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THE SPACE OF BOUNDED SEQUENCES WITH THE MIXED TOPOLOGY

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AJIT KAUR CHILANA

The space of bounded sequences with the mixed topology has some interesting properties and can be used to answer two questions on boundedly generated spaces asked by T. Ito and T. Seidman.

1. Introduction. We consider the locally convex algebra m of bounded complex sequences with pointwise addition and multiplication equipped with the mixed topology [11]. The topology τ is the same as the strict topology β [1] on $C(S)$, when S is taken to be the space of positive integers. This space has a number of interesting properties, some of which can be found in [11], [3] (Example 3), [1], and [4]. In this note we obtain some further results such as: this space is hereditary boundedly generated [6], it has an unconditional Schauder basis, and its Gelfand map is continuous.

The basic ideas are as in [7]. A locally convex space E is said to be *boundedly generated*, in short, BG if it is the closed linear span of a bounded subset; it is said to be *hereditary boundedly generated*, in short, HBG, if every closed linear subspace of the space is BG [6]. E is called *sequentially* barrelled if every $\tau_s(E, E')$-null sequence in the topological dual E' of E is equicontinuous [10]. The *sequential dual* E^+ of E is the set of sequentially continuous linear functionals f on E (i.e., $f(x_n) \to 0$ whenever $x_n \to 0$ in E) [10]. E is called *semi 1-barrelled* if every $\tau_s(E, E')$-bounded sequence in E' is equicontinuous [3]. A barrel in a locally convex algebra E that is also an idempotent set (i.e., $A \subset E$ such that $A.A \subset A$) is called an *m-barrel* and E is said to be *m-barrelled* if every m-barrel in E is a neighborhood of 0 [8]. Let E be a complex locally convex algebra. Let M denote the set of nonzero, continuous, multiplicative, linear functionals on E, provided with the weak topology induced by E. Let $C(M)$ denote the space of complex continuous functions on M with the topology of compact convergence. The Gelfand map G on E to $C(M)$ is given by $G(x)(m) = m(x)$ $(x \in E, m \in M)$. E is called strongly semi-simple if G is an algebraic isomorphism of E into $C(M)$.

Now let E denote the space m with the mixed topology τ or, equivalently, the strict topology β. A base of τ-neighborhoods of 0 is given by

$$\mathscr{U} = \{U_a = \{x = (x_n): |x_n| \leqslant a_n,\ n = 1, 2, \cdots\},\ a = (a_n),\ 0 < a_n \to \infty\} .$$

Let $B = \{x \in E: |x_n| \leqslant 1,\ n = 1, 2, \cdots\}$, $||x|| = \sup\{|x_n|: n = 1, 2, \cdots\}$ for $x \in m$. Let $e^i = (x_n)$: $x_i = 1$ and $x_n = 0$ if $i \neq n$. Let

$$l^1 = \left\{x \in m: \sum_{n=1}^{\infty} |x_n| < \infty\right\},$$

and $||x||_1 = \sum_{n=1}^{\infty} |x_n|$ $(x \in l^1)$. The α-dual of m is l^1 ([7], § 30.1) and (m, l^1) is thus a dual pair.

2. Properties of the space E.

I. *E is complete and has the Mackey topology $\tau_k(l^1)$.*

It is proved in [1] (also in [3]) that E is complete. The second part follows from [4], Theorems 2 and 4, on taking S to be the set of positive integers with the discrete topology.

II. *No topology on m compatible with duality is barrelled or m-barrelled.*

The set B is a barrel [3] and not a neighborhood in E, also it is idempotent. As E carries the strongest topology compatible with duality and barrels remain barrels under any topology compatible with duality, the result follows immediately.

III. *E is an HBG space.*

For any subspace F of E, $F = \bigcup\{n(B \cap F): n = 1, 2, \cdots\}$ and, therefore, F is the closed linear hull of a bounded subset of itself.

REMARK 1. It was asked in question (2) of [6], if there are any *HBG* spaces that are not Banach or separable Fréchet spaces. II and III give an affirmative answer.

REMARK 2. In [2] the first part of question (3) in [6] was answered in the negative and the following more general question was raised: If F is a *BG* space with dual F', then must there be a barrelled topology compatible with duality (F, F')? The example given there to prove that the answer is "No" is artificial in the sense that its completion is a Banach space and thus barrelled. By I, E is a complete space and II and III show that it serves as a better example.

