

Pacific Journal of Mathematics

A CLASS OF GENERALIZED FUNCTIONAL DIFFERENTIAL EQUATIONS

MURIL LYNN ROBERTSON

A CLASS OF GENERALIZED FUNCTIONAL DIFFERENTIAL EQUATIONS

MURIL ROBERTSON

In this paper, the equation $y' = Ay$ is solved, where A is a self-mapping of a certain set of functions. Also, a continuous dependence theorem is proven, and n th-order differential equations are considered.

1. **Definitions.** If p is a real number and $I = \{I_1, I_2, \dots\}$ is a collection of intervals so that $p \in I_1$ and $I_n \subseteq I_{n+1}$ for each positive integer n , then I is said to be a nest of intervals about p . Let $I_0 = \{p\}$ and $[a_n, b_n] = I_n$ for each nonnegative integer n . Let I^* denote the union of all the elements of I .

In general, B denotes a Banach space; and if D is a real number set, let $C[D, B]$ denote the set of continuous functions from D into B . Whenever D is an interval, $C[D, B]$ is considered a Banach space with supremum norm $|\cdot|$.

Let $C(I, B)$ denote the set of continuous functions whose domain is either I_0, I^* , or an element of I ; and whose range is a subset of B .

Suppose A is a mapping from $C(I, B)$ into $C(I, B)$ so that

(i) domain $f =$ domain Af , for all $f \in C(I, B)$,

(ii) $(Af)|_{I_k} = A(f|_{I_k})$, for all $f \in C(I, B)$ and $I_k \subseteq$ domain f , for positive k , [Note: $f|_{I_k}$ is the restriction of f to I_k .] and

(iii) there is a function M from I^* into the nonnegative reals that is Lebesgue integrable on any interval contained in I , so that $\|Af(x) - Ag(x)\| \leq M(x) \cdot |f - g|$, for all $f, g \in C[I, B]$ so that $f|_{I_{i-1}} = g|_{I_{i-1}}$ and $x \in I_i$, for each positive integer i .

Then, A is said to be an I -map with function M . Furthermore, if the phrase " $f|_{I_{i-1}} = g|_{I_{i-1}}$ " is removed from part (iii) of the previous definition, A is said to be an I -map with strong function M .

2. Main results.

THEOREM A. *Suppose A is an I -map with function M ; and $\max \left\{ \int_{a_i}^{a_{i-1}} M, \int_{b_{i-1}}^{b_i} M \right\} < 1$, for all positive integers i . Then if $q \in B$, there is a unique $y \in C[I^*, B]$ so that $y' = Ay$ and $y(p) = q$.*

Proof. Let $\{(p, q)\} = y_0$. Then y_0 is the unique function in $C[I_0, B]$ so that $y_0(x) = q + \int_p^x Ay_0$ for all $x \in I_0$. Now, suppose n is a non-negative integer so that y_n has been defined in $C[I_n, B]$ to be the unique function so that $y_n(x) = q + \int_p^x Ay_n$ for all $x \in I_n$. Then, $D =$

$\{f \in C[I_{n+1}, B] / f|_{I_n} = y_n\}$ is a complete metric space. If $f \in D$, let $Tf(x) = q + \int_p^x Af$, for all $x \in I_{n+1}$. Now if $x \in I_n$ and $f \in D$, then $Tf(x) = q + \int_p^x Af = q + \int_p^x (Af)|_{I_n} = q + \int_p^x A(f|_{I_n}) = q + \int_p^x Ay_n = y_n(x)$. Thus $(Tf)|_{I_n} = y_n$, and thus $Tf \in D$.

Suppose $f, g \in D$. Then,

$$\begin{aligned} |Tf - Tg| &= \max \{ \| Tf(x) - Tg(x) \| / x \in I_{n+1} \} \\ &= \max \left\{ \left\| \int_p^x (Af - Ag) \right\| \right\} \\ &\leq \max \left\{ \left\| \int_p^x \| Af(s) - Ag(s) \| ds \right\| \right\}. \end{aligned}$$

