A NOTE ON PRIMARY DECOMPOSITIONS OF A PSEUDOVALUATION

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Some connections are established between a primary decomposition of a pseudovaluation \( v \) on a commutative ring and a primary decomposition of the zero ideal of the associated graded ring of \( v \). The primary decomposition of a certain pseudovaluation \( v_q \) on a one-dimensional local ring \( Q \) is described in terms of the extensions of \( v_q \) to monoidal transforms of \( Q \).

1. Primary decompositions and the associated graded ring. Let \( R \) be a commutative ring with an identity. We consider a pseudovaluation \( v \) on \( R \). By this we mean that \( v \) is a mapping from \( R \) to \( P \), the set of all real numbers together with \( \infty \), such that

\[
v(0) = \infty \quad , \quad v(1) = 0,
\]

and, for \( x, y \in R \),

\[
v(xy) \geq v(x) + v(y),
\]

and

\[
v(x - y) \geq \min \{ v(x), v(y) \}.
\]

For each \( a \in P \), write \( v_a = \{ x \in R \mid v(x) \geq a \} \) and \( v_a = \{ x \in R \mid v(x) > a \} \). The associated graded ring of \( v \), introduced by Szpiro in [11], is \( G = \bigoplus_{a \in R} v_a/v_a \). We shall use \(- \) to denote the natural mapping from \( R \) into \( G \).

Let \( u \) be a pseudovaluation such that \( u \geq v \) (this means that \( u(x) \geq v(x) \) for all \( x \)). We denote by \( T(u) \) the set of all \( x \), not in \( u_w \), such that \( u(x^n) = nu(x) \) for all positive integers \( n \), and by \( S(u) \) the set of all \( x \), not in \( u_w \), such that \( u(xy) = u(x) + u(y) \) for all \( y \in R \). As in [10], we call \( u \) primary if \( T(u) = S(u) \). We denote by \( F(u, v) \) the set of all \( x \) such that either \( u(x) > v(x) \) or \( u(x) = \infty \), and we put \( T(u, v) = T(u) \setminus F(u, v) \).

Let \( I(u, v) \) be the ideal generated in \( G \) by \( F(u, v) \).

**Lemma 1.** \( \overline{F(u, v)} \) is the set of all homogeneous elements of \( I(u, v) \), and \( \overline{T(u, v)} \) is the set of all homogeneous elements of \( G/\text{rad} \ I(u, v) \). If the pseudovaluation \( u \) is primary then the ideal \( I(u, v) \) is primary.

**Proof.** Let \( r \in R \) and \( s \in F(u, v) \). Either \( r\overline{s} = 0 \) or \( r\overline{s} = \overline{r}s \). In the latter case either \( v(s) = \infty \) or

\[
v(rs) = v(r) + v(s) < u(r) + u(s) \leq u(rs).
\]
Thus, in each case, \( \bar{r}s \in F(u, v) \). If we suppose, also, that \( r \in F(u, v) \) and that \( \bar{r} \) and \( \bar{s} \) have the same degree, then either \( \bar{r} - \bar{s} = 0 \) or \( \bar{r} - \bar{s} = \bar{r} - \bar{s} \). In the latter case either \( v(r) = \infty \) or \( v(r - s) = v(r) = v(s) < \min \{u(r), u(s)\} \leq u(r - s) \).

Hence, in each case, \( \bar{r} - \bar{s} \in F(u, v) \). It is now clear that \( F(u, v) \) is the set of homogeneous elements of \( I(u, v) \).

Let \( r \in T(u, v) \) and let \( n \) be a positive integer. Then it is easy to see that \( u(r^n) = v(r^n) = nv(r) \neq \infty \); i.e., \( r^n \notin F(u, v) \). Therefore, \( \bar{r}^n = r^n \in I(u, v) \). Now suppose that \( r \in T(u, v) \). If \( r \notin T(v) \) then there exists \( m \) such that \( \bar{r}^m = 0 \). Suppose that \( r \in T(v) \). Then, by 4.1 of [10], there exists \( n \) such that \( r^n \in F(u, v) \). Hence \( \bar{r}^n = \bar{r}^n \in I(u, v) \).

