A CLASS OF GENERALIZED FUNCTIONAL DIFFERENTIAL EQUATIONS

Muril Lynn Robertson
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In this paper, the equation \( y' = Ay \) is solved, where \( A \) is a self-mapping of a certain set of functions. Also, a continuous dependence theorem is proven, and \( n \)th-order differential equations are considered.

1. Definitions. If \( p \) is a real number and \( I = \{I_1, I_2, \ldots\} \) is a collection of intervals so that \( p \in I \) and \( I_n \subseteq I_{n+1} \) for each positive integer \( n \), then \( I \) is said to be a nest of intervals about \( p \). Let \( I_0 = \{p\} \) and \( [a_n, b_n] = I_n \) for each nonnegative integer \( n \). Let \( I^* \) denote the union of all the elements of \( I \).

In general, \( B \) denotes a Banach space; and if \( D \) is a real number set, let \( C[D, B] \) denote the set of continuous functions from \( D \) into \( B \). Whenever \( D \) is an interval, \( C[D, B] \) is considered a Banach space with supremum norm \( | \cdot | \).

Let \( C(I, B) \) denote the set of continuous functions whose domain is either \( I_0 \), \( I^* \), or an element of \( I \); and whose range is a subset of \( B \).

Suppose \( A \) is a mapping from \( C(I, B) \) into \( C(I, B) \) so that

(i) domain \( f = \) domain \( Af \), for all \( f \in C(I, B) \),

(ii) \( (Af)_{|I_k} = A(f_{|I_k}) \), for all \( f \in C(I, B) \) and \( I_k \subseteq \) domain \( f \), for positive \( k \), [Note: \( f_{|I_k} \) is the restriction of \( f \) to \( I_k \), and]

(iii) there is a function \( M \) from \( I^* \) into the nonnegative reals that is Lebesgue integrable on any interval contained in \( I \), so that

\[ ||Af(x) - Ag(x)|| \leq M(x) \cdot |f - g| \]

for all \( f, g \in C[I_0, B] \) so that \( f_{|I_{i-1}} = g_{|I_{i-1}} \) and \( x \in I_i \), for each positive integer \( i \).

Then, \( A \) is said to be an \( I \)-map with function \( M \). Furthermore, if the phrase "\( f_{|I_{i-1}} = g_{|I_{i-1}} \)" is removed from part (iii) of the previous definition, \( A \) is said to be an \( I \)-map with strong function \( M \).

2. Main results.

THEOREM A. Suppose \( A \) is an \( I \)-map with function \( M \); and

\[ \max \left\{ \int_{a_i}^{b_i} M, \int_{b_i}^{b_{i+1}} M \right\} < 1 \]

for all positive integers \( i \). Then if \( q \in B \), there is a unique \( y \in C[I^*, B] \) so that \( y' = Ay \) and \( y(p) = q \).

Proof. Let \( \{(p, q) = y_0 \)\. Then \( y_0 \) is the unique function in \( C[I_0, B] \) so that \( y_0(x) = q + \int_p^x A y_0 \) for all \( x \in I_0 \). Now, suppose \( n \) is a nonnegative integer so that \( y_n \) has been defined in \( C[I_n, B] \) to be the unique function so that \( y_n(x) = q + \int_p^x A y_n \) for all \( x \in I_n \). Then, \( D = \)
\( \{ f \in C[I_{n+1}, B] | f|_{I_n} = y_n \} \) is a complete metric space. If \( f \in D \), let 
\( T_f(x) = q + \int_p^x A f, \) for all \( x \in I_{n+1} \). Now if \( x \in I_n \) and \( f \in D \), then
\[ T_f(x) = q + \int_p^x A f = q + \int_p^x (A f)|_{I_n} = q + \int_p^x A(y_n) = y_n(x). \]
Thus \( (T_f)|_{I_n} = y_n \), and thus \( T_f \in D \).

Suppose \( f, g \in D \). Then,
\[ |T_f - T_g| = \max \{ ||T_f(x) - T_g(x)||/x \in I_{n+1} \} \]
\[ = \max \left\{ \int_p^x |A f - A g| \right\} \]
\[ \leq \max \left\{ \int_p^x |A f(s) - A g(s)| ds \right\}. \]

Note that \( f|_{I_n} = g|_{I_n} \) and this implies that \( A(f)|_{I_n} = A(g)|_{I_n} \). Thus, 
\( (A f)|_{I_n} = (A g)|_{I_n} \), that is, \( A f(s) = A g(s) \) for all \( s \) in \( I_n \). So
\[ |T_f - T_g| \leq \max \left\{ \sup_{b_n} \int_a^{b_n} |A f(s) - A g(s)| ds, \sup_{b_{n+1}} \int_a^{b_{n+1}} |A f(s) - A g(s)| ds \right\}. \]
\[ \leq \max \left\{ \sup_{b_n} \int_a^{b_n} |f - g| ds, \sup_{b_{n+1}} \int_a^{b_{n+1}} |f - g| ds \right\}. \]
\[ \leq \max \left\{ \int_a^{b_{n+1}} M, \int_{b_n}^{b_{n+1}} M \right\} \cdot |f - g|. \]

