ON SATURATED FORMATIONS OF SOLVABLE LIE ALGEBRAS

ERNEST LESTER STITZINGER
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ERNEST L. STITZINGER

The concepts of formations, $\mathcal{F}$-projectors and $\mathcal{F}$-normalizers have all been developed for solvable Lie algebras. In this note, for each saturated formation $\mathcal{F}$ of solvable Lie algebras, the class $\mathcal{I}(\mathcal{F})$ of solvable Lie algebras $L$ in which each $\mathcal{F}$-normalizer of $L$ is an $\mathcal{F}$-projector is considered. This is the natural generalization of the Lie algebra analogue to $SC$ groups which were first investigated by R. Carter. It is shown that $\mathcal{I}(\mathcal{F})$ is a formation. Then some properties of $\mathcal{F}$-normalizers of $L \in \mathcal{I}(\mathcal{F})$ are considered.

All Lie algebras considered here are solvable and finite dimensional over a field $F$. $\mathcal{F}$ will always denote a saturated formation of solvable Lie algebras and $L$ will be a solvable Lie algebra. $N(L)$ is the nil-radical of $L$ and $\Phi(L)$ is the Frattini subalgebra of $L$. For definitions and properties of all these concepts see [3], [4], and [9]. For SC groups see [6].

We begin with a general lemma.

**Lemma 1.** Let $N$ be an ideal of $L$ and $D/N$ be an $\mathcal{F}$-normalizer of $L/N$. Then there exists an $\mathcal{F}$-normalizer $E$ of $L$ such that $E + N = D$.

**Proof.** Let $L$ be a minimal counterexample and we may assume that $N$ is a minimal ideal of $L$. If $D/N = L/N$, then any $\mathcal{F}$-normalizer of $L$ has the desired property, hence we may suppose that $D/N \subset L/N$. Suppose first that $N$ is $\mathcal{F}$-central in $L$. Let $N*/N = N(L/N)$ and $C = C_L(N)$. Then $N(L) = N* \cap C$. Let $M/N$ be a maximal $\mathcal{F}$-critical subalgebra of $L/N$ such that $D/N$ is an $\mathcal{F}$-normalizer of $M/N$. Now either $M$ is $\mathcal{F}$-critical in $L$ or $M$ complements a chief factor of $L$ between $N*$ and $N(L)$. In the first case, by induction, there exists an $\mathcal{F}$-normalizer $E$ of $M$ such that $E + N = D$ and $E$ is also an $\mathcal{F}$-normalizer in $L$. In the second case, $L/C \in \mathcal{F}$ and $C + N*/C$ is operator isomorphic to $N*/N* \cap C = N*/N(L)$. Hence each chief factor of $L$ between $N*$ and $N(L)$ is $\mathcal{F}$-central which contradicts $M$ being $\mathcal{F}$-abnormal.

Now suppose that $N$ is $\mathcal{F}$-eccentric and assume $N \subseteq \Phi(L)$. Let $M/N$ be as in the above paragraph. Again, by induction, there exists an $\mathcal{F}$-normalizer $E$ of $M$ such that $E + N = D$. But $N \subseteq \Phi(L)$ yields that $M$ is $\mathcal{F}$-critical in $L$ using Theorem 2.5 of [4]. Hence $E$ is an
Finally suppose that $N$ is $\mathcal{F}$-eccentric and assume $N \not\subseteq \Phi(L)$. Then $N$ is complemented by a maximal subalgebra $M$ which must be $\mathcal{F}$-critical in $L$. Now there must exist an $\mathcal{F}$-normalizer $E$ of $M$ such that $E + N = D$. Again $E$ must be an $\mathcal{F}$-normalizer of $L$ and the result is shown.

**COROLLARY.** $\mathcal{F}(\mathcal{F})$ is closed under homomorphisms.

**Proof.** Let $N$ be a minimal ideal of $L$, $L \in \mathcal{F}(\mathcal{F})$. Let $D/N$ be an $\mathcal{F}$-normalizer of $L/N$. Then $D = E + N$ for some $\mathcal{F}$-normalizer of $L$. Now $E$ is an $\mathcal{F}$-projector of $L$ and $E + N/N = D/N$ is an $\mathcal{F}$-projector of $L/N$.