Finally let \( u \) be primary, and suppose that \( r, s \) are elements of \( R \) such that \( r \in T(u, v) \), \( \bar{s} \neq 0 \), and \( \bar{r}s \in I(u, v) \). Either \( \bar{r}s = 0 \) and so \( v(r) + v(s) < v(rs) \leq u(rs) = u(r) + u(s) = v(r) + u(s) \), or \( \bar{r}s = \bar{r}s \) and so \( v(r) + v(s) = v(rs) < u(rs) = u(r) + u(s) = v(r) + u(s) \). In each case \( v(s) < u(s) \) and, hence, \( \bar{s} \in I(u, v) \). Therefore, \( I(u, v) \) is primary.

**REMARK.** The set \( S(u) \cap F(u, v) \) is contained in the set \( S_0(u, v) \) of all \( x \in F(u, v) \) such that \( u(xy) = u(x) + u(y) \) for all \( y \notin F(u, v) \). These sets and their images in \( G \) are multiplicatively closed, and \( S_0(u, v) \) is the set of all homogeneous elements of \( G \) which are relatively prime to \( I(u, v) \).

If \( W \) is a collection of pseudovaluations the lower envelope \( w_\circ = \bigwedge W \) is defined by \( w_\circ(x) = \inf \{w(x) \mid w \in W\} \). From Lemma 1 we deduce

**Theorem 1.** If \( u_1 \land u_2 \land \cdots \land u_n \) is a primary decomposition of \( v \) then \( I(u_1, v) \cap I(u_2, v) \cap \cdots \cap I(u_n, v) \) is a primary decomposition of \( 0_v \).

**Corollary.** Let \( u_1 \land u_2 \land \cdots \land u_n \) be an irredundant primary decomposition of \( v \), and suppose that \( G \) is Noetherian. Then, for each \( i \), there exists \( r_i \in R \) such that \( T(u_i, v) \) is the set of \( x \), not in \( v_\circ \), for which \( v(xr_i) = v(x) + v(r_i) \).

**Proof.** The decomposition \( 0_v = I(u_i, v) \cap \cdots \cap I(u_n, v) \) is clearly irredundant. It follows that the homogeneous elements of \( G \) not in \( T(u_i, v) \) generate a prime ideal which belongs to \( 0_v \) and which, therefore, takes the form \( 0_v : (G \bar{r}_i) \) for some homogeneous element \( \bar{r}_i \) in \( G \).
REMARK. For each positive \( b \in P \), denote by \( F(u, v, b) \) the set of all \( r \in R \) such that either \( u(r) = \infty \) or \( u(r) - v(r) \geq b \). The proof of Lemma 1 shows that \( F(u, v, b) \) is the set of homogeneous elements of the ideal \( I(u, v, b) \) which it generates in \( G \), and that \( \overline{T(u, v)} \) is the set of homogeneous elements of \( G \backslash \text{rad} I(u, v, b) \). It is easy to verify that, for a (possibly infinite) collection of pseudovaluations \( v_i \geq v, v = \bigwedge_i v_i \) if and only if, for every \( b > 0 \), \( 0 \leq \bigcap_i I(v_i, v, b) \).

For all \( b > 0 \) and \( c > 0 \),

\[
I(u, v, b)I(u, v, c) \subseteq I(u, v, b + c) .
\]

Hence each \( u \geq v \) naturally induces a (nonnegative) pseudovaluation \( v' \) on \( G \). Thus \( v = \bigwedge_i v_i \) if and only if \( \bigwedge_i v_i' \) is the trivial pseudovaluation on \( G \).

When \( v \) is homogeneous the following result may be regarded as a special case of [11, Théorème 1]. Recall that \( v \) is said to be discrete if \( v(R \backslash v_\infty) \) generates a discrete subgroup of \( R \).

**Theorem 2.** Suppose that \( v \) is a discrete pseudovaluation. If \( 0_\circ \) has a finite primary decomposition without embedded components then \( v \) has a primary decomposition.

