). But \( U'' \cap U' \) is a neighborhood of \( x \), an element of \( D \), so that there is some \( n \) and some \( \bar{x} \) such that \( \bar{x} \in U_{x(n)} \cap U' \cap U'' \). Now \( (\bar{x}, y_1) \in U_n \times V_0 \) so \( (\bar{x}, y_1) \in \text{coz } g \) while \( (\bar{x}, y_0) \in U'' \times V'' \) so \( (\bar{x}, y_1) \not\in \text{coz } g \), a contradiction. By interchanging \( y_1 \) and \( y_0 \) in the above argument one sees that it is also impossible to have \( (x, y_0) \) in \( \text{cl pos } f \) while \( (x, y_1) \not\in \text{cl pos } f \).

One also sees in an identical fashion that condition (3) holds. Thus
Let \((x, y) \in X \times Y\). If \(x \in D\) then \(x\) is a basically disconnected point of \(X\) and so, by Lemma 2.3, \((x, y)\) is a basically disconnected point of \(X \times Y\). Consequently each of \(g, h_1,\) and \(h_2\) are continuous at \((x, y)\), and so \(k\) is continuous at \((x, y)\). If \(x \in D\) and \((x, y) \notin \text{cl pos } f \cup \text{cl neg } f\) then there is a neighborhood of \((x, y)\) on which \(k\) agrees with the continuous function \(g\) so that \(k\) is continuous at \((x, y)\). If \(x \notin D\) and \((x, y) \notin \text{cl pos } f\) then \((x, y, v) \in \text{cl pos } f\) where \(V\) is the member of \(\Gamma\) in which \(\#\) lies. Therefore, \(g(x, y) = g_v(x) = 1\). Let \(\varepsilon > 0\) be given. Then there is a neighborhood \(U\) of \(x\) on which \(g_r > 1 - \varepsilon\). Let \(U' \times V'\) be a neighborhood of \((x, y)\) which misses \(\text{cl neg } f\). Then \((U \cap U') \times (V \cap V')\) is a neighborhood of \((x, y)\) on which \(k > 1 - \varepsilon\) and hence \(k\) is continuous at \((x, y)\). Similarly if \(x \in D\) and \((x, y) \in \text{cl neg } f\) and \(\varepsilon > 0\) one can find a neighborhood of \((x, y)\) on which \(k < \varepsilon\). Thus \(k\) is continuous and hence \(X \times Y\) is an \(F\)-space.

To prove the parenthetical theorem it is only necessary to note that if \(X\) is a \(U\)-space one can choose the functions \(g_r\) in the above argument to assume only the values 0 and 1. (The characteristic function of an open-and-closed set is continuous.) Consequently the function \(k\) assumes only the values 0 and 1 and the set \(A = \{(x, y) : k(x, y) = 0\}\) is an open-and-closed set containing \(\text{neg } f\) and missing \(\text{pos } f\). Thus \(X \times Y\) is a \(U\)-space.

Corollary 2.5 is the strongest result we have been able to obtain. It is shown in Example 3.2 that the conditions given on \(X\) are still not necessary in order for its product with every \(P\)-space to be an \(F\)-space.

**Corollary 2.5.** If \(X\) is a CLWL \(F\)-space (respectively \(U\)-space), and there are a subset \(D\) of \(X\) and a partition \(\Delta\) of \(X\) into open-and-closed sets such that every point of \(X \setminus D\) is a basically disconnected point and \(U \cap D\) is weakly Lindelöf for each \(U\) in \(\Delta\), and if \(Y\) is a \(P\)-space, then \(X \times Y\) is an \(F\)-space (respectively \(U\)-space).
**Proof.** Let \( f \in C^*(X \times Y) \) and for each \( U \) in \( A \) let \( k_U \in C^*(U \times Y) \) such that \( k_U = 1 \) on \( \text{pos } f \cap (U \times Y) \) and \( k_U = 0 \) on \( \text{neg } f \cap (U \times Y) \). Define \( k \) in \( C^*(X \times Y) \) by the rule \( k(x, y) = k_U(x, y) \) where \( x \in U \). The parenthetical statement is similarly proved.

Corollary 2.6 appears in [6] and Corollary 2.7 appears in [8].