**LEMMA 2.** If $L \in \mathcal{F}(\mathcal{F})$ and $C$ is an $\mathcal{F}$-projector of $L$, then $C$ is an $\mathcal{F}$-normalizer of $L$.

**Proof.** Let $N$ be a minimal ideal of $L$, $L/N \in \mathcal{F}(\mathcal{F})$ hence $C + N/N$ is an $\mathcal{F}$-normalizer of $L/N$ by induction. Hence $C + N = D + N$ for some $\mathcal{F}$-normalizer $D$ of $L$. Now $D$ is also an $\mathcal{F}$-projector of $L$ and both $C$ and $D$ are $\mathcal{F}$-projectors of $C + N$. Then $C$ and $D$ are conjugate in $C + N$ by an inner automorphism of $C + N$ induced by an element of $N$ by Lemma 1.11 of [3]. Hence $D$ and $C$ are conjugate in $L$ and the result holds.

Note that $\mathcal{F}(\mathcal{F})$ contains a large class of Lie algebras. In fact by Theorem 3 of [9] we have

**LEMMA 3.** $N.\mathcal{F} \subseteq \mathcal{F}(\mathcal{F})$.

In order to obtain that $\mathcal{F}(\mathcal{F})$ is a formation, we record a characterization of $\mathcal{F}$-projectors which is completely analogous to a result in group theory due to Bauman [5]. Since the proofs carry over virtually unchanged, we omit them.

**DEFINITION.** If $M$ is a subalgebra of $L$, then a series $0 = L_0 \subseteq \cdots \subseteq L_n = L$ is called an $M$-series if $L_i$ is an ideal in $L_{i+1}$, if $M \subseteq N_L(L_i)$ and if each $L_{i+1}/L_i$ is a nontrivial, irreducible $M$-factor of $L$.

**THEOREM 1.** If $C$ is an $\mathcal{F}$-projector of $L$ and $\{L_i\}$, $0 \leq i \leq n$, is any $C$-series of $L$, then $C$ covers $L_i/L_{i-1}$ if and only if $C + L_i/L_{i-1} \in \mathcal{F}$.

**Proof.** See proof of Theorem 1 of [5].
THEOREM 2. If \{L_i\} is a C-series of L such that C covers \(L_i/L_{i-1}\) if and only if \(C + L_i/L_{i-1} \in \mathcal{F}\), then C is an \(\mathcal{F}\)-projector of L.

Proof. See proof of Theorem 2 of [5].

We intend to use these results in a slightly different form by means of

LEMMA 4. Let M be a subalgebra of L, \(M \in \mathcal{F}\) and \(H/K\) be a nontrivial, irreducible M-factor of L. Then \(M + H/K \in \mathcal{F}\) if and only if the split extension of \(H/K\) by \(M/C_M(H/K)\) is in \(\mathcal{F}\).

Proof. Since \(M + H/H \in \mathcal{F}\), \(M + H/K\) will be in \(\mathcal{F}\) if and only if the minimal ideal \(H/K\) of \(M + H/K\) is \(\mathcal{F}\)-central in \(M + H/K\); that is, if and only if the split extension of \(H/K\) by \(M + H/C_M(H/K)\) is in \(\mathcal{F}\). But

\[
M/C_M(H/K) = M/M \cap C_{M+H}(H/K) = M + C_{M+H}(H/K)/C_{M+H}(H/K).
\]

Now the corresponding split extensions of \(H/K\) by \(M + H/C_{M+H}(H/K)\) and \(H/K\) by \(M/C_M(H/K)\) are isomorphic and the result holds.

THEOREM 3. \(\mathcal{I}(\mathcal{F})\) is a formation.

