ON SATURATED FORMATIONS OF SOLVABLE LIE ALGEBRAS

Ernest Lester Stitzinger
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ERNEST L. STITZINGER

The concepts of formations, \( \mathcal{F} \)-projectors and \( \mathcal{F} \)-normalizers have all been developed for solvable Lie algebras. In this note, for each saturated formation \( \mathcal{F} \) of solvable Lie algebras, the class \( \mathcal{F}(\mathcal{F}) \) of solvable Lie algebras \( L \) in which each \( \mathcal{F} \)-normalizer of \( L \) is an \( \mathcal{F} \)-projector is considered. This is the natural generalization of the Lie algebra analogue to SC groups which were first investigated by R. Carter. It is shown that \( \mathcal{F}(\mathcal{F}) \) is a formation. Then some properties of \( \mathcal{F} \)-normalizers of \( L \in \mathcal{F}(\mathcal{F}) \) are considered.

All Lie algebras considered here are solvable and finite dimensional over a field \( F \). \( \mathcal{F} \) will always denote a saturated formation of solvable Lie algebras and \( L \) will be a solvable Lie algebra. \( N(L) \) is the nil-radical of \( L \) and \( \Phi(L) \) is the Frattini subalgebra of \( L \). For definitions and properties of all these concepts see [3], [4], and [9]. For SC groups see [6].

We begin with a general lemma.

**Lemma 1.** Let \( N \) be an ideal of \( L \) and \( D/N \) be an \( \mathcal{F} \)-normalizer of \( L/N \). Then there exists an \( \mathcal{F} \)-normalizer \( E \) of \( L \) such that \( E + N = D \).

**Proof.** Let \( L \) be a minimal counterexample and we may assume that \( N \) is a minimal ideal of \( L \). If \( D/N = L/N \), then any \( \mathcal{F} \)-normalizer of \( L \) has the desired property, hence we may suppose that \( D/N \subset L/N \). Suppose first that \( N \) is \( \mathcal{F} \)-central in \( L \). Let \( N^*/N = N(L/N) \) and \( C = C_L(N) \). Then \( N(L) = N^* \cap C \). Let \( M/N \) be a maximal \( \mathcal{F} \)-critical subalgebra of \( L/N \) such that \( D/N \) is an \( \mathcal{F} \)-normalizer of \( M/N \). Now either \( M \) is \( \mathcal{F} \)-critical in \( L \) or \( M \) complements a chief factor of \( L \) between \( N^* \) and \( N(L) \). In the first case, by induction, there exists an \( \mathcal{F} \)-normalizer \( E \) of \( M \) such that \( E + N = D \) and \( E \) is also an \( \mathcal{F} \)-normalizer in \( L \). In the second case, \( L/C \in \mathcal{F} \) and \( C + N^*/C \) is operator isomorphic to \( N^*/N^* \cap C = N^*/N(L) \). Hence each chief factor of \( L \) between \( N^* \) and \( N(L) \) is \( \mathcal{F} \)-central which contradicts \( M \) being \( \mathcal{F} \)-abnormal.

Now suppose that \( N \) is \( \mathcal{F} \)-eccentric and assume \( N \supseteq \Phi(L) \). Let \( M/N \) be as in the above paragraph. Again, by induction, there exists an \( \mathcal{F} \)-normalizer \( E \) of \( M \) such that \( E + N = D \). But \( N \supseteq \Phi(L) \) yields that \( M \) is \( \mathcal{F} \)-critical in \( L \) using Theorem 2.5 of [4]. Hence \( E \) is an
\( \mathcal{F} \)-normalizer of \( L \) and this case is completed.

Finally suppose that \( N \) is \( \mathcal{F} \)-eccentric and assume \( N \not\in \Phi(L) \). Then \( N \) is complemented by a maximal subalgebra \( M \) which must be \( \mathcal{F} \)-critical in \( L \). Now there must exist an \( \mathcal{F} \)-normalizer \( E \) of \( M \) such that \( E + N = D \). Again \( E \) must be an \( \mathcal{F} \)-normalizer of \( L \) and the result is shown.

**Corollary.** \( \mathcal{F}(\mathcal{F}) \) is closed under homomorphisms.

**Proof.** Let \( N \) be a minimal ideal of \( L, L \in \mathcal{F}(\mathcal{F}) \). Let \( D/N \) be an \( \mathcal{F} \)-normalizer of \( L/N \). Then \( D = E + N \) for some \( \mathcal{F} \)-normalizer of \( L \). Now \( E \) is an \( \mathcal{F} \)-projector of \( L \) and \( E + N/N = D/N \) is an \( \mathcal{F} \)-projector of \( L/N \).

