DIFFERENTIABLE OPEN MAPS OF \((p + 1)\)-MANIFOLD TO \(p\)-MANIFOLD

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DIFFERENTIABLE OPEN MAPS OF 
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Let \( f: M^{p+1} \to N^p \) be a \( C^3 \) open map with \( p \geq 1 \), let \( R_{p-1}(f) \) be the critical set of \( f \), and let

\[
\dim (R_{p-1}(f) \cap f^{-1}(y)) \leq 0
\]

for each \( y \in N^p \). Then (1.1) there is a closed set \( X \subseteq M^{p+1} \) such that \( \dim f(X) \leq p - 2 \) and, for every \( x \in M^{p+1} - X \), there is a natural number \( d(x) \) with \( f \) at \( x \) locally topologically equivalent to the map

\[
\phi_{d(x)}: C \times R^{p-1} \to R \times R^{p-1}
\]

defined by

\[
\phi_{d(x)}(z, t_1, \ldots, t_{p-1}) = (\Re(z^{d(x)}), t_1, \ldots, t_{p-1})
\]

(\( \Re(z^{d(x)}) \) is the real part of the complex number \( z^{d(x)} \)).

The hypothesis on the critical set is essential \([3, (4.11)]\), but in \([4]\) we show that any real analytic open map satisfies this hypothesis, and thus this conclusion.

**Corollary 1.2.** If \( f: M^{p+1} \to N^p \) is a \( C^{p+1} \) open map with \( \dim (R_{p-1}(f)) \leq 0 \), then at each \( x \in M^{p+1} \), \( f \) is locally topologically equivalent to one of the following maps:

(a) the projection map \( \rho: R^{p+1} \to R^p \),
(b) \( \tau: C \times C \to C \times R \) defined by

\[
\tau(z, w) = (2z \cdot \bar{w}, |w|^2 - |z|^2),
\]

where \( \bar{w} \) is the complex conjugate of \( w \).
(c) \( \psi: C \to R \) defined by \( \psi_d(z) = \Re(z^d) \).

In order to read the proofs in this paper, the reader will need to have \([3]\) at hand. In particular, the terms locally topologically equivalent, branch set \( B_f \), layer map, extended embedding, and 0-regular are defined in \([3; (1.3), (1.5), (2.1), (2.3), \) and (4.1), respectively].

2. Spoke sets. The definition and lemmas of this section are given in somewhat greater generality than needed in this paper (i.e., for open maps), for use in a subsequent paper.

Let \( I^n \) be any 2-manifold (without boundary).

**Definition 2.1.** Let \( \psi \times \iota: C \times R^{p-1} \to R \times R^{p-1} \) be defined by

\[
\psi \times \iota(z, t) = (|z|, t)
\]

and \( \psi \times \iota(z, t) = (\Re(z^w), t) (w = 1, 2, \ldots) \). Thus
$B(\psi_1 \times \epsilon) = \emptyset$ and $B(\psi_w \times \epsilon) = \{0\} \times R^{p-1}$ otherwise. For $w = 0$ let $L = D^2 \times D^{p-1}$ and let $J = [-1, 1]$; for $w \geq 1$ and $\eta > 0$ sufficiently small, let

$$L = (D^2 \times D^{p-1}) \cap (\psi_w \times \epsilon)^{-1}([\eta, \eta] \times D^{p-1})$$

and let $J = [-\eta, \eta]$. These examples motivate the following definition.

Let $f: \Gamma^2 \times R^{p-1} \rightarrow R \times R^{p-1}$ be a layer map, let $J = [b_0, b_1] \subset R$, and let $W \subset R^{p-1}$ be a closed $q$-cell ($q = 0, 1, \cdots, p - 1$). Let $\{\gamma_j\}$ be a (possibly empty) collection of $2w$ disjoint closed arcs in $S'(j = 1, 2, \cdots, 2w)$; let $A = \bigcup_j \gamma_j$, and let $\zeta: S^1 \times W \rightarrow \Gamma^2 \times W$ be a layer embedding such that $B_f \cap \text{imag } \zeta = \emptyset$, $f \circ \zeta: \gamma_j \times W \approx J \times W$, and for each component $\Phi$ of $\text{Cl}[S^1 - A]$, $f(\zeta(\Phi \times W)) = \{b_i\} \times W$ ($i = 0$ or 1). A spoke set of $f$ over $J \times W$ is (i) a compact, connected subspace $L \subset f^{-1}(R \times W)$ such that (ii) $L \cap (\Gamma^2 \times \{t\})$ is a 2-cell for each $t \in W$ and (iii) for some $\zeta$ as above, the boundary $\Omega$ of $L$ with respect to $f^{-1}(R \times W)$ is imag $\zeta$. Thus if $A = \emptyset$, $f(\Omega) = \{b_i\} \times W$ ($i = 0$ or 1). (In case $A \neq \emptyset$ and $q = 1$, $L$ is homeomorphic to the hub and spokes of a wagon wheel, where $\zeta(A \times W)$ corresponds to the ends of the spokes.) The index $\xi(L) = 1 - w$.

