A CLASS OF GENERALIZED FUNCTIONAL DIFFERENTIAL EQUATIONS

Muril Lynn Robertson
A CLASS OF GENERALIZED FUNCTIONAL DIFFERENTIAL EQUATIONS

MURIL ROBERTSON

In this paper, the equation $y' = Ay$ is solved, where $A$ is a self-mapping of a certain set of functions. Also, a continuous dependence theorem is proven, and $n$th-order differential equations are considered.

1. Definitions. If $p$ is a real number and $I = \{I_1, I_2, \cdots\}$ is a collection of intervals so that $p \in I_i$ and $I_n \subseteq I_{n+1}$ for each positive integer $n$, then $I$ is said to be a nest of intervals about $p$. Let $I_0 = \{p\}$ and $[a_n, b_n] = I_n$ for each nonnegative integer $n$. Let $I^*$ denote the union of all the elements of $I$.

In general, $B$ denotes a Banach space; and if $D$ is a real number set, let $C[D, B]$ denote the set of continuous functions from $D$ into $B$. Whenever $D$ is an interval, $C[D, B]$ is considered a Banach space with supremum norm $| \cdot |$.

Let $C(I, B)$ denote the set of continuous functions whose domain is either $I_0$, $I^*$, or an element of $I$; and whose range is a subset of $B$.

Suppose $A$ is a mapping from $C(I, B)$ into $C(I, B)$ so that

(i) domain $f = domain Af$, for all $f \in C(I, B)$,

(ii) $(Af)|_{I_k} = A(f|_{I_k})$, for all $f \in C(I, B)$ and $I_k \subseteq domain f$, for positive $k$, [Note: $f|_{I_k}$ is the restriction of $f$ to $I_k$] and

(iii) there is a function $M$ from $I^*$ into the nonnegative reals that is Lebesgue integrable on any interval contained in $I$, so that $||Af(x) - Ag(x)|| \leq M(x) \cdot |f - g|$, for all $f, g \in C[I_i, B]$ so that $f|_{I_{i-1}} = g|_{I_{i-1}}$ and $x \in I_i$, for each positive integer $i$.

Then, $A$ is said to be an $I$-map with function $M$. Furthermore, if the phrase "$f|_{I_{i-1}} = g|_{I_{i-1}}$" is removed from part (iii) of the previous definition, $A$ is said to be an $I$-map with strong function $M$.

2. Main results.

THEOREM A. Suppose $A$ is an $I$-map with function $M$; and

$$\max \left\{ \sum_{i=1}^{a_i-1} M, \sum_{i=1}^{b_i-1} M \right\} < 1, \text{ for all positive integers } i.$$

Then if $q \in B$, there is a unique $y \in C[I^*, B]$ so that $y' = Ay$ and $y(p) = q$.

Proof. Let $\{(p, q) = y_0$. Then $y_0$ is the unique function in $C[I_0, B]$ so that $y_0(x) = q + \int_p^x Ay_0$ for all $x \in I_0$. Now, suppose $n$ is a non-negative integer so that $y_n$ has been defined in $C[I_n, B]$ to be the unique function so that $y_n(x) = q + \int_p^x Ay_n$ for all $x \in I_n$. Then, $D =$
\{f \in C[I_{n+1}, B] \mid f|_{I_n} = y_n\} is a complete metric space. If \( f \in D \), let
\[ T_f(x) = q + \int_p^x A f, \] for all \( x \in I_{n+1} \). Now if \( x \in I_n \) and \( f \in D \), then
\[ T_f(x) = q + \int_p^x A f = q + \int_p^x (A f)|_{I_n} = q + \int_p^x A y_n = y_n(x). \]
Thus \( (T_f)|_{I_n} = y_n \), and thus \( T_f \in D \).