**Proof.** Suppose that \( H_1 \cap H_2 \cap \cdots \cap H_k \) is the primary decomposition of \( 0_\circ \). For each \( i \), write \( \text{rad} H_i = P_i \) and denote by \( S_i \) the set of elements \( r \in R \) such that \( \overline{r} \in P_i \); then \( v(ab) = v(a) + v(b) \) for all \( a \) and \( b \) in \( S_i \), and \( S_i \) is multiplicatively closed. Mappings \( v_i \) are defined, for all \( x \in R \), by

\[
v_i(x) = \sup \{ v(xa) - v(a) | a \in S_i \} .
\]

Observe that if \( a, b \in S_i \) then

\[
v_i(x) \geq v(xab) - v(ab) \geq v(xa) - v(a) \geq v(x) .
\]

By 3.1 and 3.2 of [6], \( v_i \) is a pseudovaluation.

Let \( x \in R \backslash v_\infty \). Then there exists \( i \) such that \( \overline{x} \in H_i \). If \( c \in S_i \) then \( \overline{xc} \neq 0 \), and so \( v(xc) - v(c) = v(x) \). Thus \( v_i(x) = v(x) \), and so \( \bigwedge_i v_i = v \).

We shall now show that \( v_i \), being a typical \( v_i \), is primary. Let \( x \in S(v_i) \) and suppose that \( v_i(x) \neq \infty \). Then there exists \( y \in R \) such that \( v_i(xy) > v_i(x) + v_i(y) \). Therefore, we may choose \( a \in S_i \) such that

\[
v_i(x) = v(xa) - v(a) ,
\]

and

\[
v_i(xy) \geq v(xya) - v(a) > v_i(x) + v_i(y) .
\]
Now write $\bigcap_{i>1} P_i = K$ and choose $c \in R$ such that $\bar{c} \in K \setminus P_i$. Then $\bar{a} \bar{c} \in P_i$, and so $\bar{a} \bar{c} = \bar{a} \bar{c} \in K \setminus P_i$. We may therefore assume (by replacing $a$ by $ac$) that $\bar{a} \in K \setminus P_i$. This implies that $\bar{a}^3 = \bar{a} \bar{a} \in K \setminus P_i$. Since $v_i(x) = v(xa^3) - v(a^3) = v(xa) - v(a)$, it follows that

$$v(xa^3) = v(xa) + v(a).$$

Therefore, $\bar{xa}^3 = \bar{xa} \bar{a} \in K$, and so (replacing $a$ by $a^3$) we may also assume that $\bar{xa} \in K$. If $\bar{xa} \in P_i$, then

$$v(yxa) - v(a) = v(yxa) - v(xa) + v(xa) - v(a) \leq v(y) + v(x),$$

which is false. Therefore, $\bar{xa} \in \bigcap_{i>1} P_i$, and so, for some $n$, $(\bar{xa})^n = 0$. Since $v_i(x) \neq \infty$, we have $v(xa) \neq \infty$ and so $v((xa)^n) > nv(xa)$. Therefore, $v_i(x^n) \geq v(x^n a^n) - v(a^n) > nv(xa) - v(a^n) = nv(xa)$. Thus $x \notin T(v_i)$. Therefore, $S(v_i) = T(v_i)$; i.e., $v_i$ is primary.

2. Extensions of pseudovaluations. In this section we introduce some terminology for use in § 3, and we prove a result pertinent to [2].

We suppose the definition of a pseudovaluation $u$ to be modified as follows:

(i) $u \geq 0$.

(ii) It is not required that $u(1) = 0$ (this facilitates the statement of Lemma 2; moreover, the rings in this section need not contain an identity).

We consider a homomorphism $f$ from a commutative ring $R$ to a commutative ring $S$. If $I$ is an ideal of $S$ (resp. $R$) then $I^r$ (resp. $I^r$) will denote $f^{-1}(I)$ (resp. the ideal generated by $f(I)$ in $S$). Suppose that $v$ is a pseudovaluation on $R$. Define $v^r$ to be the mapping from $S$ to $P$ such that, for all $x \in S$,

$$v^r(x) = \sup \{a \in P \mid x \in (v_a)^r\}.$$

LEMMA 2. The mapping $v^r$ is a pseudovaluation on $S$.