Proof. \(\mathcal{I}(\mathcal{F})\) is closed under homomorphisms has been noted already. Hence let \(N_1\) and \(N_2\) be ideals of L such that \(L/N_1, L/N_2 \in \mathcal{I}(\mathcal{F})\). We may assume \(N_1 \cap N_2 = 0\) and show that \(L \in \mathcal{I}(\mathcal{F})\). Let D be an \(\mathcal{F}\)-normalizer of L. Then \(D + N_1/N_1\) is an \(\mathcal{F}\)-normalizer of \(L/N_1\), hence is an \(\mathcal{F}\)-projector of \(L/N_1\) and the corresponding statement holds for \(D + N_2/N_2\). Consider a D-series of L which passes through \(N_1\) and \(N_1 + N_2\). There is a D-series of L which passes through \(N_2\) and \(N_1 + N_2\) which is the same as the original D-series above \(N_1 + N_2\) and corresponds to the original D-series below \(N_1 + N_2\) in the natural way. In particular, a factor \(H/K\) in the new D-series which is between \(N_2\) and \(N_1 + N_2\) corresponds to \(H \cap N_1/K \cap N_i\) in the original D-series and we claim that D covers (avoids) \(H/K\) if and only if \(D\) covers (avoids) \(H \cap N/K \cap N_i\). For if \(D\) avoids \(H/K\), then \(D \cap H \subseteq K\), hence \(D \cap H \cap N_1 \subseteq K \cap N_i\) and \(D\) avoids \(H \cap N_1/K \cap N_i\). Suppose that \(D\) covers \(H/K\). Then \(H \subseteq K + D\). In order to show that \(D\) covers \(H \cap N_1/K \cap N_i\) it is sufficient to show that \(D + (K \cap N_i) \subseteq H \cap N_i\). Since \(H \subseteq K + D, D \subseteq N_i(K)\) and \(H \subseteq N_i + N_i\), it follows that \(H \subseteq K + (D \cap (N_1 + N_2))\). Using the corollary on p. 241 of [9], \(H \subseteq K + ((D \cap N_1) + (D \cap N_2)) = K + (D \cap N_1)\). Then,
since $D \cap N_i \subset N_i(K)$ it follows that $H \cap N_i \leq (K + (D \cap N_i)) \cap N_i \leq (K \cap N_i) + (D \cap N_i) \leq (K \cap N_i) + D$, hence $D$ covers $H \cap N_i/K \cap N_i$.

By Theorem 1 and Lemma 4, a factor $H/K$ above $N_i$ in the original $D$-series is covered by $D + N_i/N_i$ (hence $D$) if and only if the split extension of $H/K$ by $D + N_i/C_{D+x}(H/K)$ is in $\mathcal{F}$. That is, $H/K$ is covered by $D$ if and only if the split extension of $H/K$ by $D/C_{D}(H/K)$ is in $\mathcal{F}$.

A similar statement holds above $N_2$. Every $D$-factor in the original series is operator isomorphic to a $D$-factor above $N_1$ or above $N_2$ and, using the result of the above paragraph, in the original $D$-series a factor $H/K$ is covered by $D$ if and only if the split extension of $H/K$ by $D/C_{D}(H/K)$ is in $\mathcal{F}$. Now by Lemma 4 and Theorem 2, $D$ is an $\mathcal{F}$-projector of $L$ and $\mathcal{F}(\mathcal{F})$ is a formation.

The following example shows that $\mathcal{N}, \mathcal{N} \subset \mathcal{F}(\mathcal{N})$ and that $\mathcal{F}(\mathcal{N})$ is not closed under taking ideals. It is a variant of an example on p. 52 of [7].

**Example.** Let $F$ be a field of characteristic $p \geq 2$ and let $A$ be a vector space over $F$ with basis $e_0, \ldots, e_{p-1}$. Define linear transformations $x, y, z$ on $A$ by

$x(e_i) = ie_i$

$y(e_i) = e_{i+1}$

and

$z(e_i) = e_i$

(subscripts mod $p$). Then $[x, y] = xy - yx = y$ and $[x, z] = [y, z] = 0$. Let $B$ be the three dimensional Lie algebra generated by $x, y, z$. Let $L$ be the semi-direct sum of $A$ and $B$ with the natural product. As on p. 53 of [7], $B$ acts irreducibly on $A$ so that $A$ is a minimal ideal of $L$. Evidently $A$ is self-centralizing in $L$, hence $A$ is the unique minimal ideal of $L$ and $N(L) = A$. Hence each $\mathcal{N}$-critical maximal subalgebra of $L$ complements $A$. Furthermore, $L$ is clearly of nilpotent length three.

Consider first any $\mathcal{N}$-normalizer $E$ of $L$ which is also an $\mathcal{N}$-normalizer of $B$. Such $\mathcal{N}$-normalizer exists since $B$ is a maximal $\mathcal{N}$-critical subalgebra of $L$. By the covering-avoidance property of $\mathcal{N}$-normalizers of $B, E = ((z, x + \alpha y))$ where $\alpha \in F$. Now $B$ is of nilpotent length 2, hence $E$ is a Cartan subalgebra of $B$. Now since $z \in E$, it is easily verified that $E$ is a Cartan subalgebra of $L$.

