

Pacific Journal of Mathematics

**A CLASS OF GENERALIZED FUNCTIONAL DIFFERENTIAL
EQUATIONS**

MURIL LYNN ROBERTSON

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In this paper, the equation $y' = Ay$ is solved, where A is a self-mapping of a certain set of functions. Also, a continuous dependence theorem is proven, and n th-order differential equations are considered.

1. **Definitions.** If p is a real number and $I = \{I_1, I_2, \dots\}$ is a collection of intervals so that $p \in I_1$ and $I_n \subseteq I_{n+1}$ for each positive integer n , then I is said to be a nest of intervals about p . Let $I_0 = \{p\}$ and $[a_n, b_n] = I_n$ for each nonnegative integer n . Let I^* denote the union of all the elements of I .

In general, B denotes a Banach space; and if D is a real number set, let $C[D, B]$ denote the set of continuous functions from D into B . Whenever D is an interval, $C[D, B]$ is considered a Banach space with supremum norm $|\cdot|$.

Let $C(I, B)$ denote the set of continuous functions whose domain is either I_0, I^* , or an element of I ; and whose range is a subset of B .

Suppose A is a mapping from $C(I, B)$ into $C(I, B)$ so that

- (i) domain $f =$ domain Af , for all $f \in C(I, B)$,
- (ii) $(Af)|_{I_k} = A(f|_{I_k})$, for all $f \in C(I, B)$ and $I_k \subseteq$ domain f , for positive k , [Note: $f|_{I_k}$ is the restriction of f to I_k .] and

(iii) there is a function M from I^* into the nonnegative reals that is Lebesgue integrable on any interval contained in I , so that $\|Af(x) - Ag(x)\| \leq M(x) \cdot |f - g|$, for all $f, g \in C[I_i, B]$ so that $f|_{I_{i-1}} = g|_{I_{i-1}}$ and $x \in I_i$, for each positive integer i .

Then, A is said to be an I -map with function M . Furthermore, if the phrase " $f|_{I_{i-1}} = g|_{I_{i-1}}$ " is removed from part (iii) of the previous definition, A is said to be an I -map with strong function M .

2. Main results.

THEOREM A. *Suppose A is an I -map with function M ; and $\max \left\{ \int_{a_i}^{a_{i-1}} M, \int_{b_{i-1}}^{b_i} M \right\} < 1$, for all positive integers i . Then if $q \in B$, there is a unique $y \in C[I^*, B]$ so that $y' = Ay$ and $y(p) = q$.*

Proof. Let $\{(p, q)\} = y_0$. Then y_0 is the unique function in $C[I_0, B]$ so that $y_0(x) = q + \int_p^x Ay_0$ for all $x \in I_0$. Now, suppose n is a non-negative integer so that y_n has been defined in $C[I_n, B]$ to be the unique function so that $y_n(x) = q + \int_p^x Ay_n$ for all $x \in I_n$. Then, $D =$

$\{f \in C[I_{n+1}, B] / f|_{I_n} = y_n\}$ is a complete metric space. If $f \in D$, let $Tf(x) = q + \int_p^x Af$, for all $x \in I_{n+1}$. Now if $x \in I_n$ and $f \in D$, then $Tf(x) = q + \int_p^x Af = q + \int_p^x (Af)|_{I_n} = q + \int_p^x A(f|_{I_n}) = q + \int_p^x Ay_n = y_n(x)$. Thus $(Tf)|_{I_n} = y_n$, and thus $Tf \in D$.

Suppose $f, g \in D$. Then,

$$\begin{aligned} |Tf - Tg| &= \max \{ \| Tf(x) - Tg(x) \| / x \in I_{n+1} \} \\ &= \max \left\{ \left\| \int_p^x (Af - Ag) \right\| \right\} \\ &\leq \max \left\{ \int_p^x \| Af(s) - Ag(s) \| ds \right\}. \end{aligned}$$

Note that $f|_{I_n} = g|_{I_n}$ and this implies that $A(f|_{I_n}) = A(g|_{I_n})$. Thus, $(Af)|_{I_n} = (Ag)|_{I_n}$; that is, $Af(s) = Ag(s)$ for all s in I_n . So

$$\begin{aligned} |Tf - Tg| &\leq \max \left\{ \sup \left\{ \int_{b_n}^x \| Af(s) - Ag(s) \| ds / x \in [b_n, b_{n+1}] \right\}, \right. \\ &\quad \left. \sup \left\{ \int_x^{a_n} \| Af(s) - Ag(s) \| ds / x \in [a_{n+1}, a_n] \right\} \right\} \\ &\leq \max \left\{ \sup \left\{ \int_{b_n}^x M(s) \cdot |f - g| ds / x \in [b_n, b_{n+1}] \right\}, \right. \\ &\quad \left. \sup \left\{ \int_x^{a_n} M(s) \cdot |f - g| ds / x \in [a_{n+1}, a_n] \right\} \right\} \\ &\leq \max \left\{ \int_{a_{n+1}}^{a_n} M, \int_{b_n}^{b_{n+1}} M \right\} \cdot |f - g|. \end{aligned}$$