Hence \( T \) is a contraction map from the complete metric space \( D \) into \( D \), and thus \( T \) has a unique fixed point \( y_{n+1} \). So \( y_{n+1} \) is the unique function in \( C[I_{n+1}, B] \) so that \( y_{n+1}(x) = q + \int_p^x A y_{n+1} \) for all \( x \) in \( I_{n+1} \).
So by induction \( y_k \) is defined for each positive integer \( k \). Define \( y(x) = y_m(x) \) whenever \( x \in I_m \setminus I_{m-1} \). Then \( y \) is the desired function.

The following corollary (See [6].) shows that Theorem A guarantees the existence of solutions to some functional differential equations. Suppose \( g \) is a function from \( I^* \) to \( I^* \) so that \( g(I_n) \subseteq I_n \) for each positive integer \( n \). Such a function is said to be an \( I \)-function. Let \( A_k = \{ x \in [a_k, a_{k+1}] | g(x) \in I_k \} \) and let \( B_k = \{ x \in [b_k, b_{k+1}] | g(x) \in I_{k+1} \} \), for each positive integer \( k \). Also, suppose \( ||F(x, y) - F(x, z)|| \leq M(x) \cdot ||y - z|| \) for all \( x \in I^*, y, z \in B \); and \( M \) is Lebesgue integrable on intervals.

**COROLLARY.** If \( q \in B \), and \( \max \left\{ \int_{A_k} M, \int_{B_k} M \right\} < 1 \), for all \( k \); then there is a unique \( y \in C[I^*, B] \) so that \( y(p) = q \) and \( y'(x) = F(x, y(g(x))) \) for all \( x \in I^* \).
Proof. Let \((Af)(x) = F(x, f(g(x)))\). Then \(A\) is an \(I\)-map with function \(T\), where

\[
T(x) = \begin{cases} 
M(x), & x \in A_n \cup B_n \\
0, & x \notin A_n \cup B_n 
\end{cases} \quad \text{for } x \in I_n \setminus I_{n-1}.
\]

The proof of the following is straightforward.

**Proposition.** Suppose \(I\) is a nest of intervals about \(p\), and each of \(\alpha\) and \(\beta\) is an \(I\)-function. Then

(i) Suppose \(P\) is of bounded variation on each interval contained in \(I^*\), and let \(Af(x) = \int_{\alpha(x)}^{\beta(x)} dF \cdot f\), for \(f \in C(I, B)\) and \(x \in \text{domain } f\). Then \(A\) is an \(I\)-map with function \(M\), where \(M(x)\) is the variation of \(F\) over \(\alpha(x), \beta(x)\)\(\setminus I_{k-1}\) where \(x \in I_k \setminus I_{k-1}\).

(ii) Suppose \(K : I^* \times I^* \to \text{the scalars which is continuous, and} \)

\(Af(x) = \int_{\alpha(x)}^{\beta(x)} K(x, t) f(t) dt\), for \(f \in C(I, B)\) and \(x \in \text{domain } f\). Then \(A\)

is an \(I\)-map with function \(M\), where \(M(x) = \int_{\alpha(x), \beta(x)\setminus I_{k-1}} |K(x, t)| dt\)

for \(x \in I_k \setminus I_{k-1}\).

It is easy to show that the set of \(I\)-maps, for a fixed nest of intervals \(I\), is a near-ring under composition and addition. Thus, there are many types of differential equations that may be solved by combining \(I\)-maps of the types given in the corollary and the proposition.

3. Continuous dependence.

**Theorem B.** Suppose \(A(z, \cdot)\) is an \(I\)-map with strong function \(M\) for each \(z\) in the topological space \(K\), \(q \in B\), and \(M_k = \max \left\{ \int_M, \int_{\alpha(x)}^{\beta(x)} M \right\} < 1\), for all positive integers \(k\). Let \(y(z, \cdot)\) be the unique function, guaranteed by Theorem A, so that \(y(z, \cdot) = A(z, y(z, \cdot))\) and \(y(z, p) = q\). Then, there exists a sequence \(\{L_i\}\) so that for \(z, z_0 \in K\),

\(\|y(z, \cdot) - y(z_0, \cdot)\|_{I} \leq L_i \cdot |A(z, y(z_0, \cdot)) - A(z_0, y(z_0, \cdot))|_{I_i}\), for each \(i\).

[In the previous inequality the norm is the supremum norm over \(I_i\).]

**Indication of proof.** Define \(\{L_i\}\) as follows: Let \(L_i = \max (p - a_i, b_i - p)/(1 - M_i)\). For \(i \geq 1\), let \(L_{i+1} = \{L_i + \max (a_i - a_{i+1}, b_{i+1} - b_i)\}/(1 - M_{i+1})\).