Now in general, each $\mathcal{N}$-normalizer of $L$ is an $\mathcal{N}$-normalizer of some $\mathcal{N}$-critical maximal subalgebra $M$ of $L$ and $M$ must complement $A$. But $L$ is of nilpotent length 3 and $L/A$ is of nilpotent length 2, hence $M$ must be conjugate to $B$ by Theorem 8 of [8].
Consequently, any \( \mathcal{N} \)-normalizer of \( L \) is a Cartan subalgebra of \( L \) and \( L \in \mathcal{F}(\mathcal{N}) \).

Now the ideal \( P = A + ((x, y)) \) of \( L \) is not in \( \mathcal{F}(\mathcal{N}) \). For \( ((x)) \subseteq ((x, y)) \subset P \) is a maximal \( \mathcal{N} \)-critical chain of \( P \), hence \( ((x)) \) is an \( \mathcal{N} \)-normalizer of \( P \). However, the normalizer of \( ((x)) \) in \( P \) is \( ((x, e)) \). Hence \( L \in \mathcal{F}(\mathcal{N}) \).

We recall that each \( \mathcal{F} \)-normalizer is contained in an \( \mathcal{F} \)-projector (Theorem 6 of [9]). However, the usual converse result, namely each \( \mathcal{F} \)-projector contains an \( \mathcal{F} \)-normalizer has not been obtained, even for \( \mathcal{N}\mathcal{F}\mathcal{F} \)-Lie algebras. We now show that this result holds if \( L \in \mathcal{N}\mathcal{F}\mathcal{F} \). First we record the following result which is needed.

**Theorem 4.** Let \( L \in \mathcal{N}\mathcal{F}(\mathcal{F}) \). Then each \( \mathcal{F} \)-normalizer of \( L \) is contained in a unique \( \mathcal{F} \)-projector of \( L \).

**Proof.** Same as the proof of Theorem 9 of [9].

**Theorem 5.** Let \( L \in \mathcal{N}\mathcal{F}(\mathcal{F}) \). Then each \( \mathcal{F} \)-projector of \( L \) contains an \( \mathcal{F} \)-normalizer of \( L \).

**Proof.** Let \( N \) be a minimal ideal of \( L \) and let \( C \) be an \( \mathcal{F} \)-projector of \( L \). Then \( C + N/N \) is an \( \mathcal{F} \)-projector of \( L/N \) and \( C + N/N \) contains an \( \mathcal{F} \)-normalizer \( D/N \) of \( L/N \) by induction. Let \( T = C + N \) and let \( F \) be an \( \mathcal{F} \)-normalizer of \( L \) such that \( F + N = D \subseteq T \). Then \( F \) is contained in an \( \mathcal{F} \)-projector \( G \) of \( L \) and \( D/N \subseteq G + N/N \). Hence \( G + N = C + N \) by Theorem 4 and \( G \) and \( C \) are \( \mathcal{F} \)-projectors of \( T \). By Lemma 1.11 of [3], \( G \) and \( C \) are conjugate in \( T \) by an inner automorphism of \( T \) induced by an element of \( N \). Hence \( G \) and \( C \) are conjugate in \( L \) and the result holds.

\( \mathcal{F} \)-normalizers have the covering-avoidance property but the converse is not true in general. However, if \( L \in \mathcal{F}(\mathcal{F}) \), then the converse is true.

**Theorem 6.** Let \( L \in \mathcal{F}(\mathcal{F}) \). If \( D \) is a subalgebra of \( L \) which covers the \( \mathcal{F} \)-central chief factors of \( L \) and avoids the \( \mathcal{F} \)-eccentric chief factors of \( L \), then \( D \) is an \( \mathcal{F} \)-normalizer of \( L \).