Note that $f|_{I_n} = g|_{I_n}$ and this implies that $A(f|_{I_n}) = A(g|_{I_n})$. Thus, $(Af)|_{I_n} = (Ag)|_{I_n}$; that is, $Af(s) = Ag(s)$ for all s in I_n . So

$$\begin{aligned} |Tf - Tg| &\leq \max \left\{ \sup \left\{ \int_{b_n}^x \| Af(s) - Ag(s) \| ds / x \in [b_n, b_{n+1}] \right\}, \right. \\ &\quad \left. \sup \left\{ \int_x^{a_n} \| Af(s) - Ag(s) \| ds / x \in [a_{n+1}, a_n] \right\} \right\} \\ &\leq \max \left\{ \sup \left\{ \int_{b_n}^x M(s) \cdot |f - g| ds / x \in [b_n, b_{n+1}] \right\}, \right. \\ &\quad \left. \sup \left\{ \int_x^{a_n} M(s) \cdot |f - g| ds / x \in [a_{n+1}, a_n] \right\} \right\} \\ &\leq \max \left\{ \int_{a_{n+1}}^{a_n} M, \int_{b_n}^{b_{n+1}} M \right\} \cdot |f - g|. \end{aligned}$$

Hence T is a contraction map from the complete metric space D into D , and thus T has a unique fixed point y_{n+1} . So y_{n+1} is the unique function in $C[I_{n+1}, B]$ so that $y_{n+1}(x) = q + \int_p^x Ay_{n+1}$ for all x in I_{n+1} . So by induction y_k is defined for each positive integer k . Define $y(x) = y_m(x)$ whenever $x \in I_m \setminus I_{m-1}$. Then y is the desired function.

The following corollary (See [6].) shows that Theorem A guarantees the existence of solutions to some functional differential equations. Suppose g is a function from I^* to I^* so that $g(I_n) \subseteq I_n$ for each positive integer n . Such a function is said to be an I -function. Let $A_k = \{x \in [a_k, a_{k-1}] / g(x) \notin I_{k-1}\}$ and let $B_k = \{x \in [b_{k-1}, b_k] / g(x) \notin I_{k-1}\}$, for each positive integer k . Also, suppose $\|F(x, y) - F(x, z)\| \leq M(x) \cdot \|y - z\|$ for all $x \in I^*$, $y, z \in B$; and M is Lebesgue integrable on intervals.

COROLLARY. *If $q \in B$, and $\max \left\{ \int_{A_k} M, \int_{B_k} M \right\} < 1$, for all k ; then there is a unique $y \in C[I^*, B]$ so that $y(p) = q$ and $y'(x) = F(x, y(g(x)))$ for all $x \in I^*$.*

Proof. Let $(Af)(x) = F(x, f(g(x)))$. Then A is an I -map with function T , where

$$T(x) = \begin{cases} M(x), & x \in A_n \cup B_n \\ 0, & x \notin A_n \cup B_n \end{cases}, \text{ for } x \in I_n \setminus I_{n-1}.$$

The proof of the following is straightforward.

PROPOSITION. *Suppose I is a nest of intervals about p , and each of α and β is an I -function. Then*

(i) *Suppose P is of bounded variation on each interval contained in I^* , and let $Af(x) = \int_{\alpha(x)}^{\beta(x)} dF \cdot f$, for $f \in C(I, B)$ and $x \in \text{domain } f$. Then A is an I -map with function M , where $M(x)$ is the variation of F over $[\alpha(x), \beta(x)] \setminus I_{k-1}$ where $x \in I_k \setminus I_{k-1}$.*

(ii) *Suppose $K: I^* \times I^*$ to the scalars which is continuous, and $Af(x) = \int_{\alpha(x)}^{\beta(x)} K(x, t)f(t)dt$, for $f \in C(I, B)$ and $x \in \text{domain } f$. Then A is an I -map with function M , where $M(x) = \left| \int_{[\alpha(x), \beta(x)] \setminus I_{k-1}} |K(x, t)| dt \right|$ for $x \in I_k \setminus I_{k-1}$.*

It is easy to show that the set of I -maps, for a fixed nest of intervals I , is a near-ring under composition and addition. Thus, there are many types of differential equations that may be solved by combining I -maps of the types given in the corollary and the proposition.

3. Continuous dependence.

THEOREM B. *Suppose $A(z, \cdot)$ is an I -map with strong function M for each z in the topological space K , $q \in B$, and $M_k = \max \left\{ \int_{a_k}^{a_{k-1}} M, \int_{b_{k-1}}^{b_k} M \right\} < 1$, for all positive integers k . Let $y(z, \cdot)$ be the unique function, guaranteed by Theorem A, so that $y_2(z, \cdot) = A(z, y(z, \cdot))$ and $y(z, p) = q$. Then, there exists a sequence $\{L_i\}$ so that for $z, z_0 \in K$, $|y(z, \cdot) - y(z_0, \cdot)|_{I_i} \leq L_i \cdot |A(z, y(z_0, \cdot)) - A(z_0, y(z_0, \cdot))|_{I_i}$, for each i . [In the previous inequality the norm is the supremum norm over I_i .]*

Indication of proof. Define $\{L_i\}$ as follows: Let $L_1 = \max(p - a_1, b_1 - p)/(1 - M_1)$. For $i \geq 1$, let $L_{i+1} = \{L_i + \max(a_i - a_{i+1}, b_{i+1} - b_i)\}/(1 - M_{i+1})$.