IV. *A sequentially continuous linear functional on E is continuous and E is sequentially barrelled.*

Combining [5], Theorem III (2.8) and I above, we have $E' = E^+$. Thus $\tau = \tau_k(E^+, E)$. Proposition 4.3 in [10] then gives that E is sequentialy barrelled.

V. *E is not semi 1-barrelled.*

Consider $A = \{e^n : n = 1, 2, \cdots\} \subset l^1 = E'$. For each $x \in E$, and for $n = 1, 2, \cdots$, $|e^n(x)| = |x_n| \leqslant \|x\|$. So A is $\tau_s(E', E)$-bounded. Also the polar A° of A in E is B which is not a neighborhood. Therefore, A is not equicontinuous.

REMARK 3. It is known ([3], Proposition 9 (ii), p. 481) that a semi 1-barrelled space is sequentially barrelled. IV and V show that the reverse implication may not be true. We take this opportunity to point out that there are two Proposition 9 in [3] (!) and in Proposition 9 (ii) on p. 481 [3] it should be almost semi-1-barrelled instead of almost semi-barrelled.

VI. *E has a Schauder basis* (e^n), *which is*

(i) *bounded multiplier,*

(ii) *boundedly complete,*

(iii) *not of type* P^*,

(iv) *unconditional,*

(v) *shrinking,*

(vi) *not of type* P,

(vii) *monotone, and*

(viii) *e-Schauder* [5].

We note that m is perfect and normal ([7], § 30.1), $E' = l' = m^\times$ and $\tau = \tau_k(l^1)$. We can use [5], I (2.5) to obtain (i), (ii), and (iii). To prove (iv) we appeal to [5], I (2.4) and (i) above. The strong dual of E is $(l^1, \|\cdot\|_1)$ and it has (e^n) as a Schauder basis, so (v) is true. For (vi) note that $e^n \to 0$ in E. Also $\mathscr{U}$ satisfies the conditions for (e^n) to be monotone and $S_N U_a \subset U_a$ for all $U_a \in \mathscr{U}$ and $N = 1, 2, \cdots$. Thus (vii) and (viii) are true.

REMARK 4. It is well-known that m with the sup-norm topology is not separable and thus cannot have a basis. The above result shows the difference a change in the topology can make.

VII. *C(M) is barrelled.*

Let $0 \neq f \in l^1$ and f be multiplicative on E. Because

$$x = \lim_{n\to\infty} \sum_{j=1}^{n} x_j e^j \ ,$$

$f(x) = \lim_{n\to\infty} \sum_{j=1}^{n} x_j f(e^j)$ for each $x \in E$. So there is an n such that $f(e^n) \neq 0$. Also $f(e^n) = f(e^n e^n) = f(e^n) f(e^n)$, so we must have $f(e^n) = 1$. Now for $j \neq n$ $f(e^n) f(e^j) = f(e^n e^j) = f(0) = 0$, so $f(e^j) = 0$. Thus $f(x) = x_n$ and f can be identified with $e^n \in l^1$. So

$$M = \{e^n : n = 1, 2, \cdots\} .$$

Also $\{x\}^\circ \cap M = \{e^n\}$ if $x \in m$ be such that $x_n = 1$ and $x_j = 2$ for $j \neq n$. Hence M can be identified with the set of positive integers with the discrete topology. Therefore, $C(M)$ is the space of all complex sequences with the topology of pointwise convergence and is, thus barrelled.

REMARK 5. We note that m-barrelledness of some topology compatible with duality is sufficient in [8], Lemma 3.1 (or [9], Cor. 6.3) and even this condition is not necessary as shown by II and VII.

VIII. *E has jointly continuous multiplication.*

If $a = (a_n)$ be such that $0 < a_n \to \infty$ then for $b = (b_n)$, where $b_n = a_n^{1/2}$, $0 < b_n \to \infty$ and also $U_b U_b \subset U_a$.

IX. *The Gelfand map is continuous but not a homeomorphism.*

It is immediate from the proof of VII.

The next result shows that E does not, however, have a good functional representation.

X. *E cannot be embedded algebraically and topologically in a C(X) for X a locally compact Hausdorff space or for X a completely regular Hausdorff space.*

From the proof of VII we get that G is an isomorphism of E into $C(M)$, and thus E is strongly semi-simple. Also in view of VIII, E is a topological algebra in the sense of [9]. Combining Theorem 4.6 of [9] and IX above we get the required result.

REMARK 6. This space also helps in distinguishing some classes of topological algebras such as m-k-barrelled algebras, m-k-infrabarrelled algebras, locally boundedly multiplicatively convex algebras.

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