**Proof.** Let \( N \) be a minimal ideal of \( L \). Then \( D + N/N \) has the covering-avoidance property in \( L/N \in \mathcal{F}(\mathcal{F}) \). By induction, \( D + N/N \) is an \( \mathcal{F} \)-normalizer of \( L/N \) and \( D + N = E + N = T \) for some \( \mathcal{F} \)-normalizer \( E \) of \( L \). Since \( L \in \mathcal{F}(\mathcal{F}) \), \( E \) is an \( \mathcal{F} \)-projector of \( L \) and then also of \( T \). If \( N \) is \( \mathcal{F} \)-central in \( L \), then \( N \subseteq D \) and \( N \subseteq E \), hence \( D = E \). Suppose \( N \) is \( \mathcal{F} \)-eccentric. Then \( D \cap N = 0 = E \cap N \). Now \( T \in \mathcal{N}\mathcal{F} \), hence \( E \) is an \( \mathcal{F} \)-normalizer of \( T \) by Theorem 3 of [9]. Furthermore, in a given chief series of \( T \) passing through \( N \), \( \mathcal{F} \)
covers all chief factors above $N$ and avoids all chief factors below $N$ and the same is true for $D$. Since $E$ is an $\mathcal{F}$-normalizer of $T$, each chief factor below $N$ must be $\mathcal{F}$-eccentric and each chief factor above $N$ must be $\mathcal{F}$-central. Hence, by Theorem 4 of [9], $D$ must be an $\mathcal{F}$-normalizer of $T$. By Theorem 3 of [9], $D$ must also be an $\mathcal{F}$-projector of $T$. Now $D$ and $E$ are conjugate in $T$ (hence in $L$) by an inner automorphism induced by an element of $N$. Hence $D$ is an $\mathcal{F}$-normalizer of $L$.

Henceforth we shall be concerned with the case $\mathcal{F} = \mathcal{N}$. Here we have the following stronger form of Theorem 4.

**Theorem 7.** Let $L \in \mathcal{N}_\mathcal{F}(\mathcal{N})$ and $D$ be an $\mathcal{N}$-normalizer of $L$. Then there exists a Cartan subalgebra $C$ of $L$ which contains every subalgebra $H$ of $L$ in which $D$ is subinvariant. In particular, $D$ is contained in a unique Cartan subalgebra of $L$. $C$ is the Fitting null component of $D$ acting on $L$.

**Proof.** $D + N(L)/N(L)$ is subinvariant in $H + N(L)/N(L)$ and $D + N(L)/N(L)$ is an $\mathcal{N}$-normalizer of $L/N(L) \in \mathcal{F}(\mathcal{N})$. Hence $D + N(L)/N(L) = H + N(L)/N(L)$ is a Cartan subalgebra of $L/N(L)$. Let $T = D + N(L) = H + N(L)$ and let $S$ be the Fitting null component of $D$ acting on $T$. Evidently $N_T(S) = S$ and $H \subseteq S$. Furthermore, $S = S \cap T = S \cap (D + N(L)) = D + (S \cap N(L))$. Each element of $D$ induces a nilpotent derivation on $S$ and $S \cap N(L)$ is a nilpotent ideal of $S$. Then, using Engel’s theorem, $S$ is nilpotent. Hence $S$ is a Cartan subalgebra of $T$ and also of $L$ by Lemma 1.8 of [3]. If $K$ is another Cartan subalgebra of $L$ containing $D$, then $D$ is subinvariant in $K$, hence $K = S$. The last part of the theorem follows from the next lemma.

**Lemma 5.** Let $L$ be a solvable Lie algebra and $D$ be a nilpotent subalgebra of $L$. Let $F$ be the Fitting null component of $D$ acting on $L$. Then $D$ is subinvariant in $F$.

**Proof.** We may suppose that $F = L$. Let $A$ be a minimal ideal of $L$. Now in $D + A$, $A$ is an abelian ideal and each element of $D$ induces a nilpotent derivation of $D + A$. Hence, using Engel’s theorem, $D + A$ is nilpotent and $D$ is subinvariant in $D + A$. But $D + A/A$ satisfies the conditions in $L/A$, hence $D + A/A$ is subinvariant in $L/A$ by induction. Therefore, $D$ is subinvariant in $L$.

For Lie algebras of nilpotent length three, a result somewhat stronger than Theorem 7 holds. The proof is the same as the proof of Theorem 7, using Theorem 1 of [8] instead of the defining property of $\mathcal{F}(\mathcal{N})$, and may be omitted.
THEOREM 8. Let $L$ be of nilpotent length three (or less) and let $D$ be a nilpotent subalgebra of $L$ which can be joined to $L$ by a maximal chain of subalgebras, each self-normalizing in the next. Then there exists a Cartan subalgebra $C$ of $L$ which contains every subalgebra $H$ of $L$ in which $D$ is subinvariant. In particular, $D$ is contained in a unique Cartan subalgebra $C$ of $L$ and $C$ is the Fitting null component of $D$ acting on $L$.