EXAMPLE. Let g be an I -function and let $N > 0$. Then let K be the metric space of all I -functions that are pointwise never more than N from g . Define $A(h, y) = y(h|_{\text{dom } y})$ and $d(h_1, h_2) = \sup \{|h_1(x) - h_2(x)|/x \in I^*\}$; d is the metric.

4. *N*th order equations.

THEOREM C. *Suppose A is an I-map with function M, n is a positive integer, and $q_0, q_1, \dots, q_{n-1} \in B$. Let*

$$N_k = \max \left\{ \int_{a_k}^{a_{k-1}} \int_{s_1}^{a_{k-1}} \dots \int_{s_{n-1}}^{a_{k-1}} M(s_n) ds_n \dots ds_1, \right. \\ \left. \int_{b_{k-1}}^{b_k} \int_{b_{k-1}}^{s_1} \dots \int_{b_{k-1}}^{s_{n-1}} M(s_n) ds_n \dots ds_1 \right\}.$$

Then, if $N_k < 1$, for all positive integers k, there is a unique $y \in C[I^, B]$ so that $y^{(n)} = Ay$ and $y(p) = q_0, \dots, y^{(n-1)}(p) = q_{n-1}$.*

Indication of proof. Use induction, Theorem A, and the following lemma.

LEMMA. *Suppose H is an I-map with function S, and $q \in B$, then define $Kf(x) = q + \int_p^x Hf$, for all $f \in C(I, B)$ and $x \in \text{domain } f$. Then K is an I-map with function T, where $T(x) = \int_x^{a_{k-1}} S$, whenever $x \in (a_k, a_{k-1}]$; and $T(x) = \int_{b_{k-1}}^x S$, whenever $x \in [b_{k-1}, b_k)$.*

The proof of Theorem D is straightforward and Theorem E is a special case of Theorem D. Both of these theorems are imitations of standard theorems of ordinary differential equations.

THEOREM D. *(A generalized system of equations theorem.) Suppose B_i is a Banach space with norm $\|\cdot\|_i$, for each positive integer i between 1 and n. Let $B' = \{(x_1, x_2, \dots, x_n) | x_i \in B_i\}$. Also, let $\|(x_1, \dots, x_n)\| = \max \{\|x_i\|_i / 1 \leq i \leq n\}$, for all elements of B' . [Then B' is a Banach space.] Furthermore, suppose $H_i: C(I, B')$ to $C(I, B_i)$ for $1 \leq i \leq n$ so that*

- (1) *if $f \in C(I, B')$, domain $f = \text{domain } H_i f$,*
- (2) *if $f \in C(I, B')$, and $I_k \subseteq \text{domain } f, k > 0$, then $(H_i f)|_{I_k} = H_i(f|_{I_k})$, and*
- (3) *there is $M_i: I^*$ to the reals which is Lebesgue integrable on intervals so that if $f, g \in C[I_k, B']$, $f|_{I_{k-1}} = g|_{I_{k-1}}$, and $x \in I_k$, then $\|H_i f(x) - H_i g(x)\| \leq M_i(x) \cdot |f - g|$. Now, define $A: C(I, B')$ to $C(I, B')$ so that $Af = (H_1 f, H_2 f, \dots, H_n f)$, for all $f \in C(I, B')$.*

Then A is an I-map with function $\max \{M_i / 1 \leq i \leq n\}$.