Suppose \( f, g \in D \). Then,
\[
| T_f - T_g | = \max \left\{ \left| \int_p^x A f(s) - A g(s) \right| ds \middle| x \in I_{n+1} \right\}
\] \[ \leq \max \left\{ \left| \int_p^x A f(s) - A g(s) \right| ds \middle| x \in I_{n+1} \right\}. \]

Note that \( f|_{I_n} = g|_{I_n} \) and this implies that \( A(f|_{I_n}) = A(g|_{I_n}) \). Thus, \( (A f)|_{I_n} = (A g)|_{I_n} \); that is, \( A f(s) = A g(s) \) for all \( s \) in \( I_n \). So
\[
| T_f - T_g | \leq \max \left\{ \left| \int_p^x M(s) \cdot | f - g | ds \middle| x \in [b_n, b_{n+1}] \right|, \right. \]
\[
\left. \sup \left\{ \left| \int_p^x M(s) \cdot | f - g | ds \middle| x \in [a_{n+1}, a_n] \right| \right\} \right\}, \]
\[
\leq \max \left\{ \left| \int_p^x M(s) \cdot | f - g | ds \middle| x \in [b_n, b_{n+1}] \right|, \right. \]
\[
\left. \sup \left\{ \left| \int_p^x M(s) \cdot | f - g | ds \middle| x \in [a_{n+1}, a_n] \right| \right\} \right\}, \]
\[
\leq \max \left\{ \left| \int_p^x M(s) \cdot | f - g | ds \right| \middle| x \in [b_n, b_{n+1}] \right\}. \]

Hence \( T \) is a contraction map from the complete metric space \( D \) into \( D \), and thus \( T \) has a unique fixed point \( y_{n+1} \). So \( y_{n+1} \) is the unique function in \( C[I_{n+1}, B] \) so that \( y_{n+1}(x) = q + \int_p^x A y_{n+1} \) for all \( x \) in \( I_{n+1} \). So by induction \( y_k \) is defined for each positive integer \( k \). Define \( y(x) = y_m(x) \) whenever \( x \in I_m \setminus I_{m-1} \). Then \( y \) is the desired function.

The following corollary (See [6].) shows that Theorem A guarantees the existence of solutions to some functional differential equations. Suppose \( g \) is a function from \( I^* \) to \( I^* \) so that \( g(I_n) \subseteq I_n \) for each positive integer \( n \). Such a function is said to be an \( I \)-function. Let \( A_k = \{ x \in [a_k, a_{k-1}] / g(x) \in I_{k-1} \} \) and let \( B_k = \{ x \in [b_{k-1}, b_k] / g(x) \in I_{k-1} \} \), for each positive integer \( k \). Also, suppose \( || F(x, y) - F(x, z) || \leq M(x) \cdot || y - z || \) for all \( x \in I^* \), \( y, z \in B \); and \( M \) is Lebesgue integrable on intervals.

**Corollary.** If \( q \in B \), and \( \max \left\{ \int_{A_k} M, \int_{B_k} M \right\} < 1 \), for all \( k \); then there is a unique \( y \in C[I^*, B] \) so that \( y(p) = q \) and \( y'(x) = F(x, y(g(x))) \) for all \( x \in I^* \).
Proof. Let \((Af)(x) = F(x, f(g(x)))\). Then \(A\) is an \(I\)-map with function \(T\), where
\[
T(x) = \begin{cases} 
M(x), & x \in A_n \cup B_n \\
0, & x \notin A_n \cup B_n 
\end{cases}
\text{for } x \in I_n \setminus I_{n-1}.
\]
The proof of the following is straightforward.

**PROPOSITION.** Suppose \(I\) is a nest of intervals about \(p\), and each of \(\alpha\) and \(\beta\) is an \(I\)-function. Then

(i) Suppose \(P\) is of bounded variation on each interval contained in \(I^*\), and let \(Af(x) = \int_{\alpha(x)}^{\beta(x)} dF \cdot f\), for \(f \in C(I, B)\) and \(x \in \text{domain } f\). Then \(A\) is an \(I\)-map with function \(M\), where \(M(x)\) is the variation of \(F\) over \(\left[\alpha(x), \beta(x)\right] \setminus I_{k-1}\) where \(x \in I_k \setminus I_{k-1}\).

(ii) Suppose \(K: I^* \times I^*\) to the scalars which is continuous, and \(Af(x) = \int_{\alpha(x)}^{\beta(x)} K(x, t)f(t)dt\), for \(f \in C(I, B)\) and \(x \in \text{domain } f\). Then \(A\) is an \(I\)-map with function \(M\), where \(M(x) = \left| \int_{\left[\alpha(x), \beta(x)\right] \setminus I_{k-1}} K(x, t)dt \right|\) for \(x \in I_k \setminus I_{k-1}\).

It is easy to show that the set of \(I\)-maps, for a fixed nest of intervals \(I\), is a near-ring under composition and addition. Thus, there are many types of differential equations that may be solved by combining \(I\)-maps of the types given in the corollary and the proposition.