Proof. It is clear that $v^r(0) = \infty$.

Let $x, y \in S$, and suppose that $x \in (v_a)^r$ and $y \in (v_b)^r$ where $a, b \in P$. Then $xy \in (v_a)(v_b)^r \subseteq (v_a v_b)^r \subseteq (v_{a+b})^r$. Thus

$$v^r(xy) \geq a + b.$$

It follows that $v^r(xy) \geq v^r(x) + v^r(y)$.

Similarly, assuming that $a \geq b$, $x - y \in (v_a)^r + (v_b)^r = (v_a + v_b)^r = (v_b)^r$. Thus $v^r(x - y) \geq b$. It follows that
Let \( w \) be a pseudovaluation on \( S \). We shall denote \( wf \) by \( w^* \). It is easy to verify that \( w^* \) is a pseudovaluation on \( R \) which is primary if \( w \) is primary.

**Lemma 3.** (i) The pseudovaluation \( v \) on \( R \) satisfies \( v \leq v^* \).
(ii) The pseudovaluation \( w \) on \( S \) satisfies \( w \geq w^* \).

**Proof.** (i) Let \( x \) be an element of \( R \) such that \( v(x) = a \). Then \( f(x) \in (v^*)^e \) and so \( a \leq v^*(f(x)) = v^*(x) \).

(ii) Let \( y \) be an element of \( S \) such that \( y \in ((w^*)^e)^e \). Since \((w^*)^e \) is primary, \( y \in (w^*)^{ae} \) and so \( w(y) \geq a \). It follows that \( w(y) \geq w^*(y) \).

**Theorem 3.** \( v = v^* \) if and only if \( v_a = (v_a)^e \) for each \( a \in R \).

**Proof.** If \( v = v^* \) then, for each \( a \in P \),

\[
(v_a)^e \subseteq \{ x \in R \mid v^*(f(x)) \geq a \} = (v^*)_a = v_a,
\]

and so \((v_a)^e = v_a \). Conversely, suppose that \( v_a = (v_a)^e \) for each \( a \in R \). Let \( x \in R \) and let \( f(x) \in (v_a)^e \) where \( a < \infty \). Then \( x \in (v_a)^e = v_a \), that is \( v(x) \geq a \). It follows that \( v \geq v^* \), and hence that \( v = v^* \).

We refer to [2, p. 296, Definition 2] for the definition of a best filtration. If \( v \) has a best filtration \( \{A_i\}_{i=0}^\infty \) then, by [2, p. 297, Lemma 1], the set of all distinct \( A_i \)'s is the same as the set of all distinct \( v_a \)'s where \( a < \infty \). Thus, taking \( f \) to be an inclusion map, our theorem includes, in the case of nonnegative pseudovaluations, Theorem 2, p. 299, and Theorem 4, p. 301, of [2].

3. An example in a one-dimensional ring. Let \( Q, m \) be a one-dimensional local ring and let \( q \) be an \( m \)-primary ideal of \( Q \). We shall consider the pseudovaluation \( v = v_a \) determined by the powers of \( q \) according to the rule

\[
v_a(x) = \sup \{ n \mid x \in q^n \} .
\]

By considering the associated graded ring \( G \) of \( v \) and proceeding as in Theorem 2, we could show that \( v \) decomposes into primary pseudovaluations corresponding to the isolated primary components of \( 0_0 \) together with an “irrelevant” component. Apart from the irrelevant component this decomposition is unique (by [10]). We shall now show how the theory of monoidal transformations developed by Northcott and Kirby provides an alternative description of this
decomposition.