We may use this to find a Lie algebra analogue to Theorem 10 of [2].

THEOREM 9. Let $M$ be a self-normalizing maximal subalgebra of $L$. Suppose that $L$ is of nilpotent length three. Then each Cartan subalgebra of $M$ is of the form $M \cap C$ for some Cartan subalgebra $C$ of $L$.

Proof. Let $D$ be a Cartan subalgebra of $M$. Then $D$ is contained in a Cartan subalgebra $C$ of $L$ by Theorem 8 and Lemma 1 of [8]. Now $M \cap C$ is nilpotent and $D$ is a Cartan subalgebra of $M \cap C$. Hence $D = M \cap C$.

The final result is of a slightly different nature. We consider the following: If an $N$-normalizer $D$ of $L$ is contained in the self-normalizing maximal subalgebra $M$ of $L$, then is $D$ contained in an $N$-normalizer of $M$. The analogous question for finite groups is answered negatively in [1]. The Lie algebra case also has a negative answer as in shown in the following result. The second part of this example is also an analogue to the example of [1].

THEOREM 10. There exists a solvable Lie algebra $L \in N \in N \in N$ which has an $N$-normalizer $D$, ideal $A$ and maximal subalgebra $M$ containing $D$ such that

1. $D$ is not contained in an $N$-normalizer of $M$
2. $N_{L/A}(D + A/A) \supseteq N_L(D) + A/A$.

Proof. This example is also a variant of an example found on p. 52 of [7]. Let $F$ be a field of characteristic $p > 2$. Let $A$ be the Lie algebra over $F$ with basis $a_0, a_1, \ldots, a_{p-1}, b, c_0, c_1, \ldots, c_{p-1}$ and products $[a_i, b] = c_i$ for $i = 0, \ldots, p - 1$ and all other products of basis elements equal to 0. Define linear transformations $x, y$ on $A$ such that

\begin{align*}
x(a_i) &= a_{i+1} & y(a_i) &= ia_i \\
x(b) &= 0 & y(b) &= 0 \\
x(c_i) &= c_{i+1} & y(c_i) &= ic_i
\end{align*}
Then $x$ and $y$ are derivations of $A$ and $[y, x] = x$.
Let $B$ be the 2-dimensional Lie algebra generated by $x$ and $y$ and let $L$ be the semi-direct sum of $A$ and $B$ with the natural product.

Let $R = ((c_0, \cdots, c_{p-1}))$ and $S = ((c_0, \cdots, c_{p-1}, b))$. The same argument used in [7] shows that $R$ and $A/S$ are $\mathcal{N}$-eccentric chief factors of $L$ and $S/R$ is clearly $\mathcal{N}$-central chief factor of $L$. Let $M = ((x, y, b, c_0, \cdots, c_{p-1}))$, $M_1 = ((x, y, b))$ and $M_2 = ((y, b))$. Each of these is a maximal $\mathcal{N}$-critical subalgebra of the preceding and $M$ is maximal, $\mathcal{N}$-critical in $L$. Now $\exp a_0$ is an automorphism of $L$ since $\text{char } F \neq 2$. Then $C = M_2^{\exp a_0} = ((y, b + c_0)) \subseteq M$ and $D$ is an $\mathcal{N}$-normalizer of $L$.

Now the $\mathcal{N}$-normalizers of $M$ have dimension 2 by the covering-avoidance property of $\mathcal{N}$-normalizers, hence, if $D$ is contained in an $\mathcal{N}$-normalizer of $M$, then it is one of them. If this is the case, then, since $b \in Z(M), b \in D$ and $\dim D > 2$, a contradiction.

For the second part, note that
\[ N_{L,R}(M_2 + R/R) = ((y + R, b + R, a_0 + R)). \]
However, an element of the form $\alpha a_0 + t, \alpha \in F, t \in R$ is not in $N_L(M_2)$ unless $\alpha = 0$, since $[b, \alpha a_0 + t] = -\alpha c_0$. Hence
\[ N_L(M_2) + R/R \subseteq N_{L,R}(M_2 + R/R). \]

References


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