**Lemma 2.** If \( L \in \mathcal{F}(\mathcal{F}) \) and \( C \) is an \( \mathcal{F} \)-projector of \( L \), then \( C \) is an \( \mathcal{F} \)-normalizer of \( L \).

**Proof.** Let \( N \) be a minimal ideal of \( L \). \( L/N \in \mathcal{F}(\mathcal{F}) \) hence \( C + N/N \) is an \( \mathcal{F} \)-normalizer of \( L/N \) by induction. Hence \( C + N = D + N \) for some \( \mathcal{F} \)-normalizer \( D \) of \( L \). Now \( D \) is also an \( \mathcal{F} \)-projector of \( L \) and both \( C \) and \( D \) are \( \mathcal{F} \)-projectors of \( C + N \). Then \( C \) and \( D \) are conjugate in \( C + N \) by an inner automorphism of \( C + N \) induced by an element of \( N \) by Lemma 1.11 of [3]. Hence \( D \) and \( C \) are conjugate in \( L \) and the result holds.

Note that \( \mathcal{F}(\mathcal{F}) \) contains a large class of Lie algebras. In fact by Theorem 3 of [9] we have

**Lemma 3.** \( \mathcal{F} \mathcal{F} \subseteq \mathcal{F}(\mathcal{F}) \).

In order to obtain that \( \mathcal{F}(\mathcal{F}) \) is a formation, we record a characterization of \( \mathcal{F} \)-projectors which is completely analogous to a result in group theory due to Bauman [5]. Since the proofs carry over virtually unchanged, we omit them.

**Definition.** If \( M \) is a subalgebra of \( L \), then a series \( 0 = L_0 \subseteq \cdots \subseteq L_n = L \) is called an \( M \)-series if \( L_i \) is an ideal in \( L_{i+1} \), if \( M \subseteq N_L(L_i) \) and if each \( L_{i+1}/L_i \) is a nontrivial, irreducible \( M \)-factor of \( L \).

**Theorem 1.** If \( C \) is an \( \mathcal{F} \)-projector of \( L \) and \( \{L_i\}, 0 \leq i \leq n, \) is any \( C \)-series of \( L \), then \( C \) covers \( L_i/L_{i-1} \) if and only if \( C + L_i/L_{i-1} \in \mathcal{F} \).

**Proof.** See proof of Theorem 1 of [5].
THEOREM 2. If \( \{L_i\} \) is a C-series of \( L \) such that \( C \) covers \( L_i/L_{i-1} \)
if and only if \( C + L_i/L_{i-1} \in \mathcal{F} \), then \( C \) is an \( \mathcal{F} \)-projector of \( L \).

Proof. See proof of Theorem 2 of [5].

We intend to use these results in a slightly different form by means of

LEMMA 4. Let \( M \) be a subalgebra of \( L \), \( M \in \mathcal{F} \) and \( H/K \) be a nontrivial, irreducible \( M \)-factor of \( L \). Then \( M + H/K \in \mathcal{F} \) if and only if the split extension of \( H/K \) by \( M/C_M(H/K) \) is in \( \mathcal{F} \).

Proof. Since \( M + H/H \in \mathcal{F} \), \( M + H/K \) will be in \( \mathcal{F} \) if and only if the minimal ideal \( H/K \) of \( M + H/K \) is \( \mathcal{F} \)-central in \( M + H/K \); that is, if and only if the split extension of \( H/K \) by \( M/C_M(H/K) \) is in \( \mathcal{F} \). But

\[
M/C_M(H/K) = M/M \cap C_{M+n}(H/K) \cong M + C_{M+n}(H/K)/C_{M+n}(H/K)
= M + H/C_{M+n}(H/K).
\]

Now the corresponding split extensions of \( H/K \) by \( M + H/C_{M+n}(H/K) \) and \( H/K \) by \( M/C_M(H/K) \) are isomorphic and the result holds.

THEOREM 3. \( \mathcal{F}(\mathcal{F}) \) is a formation.