**Corollary 2.6.** If \( X \) is a weakly Lindelöf \( F \)-space (respectively \( U \)-space) and \( Y \) is a \( P \)-space then \( X \times Y \) is an \( F \)-space (respectively \( U \)-space).

**Corollary 2.7.** If \( X \) is a compact \( F \)-space and \( Y \) is a \( P \)-space then \( X \times Y \) is an \( F \)-space.

3. **Examples.** The first example establishes that the condition that a \( U \)-space be CLWL is not sufficient to guarantee that its product with each \( P \)-space is an \( F \)-space.

**Examples 3.1.** A CLWL \( U \)-space \( X \) and a \( P \)-space \( Y \) such that \( X \times Y \) is not an \( F \)-space.

Let \( \omega_2 + 1 \) have the order topology and let \( D = \{ \sigma \in \omega_2 + 1: \sigma \) is not the supremum of countably many predecessors\} with the relative topology from \( \omega_2 + 1 \). (The space \( D \) differs from the space of [5, 9L] only by the inclusion of the endpoint, \( \omega_2 \).) Since we have deleted all non-\( P \)-points of \( \omega_2 + 1 \) we have that \( D \) is a \( P \)-space. Following the hints in [5, 9L] one easily sees that elements of \( C^*(D \setminus \{\omega_2\}) \) are constant on a tail.

Let \( p \) be a free ultrafilter on \( N \), the set of natural numbers. Let \( E = N \cup (\omega_3 + 1) \) where every point of \( E \) is isolated except \( \omega_3 \). Let basic neighborhoods of \( \omega_3 \) be of the form \( Z \cup [\gamma, \omega_3) \) where \( Z \in p \) and \( \gamma < \omega_3 \). (We shall use the interval notation to indicate subsets of \( \omega_2 + 1 \) and \( \omega_3 + 1 \). Thus the interval \([0, \gamma]\) in \( E \) is \( \{ \sigma \in \omega_3 + 1: 0 \leq \sigma < \gamma \} \) and does not include points of \( N \).)

Let \( X = (E \times D) \setminus ((N \cup \{\omega_3\}) \times \{\omega_3\}) \) and let \( X \) have the relative topology. The reader will observe that the space \( X \) bears a strong resemblance to the space constructed in [4, 8.14]. Both \( E \) and \( D \) are Hausdorff spaces with bases of open-and-closed sets so \( X \) is a completely regular Hausdorff space. It is easily verified that \( E \) is CLWL, that the product of a CLWL space with a \( P \)-space is CLWL and that open subspaces of CLWL spaces are CLWL. Consequently, since \( D \) is a \( P \)-space, one has that \( X \) is CLWL.
Note that \( E \) satisfies the hypotheses of Theorem 2.4 and so \( E \times D \) is a \( U \)-space. Consequently, to show that \( X \) is a \( U \)-space it suffices to show that \( X \) is \( C^* \)-embedded in \( E \times D \). To this end let \( f \in C^*(X) \). For each \( n \) in \( \mathbb{N} \) there exists \( \gamma_n < \omega_2 \) such that \( f \) is constant on \( \{n\} \times (\{\gamma_n\}, \omega_2] \cap D) \). (We have observed that continuous functions on \( D \setminus \{\omega_2\} \) are constant on a tail.) Define the extension \( f^* \) of \( f \) to have this constant value at \( (n, \omega_2) \). Similarly there is some \( \gamma_0 < \omega_2 \) such that \( f \) is constant on \( (\omega_3, \omega_2] \cap D \) and we may define \( f^* \) to have this constant value at \( (\omega_3, \omega_2) \). The extension \( f^* \) of \( f \) is clearly continuous at every point of \( E \times D \) except possibly \( (\omega_3, \omega_2) \).

For each \( \sigma \) in \( D \setminus \{\omega_2\} \) there is an \( \alpha_\sigma < \omega_3 \) such that \( f \) is constant on \( [\alpha_\sigma, \omega_3] \times \{\gamma\} \) (since \( \omega_2 \) is a \( P \)-point of \( E \setminus N \)). Let \( \gamma = \sup \{\gamma_n : n \in \mathbb{N} \cup \{0\}\} \) and let \( \alpha = \sup \{\alpha_\sigma : \sigma \in D \setminus \{\omega_2\}\} \).