Hence T is a contraction map from the complete metric space D into D , and thus T has a unique fixed point y_{n+1} . So y_{n+1} is the unique function in $C[I_{n+1}, B]$ so that $y_{n+1}(x) = q + \int_p^x Ay_{n+1}$ for all x in I_{n+1} . So by induction y_k is defined for each positive integer k . Define $y(x) = y_m(x)$ whenever $x \in I_m \setminus I_{m-1}$. Then y is the desired function.

The following corollary (See [6].) shows that Theorem A guarantees the existence of solutions to some functional differential equations. Suppose g is a function from I^* to I^* so that $g(I_n) \subseteq I_n$ for each positive integer n . Such a function is said to be an I -function. Let $A_k = \{x \in [a_k, a_{k-1}] / g(x) \notin I_{k-1}\}$ and let $B_k = \{x \in [b_{k-1}, b_k] / g(x) \notin I_{k-1}\}$, for each positive integer k . Also, suppose $\|F(x, y) - F(x, z)\| \leq M(x) \cdot \|y - z\|$ for all $x \in I^*$, $y, z \in B$; and M is Lebesgue integrable on intervals.

COROLLARY. *If $q \in B$, and $\max \left\{ \int_{A_k} M, \int_{B_k} M \right\} < 1$, for all k ; then there is a unique $y \in C[I^*, B]$ so that $y(p) = q$ and $y'(x) = F(x, y(g(x)))$ for all $x \in I^*$.*

Proof. Let $(Af)(x) = F(x, f(g(x)))$. Then A is an I -map with function T , where

$$T(x) = \begin{cases} M(x), & x \in A_n \cup B_n \\ 0, & x \notin A_n \cup B_n \end{cases}, \text{ for } x \in I_n \setminus I_{n-1}.$$

The proof of the following is straightforward.

PROPOSITION. *Suppose I is a nest of intervals about p , and each of α and β is an I -function. Then*

(i) *Suppose P is of bounded variation on each interval contained in I^* , and let $Af(x) = \int_{\alpha(x)}^{\beta(x)} dF \cdot f$, for $f \in C(I, B)$ and $x \in \text{domain } f$. Then A is an I -map with function M , where $M(x)$ is the variation of F over $[\alpha(x), \beta(x)] \setminus I_{k-1}$ where $x \in I_k \setminus I_{k-1}$.*

(ii) *Suppose $K: I^* \times I^*$ to the scalars which is continuous, and $Af(x) = \int_{\alpha(x)}^{\beta(x)} K(x, t)f(t)dt$, for $f \in C(I, B)$ and $x \in \text{domain } f$. Then A is an I -map with function M , where $M(x) = \left| \int_{[\alpha(x), \beta(x)] \setminus I_{k-1}} |K(x, t)| dt \right|$ for $x \in I_k \setminus I_{k-1}$.*

It is easy to show that the set of I -maps, for a fixed nest of intervals I , is a near-ring under composition and addition. Thus, there are many types of differential equations that may be solved by combining I -maps of the types given in the corollary and the proposition.

3. Continuous dependence.

THEOREM B. *Suppose $A(z, \cdot)$ is an I -map with strong function M for each z in the topological space $K, q \in B$, and $M_k = \max \left\{ \int_{a_k}^{a_{k-1}} M, \int_{b_{k-1}}^{b_k} M \right\} < 1$, for all positive integers k . Let $y(z, \cdot)$ be the unique function, guaranteed by Theorem A, so that $y_2(z, \cdot) = A(z, y(z, \cdot))$ and $y(z, p) = q$. Then, there exists a sequence $\{L_i\}$ so that for $z, z_0 \in K$, $|y(z, \cdot) - y(z_0, \cdot)|_{I_i} \leq L_i \cdot |A(z, y(z_0, \cdot)) - A(z_0, y(z_0, \cdot))|_{I_i}$, for each i . [In the previous inequality the norm is the supremum norm over I_i .]*

Indication of proof. Define $\{L_i\}$ as follows: Let $L_1 = \max(p - a_1, b_1 - p)/(1 - M_1)$. For $i \geq 1$, let $L_{i+1} = \{L_i + \max(a_i - a_{i+1}, b_{i+1} - b_i)\}/(1 - M_{i+1})$.