Proof. \( \mathcal{F}(\mathcal{F}) \) is closed under homomorphisms has been noted already. Hence let \( N_1 \) and \( N_2 \) be ideals of \( L \) such that \( L/N_1, L/N_2 \in \mathcal{F}(\mathcal{F}) \). We may assume \( N_1 \cap N_2 = 0 \) and show that \( L \in \mathcal{F}(\mathcal{F}) \). Let \( D \) be an \( \mathcal{F} \)-normalizer of \( L \). Then \( D + N_1/N_1 \) is an \( \mathcal{F} \)-normalizer of \( L/N_1 \), hence is an \( \mathcal{F} \)-projector of \( L/N_1 \) and the corresponding statement holds for \( D + N_2/N_2 \). Consider a \( D \)-series of \( L \) which passes through \( N_1 \) and \( N_1 + N_2 \). There is a \( D \)-series of \( L \) which passes through \( N_2 \) and \( N_1 + N_2 \) which is the same as the original \( D \)-series above \( N_1 + N_2 \) and corresponds to the original \( D \)-series below \( N_1 + N_2 \) in the natural way. In particular, a factor \( H/K \) in the new \( D \)-series which is between \( N_2 \) and \( N_1 + N_2 \) corresponds to \( H \cap N_1/K \cap N_1 \) in the original \( D \)-series and we claim that \( D \) covers (avoids) \( H/K \) if and only if \( D \) covers (avoids) \( H \cap N_1/K \cap N_1 \). For if \( D \) avoids \( H/K \), then \( D \cap H \subseteq K \), hence \( D \cap H \cap N_1 \subseteq K \cap N_1 \) and \( D \) avoids \( H \cap N_1/K \cap N_1 \). Suppose that \( D \) covers \( H/K \). Then \( H \subseteq D + K \). In order to show that \( D \) covers \( H \cap N_1/K \cap N_1 \) it is sufficient to show that \( D + (K \cap N_1) \cong H \cap N_1 \). Since \( H \subseteq K + D, D \subseteq N_1(K) \) and \( H \subseteq N_1 + N_2 \), it follows that \( H \subseteq K + (D \cap (N_1 + N_2)) \). Using the corollary on p. 241 of [9], \( H \subseteq K + ((D \cap N_1) + (D \cap N_2)) = K + (D \cap N_1) \). Then,
since $D \cap N_1 \subseteq N_2(K)$ it follows that $H \cap N_1 \subseteq (K + (D \cap N_1)) \cap N_1 \subseteq (K \cap N_1) + (D \cap N_1) \subseteq (K \cap N_1) + D$, hence $D$ covers $H \cap N_i/K \cap N_1$.

By Theorem 1 and Lemma 4, a factor $H/K$ above $N_1$ in the original $D$-series is covered by $D + N_1/N_1$ (hence $D$) if and only if the split extension of $H/K$ by $D + N_1/C_{D + N_1}(H/K)$ is in $\mathcal{F}$. That is, $H/K$ is covered by $D$ if and only if the split extension of $H/K$ by $D/C_D(H/K)$ is in $\mathcal{F}$. A similar statement holds above $N_2$. Every $D$-factor in the original series is operator isomorphic to a $D$-factor above $N_1$ or above $N_2$ and, using the result of the above paragraph, in the original $D$-series a factor $H/K$ is covered by $D$ if and only if the split extension of $H/K$ by $D/C_D(H/K)$ is in $\mathcal{F}$. Now by Lemma 4 and Theorem 2, $D$ is an $\mathcal{F}$-projector of $L$ and $\mathcal{F}(\mathcal{N})$ is a formation.

The following example shows that $\mathcal{N}\mathcal{N}^+ \subset \mathcal{F}(\mathcal{N})$ and that $\mathcal{F}(\mathcal{N})$ is not closed under taking ideals. It is a variant of an example on p. 52 of [7].

**Example.** Let $F$ be a field of characteristic $p \geq 2$ and let $A$ be a vector space over $F$ with basis $e_0, \ldots, e_{p-1}$. Define linear transformations $x, y, z$ on $A$ by

\[
x(e_i) = i e_i
\]

\[
y(e_i) = e_{i+1}
\]

and

\[
z(e_i) = e_i
\]

(subscripts mod $p$). Then $[x, y] = xy - yx = y$ and $[x, z] = [y, z] = 0$. Let $B$ be the three dimensional Lie algebra generated by $x, y, z$. Let $L$ be the semi-direct sum of $A$ and $B$ with the natural product. As on p. 53 of [7], $B$ acts irreducibly on $A$ so that $A$ is a minimal ideal of $L$. Evidently $A$ is self-centralizing in $L$, hence $A$ is the unique minimal ideal of $L$ and $N(L) = A$. Hence each $\mathcal{N}$-critical maximal subalgebra of $L$ complements $A$. Furthermore, $L$ is clearly of nilpotent length three.