Let $A$ denote the intersection of the primary components of $0_0$ of rank nought, and write $Q/A = Q'$ and $qQ' = q'$. Then not every element of $mQ'$ is a zero divisor. Let $R$ be the $q'$-resolute of $Q'$, for the definition of which see p. 136 of [4]; let $Q_i, \ldots, Q_r$ be the monoidal transforms of $Q'$ with respect to $q'$, i.e., the rings of quotients of $R$ with respect to the maximal ideals $p_1, \ldots, p_r$ of $R$; and, for $i = 1, \ldots, r$, let $f_i$ be the composition of the natural homomorphisms $Q \to Q' \to Q_i$. Using the symbols $e_i$ and $c_i$ to relate to $f_i$ in the same way that $e$ and $c$ were related to $f$ in §2, we observe that $\psi_i$ is the pseudovaluation on $Q_i$ determined by the powers of the ideal $q^i$. However, by [4, Theorems 1 and 8, and Lemma 3] $q^i$ is a principal ideal of $Q_i$. Therefore, by an example in §3 of [10], $\psi_i$ is primary, and so $\psi^{(\psi_i)}$ is primary.

Now, denoting by $q_i$ the restriction to $R_i$ of $q^i$, $\rad q_i = \psi_i$ and $q_1 \cap q_2 \cap \cdots \cap q_r$ is the primary decomposition of $3\psi$ (by the corollary on p. 142 of [4] and since $\Im \psi \subseteq \rad \psi$). Therefore, for all $n$,

$$R\psi^n = q_1^n \cap q_2^n \cap \cdots \cap q_r^n.$$  

By an argument on p. 88 of [8], $R\psi^n = q^n$ for all sufficiently large $n$. Therefore, for $n \geq h$ say,

$$q^n + A = (\psi^{(\psi_i)} \cap \cdots \cap \psi^{(\psi_r)})_n.$$

However, we may choose $h$ such that $A \cap q^h = 0_0$ and, hence, for $n \geq h$, $q^h \cap (q^n + A) = q^n$. Therefore, using $e_i$ and $c_i$ to relate to the natural map $f_0$ from $Q$ to $Q/q^h$, we have, for all $n$,

$$q^n = (\psi^{(\psi_0)})_n \cap (\psi^{(\psi_1)} \cap \cdots \cap \psi^{(\psi_r)})_n.$$

Finally we show that $\psi_0 = \psi$ say, is primary. If $x \in f_0(m)$ then, for some $k$, $w(x^k) = \infty$ and so $x \in T(w)$. On the other hand, if $x$ is a unit of $Q/q^h$ then $w(x) = 0$ and, for any $y$,

$$w(xy) = w(y) = w(y) + w(x);$$

i.e., $x \in S(w)$. Thus $T(w) = S(w)$.

It is now clear that

**Theorem 4.** In the notation developed above

$$\psi^{(\psi_0)} \cap \psi^{(\psi_1)} \cap \cdots \cap \psi^{(\psi_r)}$$

is a primary decomposition of $\psi$.

It is easy to extend this theorem and obtain a primary decomposition of the pseudovaluation $\psi$, determined by an ideal $I$ of rank 1
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in a 1-dimensional Noetherian ring $R$. Let $M_1, \ldots, M_m$ be the associated prime ideals (necessarily maximal) of $I$, and, for $j = 1, \ldots, m$, let $g_j$ be the natural homomorphism from $R$ to the ring $R_j$ of quotients of $R$ with respect to $M_j$. For each positive integer $n$,

$$I^n = \bigcap_j (I^n)^{e_j c_j},$$

where $e_j, c_j$ relate to $g_j$, and so

$$v_i = \bigwedge_j v_i^{e_j c_j},$$

which yields a primary decomposition of $v_i$ on application of Theorem 4 to each $v_i^{e_i}$.

We conclude by describing a result, in the same vein as the foregoing, which is implicit, as a special case, in [9]. Suppose that our ring $R$ is a domain; let $\bar{v}_i$ denote the least homogeneous pseudovaluation $\geq v_i$; and let $\bar{R}_1, \ldots, \bar{R}_h$ be the rings of quotients with respect to the maximal ideals of the integral closure of $R$ which contain $I$. Then $\bar{v}_i$ decomposes into valuations

$$\bar{v}_i = \bigwedge_i (\bar{v}_i^{e_i})^{c_i}$$

where, for each $i$, $e_i, c_i$ refer to the natural mapping $R \to \bar{R}_i$.

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