EXAMPLE. Let g be an I -function and let $N > 0$. Then let K be the metric space of all I -functions that are pointwise never more than N from g . Define $A(h, y) = y(h|_{\text{dom } y})$ and $d(h_1, h_2) = \sup \{|h_1(x) - h_2(x)|/x \in I^*\}$; d is the metric.

4. *N*th order equations.

THEOREM C. *Suppose A is an I-map with function M, n is a positive integer, and $q_0, q_1, \dots, q_{n-1} \in B$. Let*

$$N_k = \max \left\{ \int_{a_k}^{a_{k-1}} \int_{s_1}^{a_{k-1}} \dots \int_{s_{n-1}}^{a_{k-1}} M(s_n) ds_n \dots ds_1, \right. \\ \left. \int_{b_{k-1}}^{b_k} \int_{b_{k-1}}^{s_1} \dots \int_{b_{k-1}}^{s_{n-1}} M(s_n) ds_n \dots ds_1 \right\}.$$

Then, if $N_k < 1$, for all positive integers k, there is a unique $y \in C[I^, B]$ so that $y^{(n)} = Ay$ and $y(p) = q_0, \dots, y^{(n-1)}(p) = q_{n-1}$.*

Indication of proof. Use induction, Theorem A, and the following lemma.

LEMMA. *Suppose H is an I-map with function S, and $q \in B$, then define $Kf(x) = q + \int_p^x Hf$, for all $f \in C(I, B)$ and $x \in \text{domain } f$. Then K is an I-map with function T, where $T(x) = \int_{a_k}^{a_{k-1}} S$, whenever $x \in (a_k, a_{k-1}]$; and $T(x) = \int_{b_{k-1}}^x S$, whenever $x \in [b_{k-1}, b_k)$.*

The proof of Theorem D is straightforward and Theorem E is a special case of Theorem D. Both of these theorems are imitations of standard theorems of ordinary differential equations.

THEOREM D. *(A generalized system of equations theorem.) Suppose B_i is a Banach space with norm $\|\cdot\|_i$, for each positive integer i between 1 and n. Let $B' = \{(x_1, x_2, \dots, x_n) | x_i \in B_i\}$. Also, let $\|(x_1, \dots, x_n)\| = \max \{\|x_i\|_i / 1 \leq i \leq n\}$, for all elements of B' . [Then B' is a Banach space.] Furthermore, suppose $H_i: C(I, B')$ to $C(I, B_i)$ for $1 \leq i \leq n$ so that*

- (1) *if $f \in C(I, B')$, domain $f = \text{domain } H_i f$,*
- (2) *if $f \in C(I, B')$, and $I_k \subseteq \text{domain } f, k > 0$, then $(H_i f)|_{I_k} = H_i(f|_{I_k})$, and*
- (3) *there is $M_i: I^*$ to the reals which is Lebesgue integrable on intervals so that if $f, g \in C[I_k, B']$, $f|_{I_{k-1}} = g|_{I_{k-1}}$, and $x \in I_k$, then $\|H_i f(x) - H_i g(x)\| \leq M_i(x) \cdot |f - g|$. Now, define $A: C(I, B')$ to $C(I, B')$ so that $Af = (H_1 f, H_2 f, \dots, H_n f)$, for all $f \in C(I, B')$.*

Then A is an I-map with function $\max \{M_i / 1 \leq i \leq n\}$.

THEOREM E. *Suppose B' is as in Theorem D, with $B = B_i$, for all i. Also, suppose $H = H_n$ and $M = M_n$, where H_n and M_n are as in Theorem D. Suppose $q_0, \dots, q_{n-1} \in B$ and*

$$\max \left\{ \int_{a_k}^{a_{k-1}} \max \{1, M\}, \int_{b_{k-1}}^{b_k} \max \{1, M\} \right\} < 1, \text{ for all } k > 0.$$

Then, there is a unique $y \in C[I^*, B]$ so that

$$y^{(n)} = H((y, y^{(1)}, \dots, y^{(n-1)})) \text{ and } y^{(i)} = q_i, \text{ for } 0 \leq i \leq n - \iota.$$

EXAMPLE. Suppose each g_i is an I -function, then for appropriate functions F_i , Theorem E guarantees the existence of a solution to

$$y^{(n)}(x) = \sum_{k=1}^n F_k(x, y^{(n-k)}(g_k(x))), \text{ for all } x \in I^*.$$

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Received May 10, 1972. This research was supported in part by a National Aeronautics and Space Administration Traineeship, and is part of the author's Ph. D. thesis, which was directed by J. W. Neuberger, Emory University.

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The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$48.00 a year (6 Vols., 12 issues). Special rate: \$24.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

* C. DePrima will replace J. Dugundji until August 1974.

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Pacific Journal of Mathematics

Vol. 47, No. 2

February, 1973

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