Consider first any $\mathcal{N}$-normalizer $E$ of $L$ which is also an $\mathcal{N}$-normalizer of $B$. Such $\mathcal{N}$-normalizer exists since $B$ is a maximal $\mathcal{N}$-critical subalgebra of $L$. By the covering-avoidance property of $\mathcal{N}$-normalizers of $B$, $E = ((z, x + \alpha y))$ where $\alpha \in F$. Now $B$ is of nilpotent length 2, hence $E$ is a Cartan subalgebra of $B$. Now since $z \in E$, it is easily verified that $E$ is a Cartan subalgebra of $L$.

Now in general, each $\mathcal{N}$-normalizer of $L$ is an $\mathcal{N}$-normalizer of some $\mathcal{N}$-critical maximal subalgebra $M$ of $L$ and $M$ must complement $A$. But $L$ is of nilpotent length 3 and $L/A$ is of nilpotent length 2, hence $M$ must be conjugate to $B$ by Theorem 8 of [8].
Consequently, any $\mathcal{N}$-normalizer of $L$ is a Cartan subalgebra of $L$ and $L \in \mathcal{T}(\mathcal{N})$.

Now the ideal $P = A + ((x, y))$ of $L$ is not in $\mathcal{T}(\mathcal{N})$. For $((x)) \subset ((x, y)) \subset P$ is a maximal $\mathcal{N}$-critical chain of $P$, hence $((x))$ is an $\mathcal{N}$-normalizer of $P$. However, the normalizer of $((x))$ in $P$ is $((x, e_3))$. Hence $L \in \mathcal{T}(\mathcal{N})$.

We recall that each $\mathcal{F}$-normalizer is contained in an $\mathcal{F}$-projector (Theorem 6 of [9]). However, the usual converse result, namely each $\mathcal{F}$-projector contains an $\mathcal{F}$-normalizer has not been obtained, even for $\mathcal{N}\mathcal{N}\mathcal{F}$-Lie algebras. We now show that this result holds if $L \in \mathcal{N} \mathcal{F}(\mathcal{F})$. First we record the following result which is needed.

**Theorem 4.** Let $L \in \mathcal{N} \mathcal{F}(\mathcal{F})$. Then each $\mathcal{F}$-normalizer of $L$ is contained in a unique $\mathcal{F}$-projector of $L$.

**Proof.** Same as the proof of Theorem 9 of [9].

**Theorem 5.** Let $L \in \mathcal{N} \mathcal{F}(\mathcal{F})$. Then each $\mathcal{F}$-projector of $L$ contains an $\mathcal{F}$-normalizer of $L$.

**Proof.** Let $N$ be a minimal ideal of $L$ and let $C$ be an $\mathcal{F}$-projector of $L$. Then $C + N/N$ is an $\mathcal{F}$-projector of $L/N$ and $C + N/N$ contains an $\mathcal{F}$-normalizer $D/N$ of $L/N$ by induction. Let $T = C + N$ and let $F$ be an $\mathcal{F}$-normalizer of $L$ such that $F + N = D \subseteq T$. Then $F$ is contained in an $\mathcal{F}$-projector $G$ of $L$ and $D/N \subseteq G + N/N$. Hence $G + N = C + N$ by Theorem 4 and $G$ and $C$ are $\mathcal{F}$-projectors of $T$. By Lemma 1.11 of [3], $G$ and $C$ are conjugate in $T$ by an inner automorphism of $T$ induced by an element of $N$. Hence $G$ and $C$ are conjugate in $L$ and the result holds.

$\mathcal{F}$-normalizers have the covering-avoidance property but the converse is not true in general. However, if $L \in \mathcal{F}(\mathcal{F})$, then the converse is true.

**Theorem 6.** Let $L \in \mathcal{F}(\mathcal{F})$. If $D$ is a subalgebra of $L$ which covers the $\mathcal{F}$-central chief factors of $L$ and avoids the $\mathcal{F}$-eccentric chief factors of $L$, then $D$ is an $\mathcal{F}$-normalizer of $L$.