Let \( \varepsilon > 0 \) be given and let \( n \in \mathbb{N} \) such that \( |f(m, \gamma + 1) - f(\omega_3, \gamma + 1)| < \varepsilon \) whenever \( m \in \mathbb{N} \) and \( m > n \). Then on \( (\{\alpha, \omega_2\} \cup \{m : m \in \mathbb{N} \) and \( m > n\}) \times \{\gamma, \omega_2\} \) \( f^* \) differs from \( f^*(\omega_3, \omega_2) \) by less than \( \varepsilon \). Consequently \( f^* \) is continuous as desired.

Now, let \( Y = \omega_2 + 1 \), where every point is isolated except \( \omega_2 \), whose basic neighborhoods are as in the interval topology. Since \( \omega_2 \) is not the supremum of countably many predecessors we have that \( Y \) is a \( P \)-space.

We claim that \( X \times Y \) is not an \( F \)-space. To see this define \( f \) in \( C^*(X \times Y) \) by the rule \( f((n, \tau), \gamma) = 1/n \) if \( n \in \mathbb{N}, \gamma \) is even and \( \tau > \gamma \), \( f((n, \tau), \gamma) = -1/n \) if \( n \in \mathbb{N}, \gamma \) is odd and \( \tau > \gamma \) and \( f = 0 \) elsewhere. (An ordinal is even if it is a limit ordinal or the sum of a limit ordinal and an even finite ordinal.) For each \( \gamma < \omega_2 \) \( f \) is clearly continuous on the open subset \( X \times \{\gamma\} \) of \( X \times Y \). Also, for each \( \tau < \omega_2 \) \( f \) is identically 0 on the open subset \( (E \times ([0, \tau[ \cap D)) \times \tau, \omega_2) \) of \( X \times Y \). Finally, for each \( \delta < \omega_3 \), \( f \) is identically 0 on the open subset \( ([\delta] \times D) \times Y \) of \( X \times Y \). Thus \( f \) is continuous on all of \( X \times Y \).

Now let \( U \) and \( V \) be open sets with \( \text{cl pos } f \subseteq U \) and \( \text{cl neg } f \subseteq U \). We claim that \( \text{cl } U \cap \text{cl } V \neq \emptyset \) and consequently that \( \text{pos } f \) and \( \text{neg } f \) are not completely separated. Let \( \gamma \) be even, with \( \gamma < \omega_2 \). For each \( \tau \in D \) such that \( \tau > \gamma \) one has \( ((\omega_3, \tau), \gamma) \in \text{cl pos } f \) so there is some \( \gamma_\tau < \omega_3 \) such that \( ([\gamma_\tau, \omega_3] \times \{\tau\}) \times \{\gamma\} \subseteq U \). Let \( \mu_\tau = \sup \{\gamma_\tau : \tau \in \gamma, \omega_3] \cap D\} \). Then \( ([\mu_\tau, \omega_3] \times \{\omega_2\}) \times \{\gamma\} \subseteq \text{cl } U \). Similarly, for each odd \( \gamma < \omega_2 \) there is some \( \mu_\gamma < \omega_3 \) such that \( ([\mu_\gamma, \omega_3 \times \{\omega_2\}) \times \{\gamma\} \subseteq \text{cl } V \).
Let \( \mu = \sup \{ \mu \gamma : \gamma \in \omega_2 \} \). Then \( \mu < \omega_3 \) and \((\mu + 1, \omega_2), \omega_2) \in \text{cl} \ U \cap \text{cl} \ V \) as desired.

The following example shows that the sufficient condition obtained in Corollary 2.5 is not necessary.

**Example 3.2.** A \( U \)-space which does not satisfy the hypotheses of Corollary 2.5 but whose product with each \( P \)-space is a \( U \)-space.

Let \( p \) be a free ultrafilter on \( N \). Let \( B = N \cup (\omega_2 + 1) \) with every point of \( B \) isolated except \( \omega_2 \) whose basic neighborhoods are of the form \( Z \cup \{ \sigma: \gamma < \sigma \leq \omega_2 \} \) where \( Z \in p \) and \( \sigma < \omega_2 \). (This is the space \( T \) of [7].) Note that \( B \) is a \( \text{CLWL} \) \( U \)-space with only one non basically disconnected point. Consequently by Theorem 2.4, its product with any \( P \)-space is a \( U \)-space.