THEOREM E. *Suppose B' is as in Theorem D, with $B = B_i$, for all i. Also, suppose $H = H_n$ and $M = M_n$, where H_n and M_n are as in Theorem D. Suppose $q_0, \dots, q_{n-1} \in B$ and*

$$\max \left\{ \int_{a_k}^{a_{k-1}} \max \{1, M\}, \int_{b_{k-1}}^{b_k} \max \{1, M\} \right\} < 1, \text{ for all } k > 0 .$$

Then, there is a unique $y \in C[I^*, B]$ so that

$$y^{(n)} = H((y, y^{(1)}, \dots, y^{(n-1)})) \text{ and } y^{(i)} = q_i, \text{ for } 0 \leq i \leq n - 1 .$$

EXAMPLE. Suppose each g_i is an I -function, then for appropriate functions F_i , Theorem E guarantees the existence of a solution to

$$y^{(n)}(x) = \sum_{k=1}^n F_k(x, y^{(n-k)}(g_k(x))), \text{ for all } x \in I^* .$$

REFERENCES

1. David R. Anderson, *An existence theorem for a solution of $f'(x) = F(x, f(g(x)))$* , SIAM Review, **8** (1966), 359-362.
2. Gregory M. Dunkel, *On nested functional differential equations*, SIAM J. of Appl. Math., **18** (1970), 514-525.
3. W. B. Fite, *Properties of the solutions of certain functional differential equations*, Trans. Amer. Math. Soc., **22** (1921), 311-319.
4. Jack Hale, *Functional Differential Equations*, Springer-Verlag, New York, 1971.
5. Robert J. Oberg, *On the local existence of solutions of certain functional differential equations*, Proc. Amer. Math. Soc., **20** (1969), 295-302.
6. Muril Robertson, *The equation $y'(t) = F(t, y(g(t)))$* , Pacific J. Math., **43** (1972), 483-491.
7. Y. T. Siu, *On the solution of the equation $f'(x) = \lambda f(g(x))$* , Math. Z., **90** (1965), 391-392.

Received May 10, 1972. This research was supported in part by a National Aeronautics and Space Administration Traineeship, and is part of the author's Ph. D. thesis, which was directed by J. W. Neuberger, Emory University.

AUBURN UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

D. GILBARG AND J. MILGRAM

Stanford University
Stanford, California 94305

J. DUGUNDJI*

Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT

University of Washington
Seattle, Washington 98105

RICHARD ARENS

University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$48.00 a year (6 Vols., 12 issues). Special rate: \$24.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

* C. DePrima will replace J. Dugundji until August 1974.

Copyright © 1973 by
Pacific Journal of Mathematics
All Rights Reserved

David Parham Bellamy, <i>Composants of Hausdorff indecomposable continua; a mapping approach</i>	303
Colin Bennett, <i>A Hausdorff-Young theorem for rearrangement-invariant spaces</i>	311
Roger Daniel Bleier and Paul F. Conrad, <i>The lattice of closed ideals and a^*-extensions of an abelian l-group</i>	329
Ronald Elroy Bruck, Jr., <i>Nonexpansive projections on subsets of Banach spaces</i>	341
Robert C. Busby, <i>Centralizers of twisted group algebras</i>	357
M. J. Canfell, <i>Dimension theory in zero-set spaces</i>	393
John Dauns, <i>One sided prime ideals</i>	401
Charles F. Dunkl, <i>Structure hypergroups for measure algebras</i>	413
Ronald Francis Gariepy, <i>Geometric properties of Sobolev mappings</i>	427
Ralph Allen Gellar and Lavon Barry Page, <i>A new look at some familiar spaces of intertwining operators</i>	435
Dennis Michael Girard, <i>The behavior of the norm of an automorphism of the unit disk</i>	443
George Rudolph Gordh, Jr., <i>Terminal subcontinua of hereditarily unicoherent continua</i>	457
Joe Alston Guthrie, <i>Mapping spaces and cs-networks</i>	465
Neil Hindman, <i>The product of F-spaces with P-spaces</i>	473
M. A. Labbé and John Wolfe, <i>Isomorphic classes of the spaces $C_\sigma(S)$</i>	481
Ernest A. Michael, <i>On k-spaces, k_R-spaces and $k(X)$</i>	487
Donald Steven Passman, <i>Primitive group rings</i>	499
C. P. L. Rhodes, <i>A note on primary decompositions of a pseudovaluation</i>	507
Muril Lynn Robertson, <i>A class of generalized functional differential equations</i>	515
Ruth Silverman, <i>Decomposition of plane convex sets. I</i>	521
Ernest Lester Stitzinger, <i>On saturated formations of solvable Lie algebras</i>	531
B. Andreas Troesch, <i>Sloshing frequencies in a half-space by Kelvin inversion</i>	539
L. E. Ward, <i>Fixed point sets</i>	553
Michael John Westwater, <i>Hilbert transforms, and a problem in scattering theory</i>	567
Misha Zafran, <i>On the spectra of multipliers</i>	609