**Proof.** Let $N$ be a minimal ideal of $L$. Then $D + N/N$ has the covering-avoidance property in $L/N \in \mathcal{T}(\mathcal{F})$. By induction, $D + N/N$ is an $\mathcal{F}$-normalizer of $L/N$ and $D + N = E + N = T$ for some $\mathcal{F}$-normalizer $E$ of $L$. Since $L \in \mathcal{T}(\mathcal{F})$, $E$ is an $\mathcal{F}$-projector of $L$ and then also of $T$. If $N$ is $\mathcal{F}$-central in $L$, then $N \subseteq D$ and $N \subseteq E$, hence $D = E$. Suppose $N$ is $\mathcal{F}$-eccentric. Then $D \cap N = 0 = E \cap N$. Now $T \in \mathcal{N} \mathcal{F}$, hence $E$ is an $\mathcal{F}$-normalizer of $T$ by Theorem 3 of [9]. Furthermore, in a given chief series of $T$ passing through $N$, $E$
covers all chief factors above $N$ and avoids all chief factors below $N$ and the same is true for $D$. Since $E$ is an $\mathcal{T}$-normalizer of $T$, each chief factor below $N$ must be $\mathcal{T}$-eccentric and each chief factor above $N$ must be $\mathcal{T}$-central. Hence, by Theorem 4 of [9], $D$ must be an $\mathcal{T}$-normalizer of $T$. By Theorem 3 of [9], $D$ must also be an $\mathcal{T}$-projector of $T$. Now $D$ and $E$ are conjugate in $T$ (hence in $L$) by an inner automorphism induced by an element of $N$. Hence $D$ is an $\mathcal{T}$-normalizer of $L$.

Henceforth we shall be concerned with the case $\mathcal{T} = \mathcal{N}$. Here we have the following stronger form of Theorem 4.

**Theorem 7.** Let $L \in \mathcal{N}/\mathcal{T}(\mathcal{N})$ and $D$ be an $\mathcal{N}$-normalizer of $L$. Then there exists a Cartan subalgebra $C$ of $L$ which contains every subalgebra $H$ of $L$ in which $D$ is subinvariant. In particular, $D$ is contained in a unique Cartan subalgebra of $L$. $C$ is the Fitting null component of $D$ acting on $L$.

**Proof.** $D + N(L)/N(L)$ is subinvariant in $H + N(L)/N(L)$ and $D + N(L)/N(L)$ is an $\mathcal{N}$-normalizer of $L/N(L) \in \mathcal{T}(\mathcal{N})$. Hence $D + N(L)/N(L) = H + N(L)/N(L)$ is a Cartan subalgebra of $L/N(L)$. Let $T = D + N(L) = H + N(L)$ and let $S$ be the Fitting null component of $D$ acting on $T$. Evidently $N_T(S) = S$ and $H \subseteq S$. Furthermore, $S = S \cap T = S \cap (D + N(L)) = D + (S \cap N(L))$. Each element of $D$ induces a nilpotent derivation on $S$ and $S \cap N(L)$ is a nilpotent ideal of $S$. Then, using Engel's theorem, $S$ is nilpotent. Hence $S$ is a Cartan subalgebra of $T$ and also of $L$ by Lemma 1.8 of [3]. If $K$ is another Cartan subalgebra of $L$ containing $D$, then $D$ is subinvariant in $K$, hence $K = S$. The last past of the theorem follows from the next lemma.

**Lemma 5.** Let $L$ be a solvable Lie algebra and $D$ be a nilpotent subalgebra of $L$. Let $F$ be the Fitting null component of $D$ acting on $L$. Then $D$ is subinvariant in $F$.

**Proof.** We may suppose that $F = L$. Let $A$ be a minimal ideal of $L$. Now in $D + A$, $A$ is an abelian ideal and each element of $D$ induces a nilpotent derivation of $D + A$. Hence, using Engel's theorem, $D + A$ is nilpotent and $D$ is subinvariant in $D + A$. But $D + A/A$ satisfies the conditions in $L/A$, hence $D + A/A$ is subinvariant in $L/A$ by induction. Therefore, $D$ is subinvariant in $L$.