Let \( C = \omega_2 + 1 \) with every point of \( C \) isolated except \( \omega_2 \) and with basic neighborhoods of \( \omega_2 \) as in the interval topology. Then \( C \) is a \( P \)-space. Let \( X = B \times C \). Then \( X \) is a \( U \)-space. If \( Y \) is any \( P \)-space then \( X \times Y \) is homeomorphic to \( B \times (C \times Y) \) and \( C \times Y \) is a \( P \)-space so \( X \times Y \) is a \( U \)-space, by Theorem 2.4.

Suppose \( X \) satisfies the hypotheses of Corollary 2.5 and let \( D \) and \( A \) be as given there. There is some member \( U \) of \( A \) such that \((\omega_2, \omega_1) \in U \). Note also that \( D \supseteq \{ \omega \} \times C \) since \((\omega, \gamma) \) is a non basically disconnected point of \( X \) whenever \( \gamma \in C \). Since \( U \) is open there is some \( \delta \) in \( C \) such that \( \delta < \omega_2 \) and \( \{ \omega \} \times \{ \gamma \in C: \delta < \gamma \} \subseteq U \cap D \). Let \( \mu \in C \) such that \( \mu < \omega_2 \) and \( \{ \gamma \in C: \delta < \gamma < \mu \} \) is uncountable. Let \( \Gamma = \{ B \times \{ \gamma \}: \gamma \leq \mu \} \cup \{ B \times \{ \gamma \in C: \gamma > \mu \} \} \). Let \( \Pi = \{ V \cap (U \cap D): V \in \Gamma \} \). Then \( \Pi \) is an open over of \( U \cap D \), no countable subfamily of which has dense union in \( U \cap D \). This is a contradiction since \( U \cap D \) is weakly Lindelöf.

The author wishes to express his gratitude to the referee for his constructive and thoughtful criticism.

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Received June 30, 1972.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

* C. DePrima will replace J. Dugundji until August 1974.

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David Parham Bellamy, *Composants of Hausdorff indecomposable continua; a mapping approach* ................................................................. 303

Colin Bennett, *A Hausdorff-Young theorem for rearrangement-invariant spaces* .................................................................................. 311

Roger Daniel Bleier and Paul F. Conrad, *The lattice of closed ideals and \( a^* \)-extensions of an abelian \( l \)-group* .................................. 329

Ronald Elroy Bruck, Jr., *Nonexpansive projections on subsets of Banach spaces* ............................................................................ 341

Robert C. Busby, *Centralizers of twisted group algebras* .................. 357

M. J. Canfell, *Dimension theory in zero-set spaces* ........................ 393

John Dauns, *One sided prime ideals* .................................................. 401

Charles F. Dunkl, *Structure hypergroups for measure algebras* .... 413

Ronald Francis Gariepy, *Geometric properties of Sobolev mappings* 427


Dennis Michael Girard, *The behavior of the norm of an automorphism of the unit disk* ................................................................. 443

George Rudolph Gordh, Jr., *Terminal subcontinua of hereditarily unicoherent continua* .............................................................. 457


Neil Hindman, *The product of \( F \)-spaces with \( P \)-spaces* ................. 473

M. A. Labbé and John Wolfe, *Isomorphic classes of the spaces \( C_\sigma(S) \)* ................................................................. 481

Ernest A. Michael, *On \( k \)-spaces, \( k_R \)-spaces and \( k(X) \)* ........... 487

Donald Steven Passman, *Primitive group rings* ................................ 499

C. P. L. Rhodes, *A note on primary decompositions of a pseudovaluation* ............................................................................ 507

Muriel Lynn Robertson, *A class of generalized functional differential equations* .............................................................. 515

Ruth Silverman, *Decomposition of plane convex sets. I* ................. 521

Ernest Lester Stitzinger, *On saturated formations of solvable Lie algebras* ............................................................................ 531

B. Andreas Troesch, *Sloshing frequencies in a half-space by Kelvin inversion* ................................................................. 539

L. E. Ward, *Fixed point sets* .............................................................. 553

Michael John Westwater, *Hilbert transforms, and a problem in scattering theory* ............................................................................ 567

Misha Zafran, *On the spectra of multipliers* ....................................... 609