**Example.** Let \(g\) be an \(I\)-function and let \(N > 0\). Then let \(K\) be the metric space of all \(I\)-functions that are pointwise never more that \(N\) from \(g\). Define \(A(h, y) = y(h|_{\text{dom } s})\) and \(d(h, h) = \sup \{|h, (x) - h_0(x)|/x \in I^*\}; d\) is the metric.
4. Nth order equations.

**Theorem C.** Suppose \( A \) is an \( I \)-map with function \( M, n \) is a positive integer, and \( q_0, q, \ldots, q_{n-1} \in B \). Let

\[
N_k = \max \left\{ \int_{s_k}^{s_{k-1}} \cdots \int_{s_1}^{s_0} M(s) ds_n \cdots ds_1, \right. \\
\left. \int_{b_{k-1}}^{b_k} \cdots \int_{b_1}^{b_0} M(s) ds_n \cdots ds_1 \right\}.
\]

Then, if \( N_k < 1 \), for all positive integers \( k \), there is a unique \( y \in C[I^*, B] \) so that \( y'(n) = Ay \) and \( y(p) = q, \ldots, y^{(n-1)}(p) = q_{n-1} \).

**Indication of proof.** Use induction, Theorem A, and the following lemma.

**Lemma.** Suppose \( H \) is an \( I \)-map with function \( S, \) and \( q \in B \), then define \( Kf(x) = q + \int_p^x Hf, \) for all \( f \in C(I, B) \) and \( x \in \text{domain } f \). Then \( K \) is an \( I \)-map with function \( T, \) where \( T(x) = \int_{x}^{a_k-1} S, \) whenever \( x \in (a_k, a_{k-1}] \); and \( T(x) = \int_{b_k-1}^{b_{k-1}} S, \) whenever \( x \in [b_{k-1}, b_k). \)

The proof of Theorem D is straightforward and Theorem E is a special case of Theorem D. Both of these theorems are imitations of standard theorems of ordinary differential equations.

**Theorem D.** (A generalized system of equations theorem.) Suppose \( B_i \) is a Banach space with norm \( \| \cdot \|_i \), for each positive integer \( i \) between 1 and \( n \). Let \( B' = \{(x_1, x_2, \ldots, x_n)/x_i \in B_i\} \). Also, let \( \| (x_1, \ldots, x_n) \| = \max \{\|x_i\|/1 \leq i \leq n\} \), for all elements of \( B' \). [Then \( B' \) is a Banach space.] Furthermore, suppose \( H_i: C(I, B_i) \to C(I, B_i) \) for \( 1 \leq i \leq n \) so that

1. if \( f \in C(I, B'), \) domain \( f = \text{domain } H_if, \)
2. if \( f \in C(I, B'), \) and \( I_k \subseteq \text{domain } f, k > 0, \) then \( (H_if)|_{I_k} = H_i(f|_{I_k}), \) and
3. there is \( M_i: I^* \) to the reals which is Lebesgue integrable on intervals so that if \( f, g \in C[I_k, B'], f|_{I_k-1} = g|_{I_k-1}, \) and \( x \in I_k, \) then

\[
\| H_if(x) - H_ig(x) \| \leq M_i(x) \cdot |f - g|. \]

Now, define \( A: C(I, B') \to C(I, B') \) so that \( Af = (H_1f, H_2f, \ldots, H_nf), \) for all \( f \in C(I, B'). \)

Then \( A \) is an \( I \)-map with function \( \max \{M_i/1 \leq i \leq n\}. \)

**Theorem E.** Suppose \( B' \) is as in Theorem D, with \( B = B_i, \) for all \( i. \) Also, suppose \( H = H_n \) and \( M = M_n, \) where \( H_n \) and \( M_n \) are as in Theorem D. Suppose \( q_0, \ldots, q_{n-1} \in B \) and
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\[
\max \left\{ \int_{a_k}^{a_{k-1}} \max \{1, M\}, \int_{b_k}^{b_{k-1}} \max \{1, M\} \right\} < 1, \text{ for all } k > 0.
\]

Then, there is a unique \( y \in C[I^*, B] \) so that

\[
y^{(n)} = H((y, y^{(1)}, \ldots, y^{(n-1)})) \quad \text{and} \quad y^{(i)} = q_i, \quad \text{for } 0 \leq i \leq n - \tau.
\]

**EXAMPLE.** Suppose each \( g_i \) is an \( I \)-function, then for appropriate functions \( F_i \), Theorem E guarantees the existence of a solution to

\[
y^{(n)}(x) = \sum_{k=1}^{n} F_k(x, y^{(n-k)}(g_k(x))), \quad \text{for all } x \in I^*.
\]

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Received May 10, 1972. This research was supported in part by a National Aeronautics and Space Administration Traineeship, and is part of the author’s Ph. D. thesis, which was directed by J. W. Neuberger, Emory University.

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