For Lie algebras of nilpotent length three, a result somewhat stronger than Theorem 7 holds. The proof is the same as the proof of Theorem 7, using Theorem 1 of [8] instead of the defining property of $\mathcal{T}(\mathcal{N})$, and may be omitted.
**Theorem 8.** Let $L$ be of nilpotent length three (or less) and let $D$ be a nilpotent subalgebra of $L$ which can be joined to $L$ by a maximal chain of subalgebras, each self-normalizing in the next. Then there exists a Cartan subalgebra $C$ of $L$ which contains every subalgebra $H$ of $L$ in which $D$ is subinvariant. In particular, $D$ is contained in a unique Cartan subalgebra $C$ of $L$ and $C$ is the Fitting null component of $D$ acting on $L$.

We may use this to find a Lie algebra analogue to Theorem 10 of [2].

**Theorem 9.** Let $M$ be a self-normalizing maximal subalgebra of $L$. Suppose that $L$ is of nilpotent length three. Then each Cartan subalgebra of $M$ is of the form $M \cap C$ for some Cartan subalgebra $C$ of $L$.

**Proof.** Let $D$ be a Cartan subalgebra of $M$. Then $D$ is contained in a Cartan subalgebra $C$ of $L$ by Theorem 8 and Lemma 1 of [8]. Now $M \cap C$ is nilpotent and $D$ is a Cartan subalgebra of $M \cap C$. Hence $D = M \cap C$.

The final result is of a slightly different nature. We consider the following: If an $N$-normalizer $D$ of $L$ is contained in the self-normalizing maximal subalgebra $M$ of $L$, then is $D$ contained in an $N$-normalizer of $M$. The analogous question for finite groups is answered negatively in [1]. The Lie algebra case also has a negative answer as shown in the following result. The second part of this example is also an analogue to the example of [1].

**Theorem 10.** There exists a solvable Lie algebra $L \in \mathcal{N}$ which has an $N$-normalizer $D$, ideal $A$ and maximal subalgebra $M$ containing $D$ such that

1. $D$ is not contained in an $N$-normalizer of $M$
2. $N_{L/A}(D + A/A) \supseteq N_L(D) + A/A$.

**Proof.** This example is also a variant of an example found on p. 52 of [7]. Let $F$ be a field of characteristic $p > 2$. Let $A$ be the Lie algebra over $F$ with basis $a_0, a_1, \ldots, a_{p-1}, b, c_0, c_1, \ldots, c_{p-1}$ and products $[a_i, b] = c_i$ for $i = 0, \ldots, p - 1$ and all other products of basis elements equal to 0. Define linear transformations $x, y$ on $A$ such that

$$\begin{align*}
x(a_i) &= a_{i+1} \\
x(b) &= 0 \\
x(c_i) &= c_{i+1}
\end{align*}$$

$$\begin{align*}
y(a_i) &= ia_i \\
y(b) &= 0 \\
y(c_i) &= ic_i
\end{align*}$$
(everything mod $p$). Then $x$ and $y$ are derivations of $A$ and $[y, x] = x$.
Let $B$ be the 2-dimensional Lie algebra generated by $x$ and $y$ and let $L$ be the semi-direct sum of $A$ and $B$ with the natural product.
Let $R = ((c_0, \ldots, c_{p-1}))$ and $S = ((c_0, \ldots, c_{p-1}, b))$. The same argument used in [7] shows that $R$ and $A/S$ are $N$-eccentric chief factors of $L$ and $S/R$ is clearly and $N$-central chief factor of $L$. Let $M = ((x, y, b, c_0, \ldots, c_{p-1}))$, $M_1 = ((x, y, b))$ and $M_2 = ((y, b))$. Each of these is a maximal $N$-critical subalgebra of the preceding and $M$ is maximal, $N$-critical in $L$. Now $\exp a_0$ is an automorphism of $L$ since char $F \neq 2$. Then $C = M_2^{\exp a_0} = ((y, b + c_0)) \subseteq M$ and $D$ is an $N$-normalizer of $L$.
Now the $N$-normalizers of $M$ have dimension 2 by the covering-avoidance property of $N$-normalizers, hence, if $D$ is contained in an $N$-normalizer of $M$, then it is one of them. If this is the case, then, since $b \in Z(M)$, $b \in D$ and dim $D > 2$, a contradiction.
For the second part, note that

$$N_{L/R}(M_2 + R/R) = ((y + R, b + R, a_0 + R)).$$

However, an element of the form $\alpha a_0 + t, \alpha \in F, t \in R$ is not in $N_L(M_2)$ unless $\alpha = 0$, since $[b, \alpha a_0 + t] = -\alpha c_0$. Hence

$$N_L(M_2) + R/R \subset N_{L/R}(M_2 + R/R).$$

**REFERENCES**


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