3. Continuous dependence.

**THEOREM B.** Suppose \(A(z, \cdot)\) is an \(I\)-map with strong function \(M\) for each \(z\) in the topological space \(K\), \(q \in B\), and \(M_k = \max \left\{ \int_{a_k}^{b_k} M, \int_{b_k}^{a_k} M \right\} < 1\), for all positive integers \(k\). Let \(y(z, \cdot)\) be the unique function, guaranteed by Theorem A, so that \(y(z, \cdot) = A(z, y(z, \cdot))\) and \(y(z, p) = q\). Then, there exists a sequence \(\{L_i\}\) so that for \(z, z_0 \in K\), \(|y(z, \cdot) - y(z_0, \cdot)|_{L_i} \leq L_i \cdot |A(z, y(z_0, \cdot)) - A(z_0, y(z_0, \cdot))|_{L_i}\), for each \(i\). [In the previous inequality the norm is the supremum norm over \(I_i\).]

**Indication of proof.** Define \(\{L_i\}\) as follows: Let \(L_i = \max (p - a_i, b_i - p)/(1 - M_i)\). For \(i \geq 1\), let \(L_{i+1} = \left\{ L_i + \max (a_i - a_{i+1}, b_i - b_{i+1}) \right\} / (1 - M_{i+1})\).

**EXAMPLE.** Let \(g\) be an \(I\)-function and let \(N > 0\). Then let \(K\) be the metric space of all \(I\)-functions that are pointwise never more that \(N\) from \(g\). Define \(A(h, y) = y(h|_{\text{dom} y})\) and \(d(h_i, h_0) = \sup \{|h_i(x) - h_2(x)|/x \in I^*\}; d\) is the metric.
4. Nth order equations.

**THEOREM C.** Suppose \( A \) is an I-map with function \( M, n \) is a positive integer, and \( q_0, q_1, \ldots, q_{n-1} \in B \). Let

\[
N_k = \max \left\{ \int_{s_{k-1}}^{s_k} \cdots \int_{s_{n-1}}^{s_n} M(s_n) ds_n \cdots ds_1, \right. \\
\left. \int_{s_{k-1}}^{s_k} \cdots \int_{s_{n-1}}^{s_n} M(s_n) ds_n \cdots ds_1 \right\}.
\]

Then, if \( N_k < 1 \), for all positive integers \( k \), there is a unique \( y \in C[I^*, B] \) so that \( y^{(n)} = Ay \) and \( y(p) = q_0, \ldots, y^{(n-1)}(p) = q_{n-1} \).

**Indication of proof.** Use induction, Theorem A, and the following lemma.

**LEMMA.** Suppose \( H \) is an I-map with function \( S \), and \( q \in B \), then define \( Kf(x) = q + \int_x f Hf, \) for all \( f \in C(I, B) \) and \( x \in \text{domain } f \). Then \( K \) is an I-map with function \( T \), where \( T(x) = \int_x^S S \), whenever \( x \in (a_k, a_{k-1}] \); and \( T(x) = \int_{b_{k-1}}^{b_k} S \), whenever \( x \in [b_{k-1}, b_k) \).

The proof of Theorem D is straightforward and Theorem E is a special case of Theorem D. Both of these theorems are imitations of standard theorems of ordinary differential equations.

**THEOREM D.** (A generalized system of equations theorem.) Suppose \( B_i \) is a Banach space with norm \( \| \cdot \|_i \), for each positive integer \( i \) between 1 and \( n \). Let \( B' = \{ (x_1, x_2, \ldots, x_n) | x_i \in B_i \} \). Also, let \( \|(x_1, \ldots, x_n)\| = \max \{ \|x_i\|/1 \leq i \leq n \} \), for all elements of \( B' \). [Then \( B' \) is a Banach space.] Furthermore, suppose \( H_i : C(I, B') \) to \( C(I, B_i) \) for \( 1 \leq i \leq n \) so that

1. if \( f \in C(I, B') \), domain \( f \) = domain \( H_if \),
2. if \( f \in C(I, B') \), and \( I_k \subseteq \text{domain } f, k > 0 \), then \( (H_if)|_{I_k} = H_i(f|_{I_k}) \), and
3. there is \( M_i : I^* \to \text{the reals which is Lebesgue integrable on intervals so that if } f, g \in C[I_k, B'], f|_{I_{k-1}} = g|_{I_{k-1}}, \text{ and } x \in I_k, \text{ then } \|H_if(x) - H_ig(x)\| \leq M_i(x) \cdot \|f - g\|. \)

Now, define \( A : C(I, B') \) to \( C(I, B') \) so that \( Af = (H_1f, H_2f, \ldots, H_nf) \), for all \( f \in C(I, B') \).

Then \( A \) is an I-map with function \( \max \{ M_i/1 \leq i \leq n \} \).

**THEOREM E.** Suppose \( B' \) is as in Theorem D, with \( B = B_i \), for all \( i \). Also, suppose \( H = H_n \) and \( M = M_n \), where \( H_n \) and \( M_n \) are as in Theorem D. Suppose \( q_0, \ldots, q_{n-1} \in B \) and
max \left\{ \frac{a_{k-1}}{b_k} \max \{1, M\}, \frac{b_k}{b_{k-1}} \max \{1, M\} \right\} < 1, \text{ for all } k > 0.

Then, there is a unique } y \in C[I^*, B] \text{ so that } y^{(n)} = H((y, y^{(1)}, \ldots, y^{(n-1)})) \text{ and } y^{(i)} = q_i, \text{ for } 0 \leq i \leq n - \epsilon.

**EXAMPLE.** Suppose each } g_i \text{ is an } I\text{-function, then for appropriate functions } F_i, \text{ Theorem E guarantees the existence of a solution to }

\[ y^{(n)}(x) = \sum_{k=1}^{n} F_k(x, y^{(n-k)}(g_k(x))), \text{ for all } x \in I^*. \]

**REFERENCES**

1. David R. Anderson, *An existence theorem for a solution of* \( f'(x) = F(x, f(g(x))) \), SIAM Review, 8 (1966), 359-362.
6. Muril Robertson, *The equation* \( y'(t) = F(t, y(g(t))) \), Pacific J. Math., 43 (1972), 483-491.
7. Y. T. Siu, *On the solution of the equation* \( f'(x) = \lambda f(g(x)) \), Math. Z., 90 (1965), 391-392.

Received May 10, 1972. This research was supported in part by a National Aeronautics and Space Administration Traineeship, and is part of the author’s Ph. D. thesis, which was directed by J. W. Neuberger, Emory University.

**AUBURN UNIVERSITY**
David Parham Bellamy, *Composants of Hausdorff indecomposable continua; a mapping approach* .................................................. 303

Colin Bennett, *A Hausdorff-Young theorem for rearrangement-invariant spaces* .......................................................... 311

Roger Daniel Bleier and Paul F. Conrad, *The lattice of closed ideals and a*-extensions of an abelian l-group* .................. 329

Ronald Elroy Bruck, Jr., *Nonexpansive projections on subsets of Banach spaces* ......................................................... 341

Robert C. Busby, *Centralizers of twisted group algebras* ....................... 357

M. J. Canfell, *Dimension theory in zero-set spaces* .................................................. 393

John Dauns, *One sided prime ideals* ................................................................. 401

Charles F. Dunkl, *Structure hypergroups for measure algebras* .............. 413

Ronald Francis Gariety, *Geometric properties of Sobolev mappings* ...... 427


Dennis Michael Girard, *The behavior of the norm of an automorphism of the unit disk* ..................................................... 443

George Rudolph Gorth, Jr., *Terminal subcontinua of hereditarily unicoherent continua* .................................................. 457


Neil Hindman, *The product of F-spaces with P-spaces* ............................ 473

M. A. Labbé and John Wolfe, *Isomorphic classes of the spaces Cσ(S)* ........ 481

Ernest A. Michael, *On k-spaces, kR-spaces and k(X)* .............................. 487

Donald Steven Passman, *Primitive group rings* ........................................... 499

C. P. L. Rhodes, *A note on primary decompositions of a pseudovaluation* ............................................................................ 507

Mural Lynn Robertson, *A class of generalized functional differential equations* ................................................................. 515

Ruth Silverman, *Decomposition of plane convex sets. I* ......................... 521

Ernest Lester Stitzinger, *On saturated formations of solvable Lie algebras* ............................................................................. 531

B. Andreas Trosch, *Sloshing frequencies in a half-space by Kelvin inversion* .............................................................................. 539

L. E. Ward, *Fixed point sets* ........................................................................ 553

Michael John Westwater, *Hilbert transforms, and a problem in scattering theory* .............................................................. 567

Misha Zafran, *On the spectra of multipliers* ........................................... 609