**Lemma 2.2.** Let $f: \Gamma^2 \times R^{p-1} \rightarrow R \times R^{p-1}$ be a layer map with $\dim (B_f \cap (\Gamma^2 \times \{t\})) = \dim (f(B_f) \cap (R \times \{t\})) \leq 0$ for each $t \in R^{p-1}$, let $E \subset B_f$ be compact, let $a \in R^{p-1}$, and let $\varepsilon > 0$. Then there are a closed $(p - 1)$-cell neighborhood $W$ of $a$, closed intervals $J_j(j = 1, 2, \cdots, m)$, and spoke sets $L_j$ over $J_j \times W$ such that

1. $E \cap L_j \neq \emptyset$ and $E \cap (\Gamma^2 \times W) \subset \bigcup_j (L_j - \Omega_j)$,
2. the $L_j - \Omega_j$ are mutually disjoint, and
3. each diam $L_j < \varepsilon$.

**Proof.** Let $F$ be a compact neighborhood of $E$ in $\Gamma^2 \times R^{p-1}$, let $\{(U_a)\}$ be a cover of $\Gamma^2$ by interiors of closed 2-cells, and let $\delta$ be the Lebesgue number of $\{(U_a \times R^{p-1})\}$ as a cover of $F$. We may suppose that $\varepsilon < \min(\delta, d(E, \text{bdy } F))$. Thus

1. for each $\Psi \subset F$ with diam $\Psi < \varepsilon$, there is a closed 2-cell $U$ with $\Psi \subset (\text{int } U) \times R^{p-1}$.

Given $y \in R$ with $(y, a) \in f(E)$ and $X = E \cap f^{-1}(y, a)$, let $Q$ be the finite set and $\nu: Q \times D \rightarrow \Gamma^2 \times R^{p-1}$ be the extended embedding with imag $\nu \cap B_f = \emptyset$ given by [3, (2.5)] for $X$ and $\varepsilon$. According to that lemma each component $K$ of $f^{-1}(\text{int } D)$-imag $\nu$ meeting $X$ has diam $K < \varepsilon$, and each is open. Since $X = E \cap f^{-1}(y, a)$ and $E$ is compact, one may prove (by contradiction) that it is possible to
select the $p$-cell neighborhood $D$ of $(y,a)$ in $\mathbb{R} \times \mathbb{R}^{p-1}$ sufficiently small that each component $K$ of $f^{-1}(\text{int } D) - \text{imag } \nu$ meeting $E$ has diam $K < \varepsilon$. Summarizing,

(2) each component $K$ of $f^{-1}(\text{int } D) - \text{imag } \nu$ with $K \cap E \neq \emptyset$ has diam $K < \varepsilon$, so that $K \subset \text{int } F$.

Choose a closed interval $J(y) \subset R$ with $y \in \text{int } J(y)$,

$$J(y) \times \{a\} \subset \text{int } D,$$

and end points $b_0(y), b_1(y)$ with $(b_0(y), a), (b_1(y), a) \in f(B_j)$. Since $f(F \cap B_j)$ is closed, there is a closed $(p - 1)$-cell neighborhood $W(y)$ of $a$ in $\mathbb{R}^{p-1}$ such that $(\partial J(y) \times W(y)) \cap f(F \cap B_j) = \emptyset$ and

$$J(y) \times W(y) \subset D.$$

Let $\nu(y)$ be the corresponding extended embedding (restricted) over $J \times W$.

There are $y_1, y_2, \cdots, y_u \in R$ with $(y_i, a) \in f(E)$ and

$$f(E) \cap (R \times \{a\}) \subset \bigcup_i \text{int } (J(y_i)) \times \{a\}.$$

The points $\{b_i(y_j): i = 0, 1; j = 1, 2, \cdots, u\}$ are the end points of a finite set of closed intervals with mutually disjoint interiors; let $J_h(h = 1, 2, \cdots, r)$ be those intervals with $(J_h \times \{a\}) \cap f(E) \neq \emptyset$.

Let $W$ be a closed $(p - 1)$-cell neighborhood of $a \in \mathbb{R}^{p-1}$ with $W \subset \bigcap_i W(y_i)$. Then $(\partial J_h \times W) \cap f(F \cap B_j) = \emptyset$ and

$$f(E) \cap (R \times W) \subset \bigcup_h ((\text{int } J_h) \times W) \quad (h = 1, 2, \cdots, r).$$

Since each $J_h$ is contained in some $J(y_j)$, restriction of $\nu(y_j)$ yields an extended embedding $\nu_h$ over $J_h \times W$.

Let $J = [b_0, b_1]$ be one of these intervals $J_h$, let

$$\nu: (Q \times J) \times W \longrightarrow I^2 \times R^{p-1}$$

be the layer embedding $\nu_h$, and let $P \subset F$ be a component of

$$f^{-1}([b_i] \times W) - \text{imag } \nu.$$

Since $([b_i] \times W) \cap f(F \cap B_j) = \emptyset$, $f^{-1}([b_i] \times W) \cap \text{int } F$ is a $p$-manifold, $\bar{P}$ is a compact connected $p$-manifold with boundary, and $[3, (1.9)]$

$$f|\bar{P}: \bar{P} \rightarrow [b_i] \times W$$

is a bundle map. Thus $[11; p. 53, (11.4)]$ it is a product bundle map, and since $f$ is a layer map

(3) there is a layer embedding $\lambda: A^1 \times W \rightarrow I^2 \times W$, where $\lambda(A^1 \times W) = \bar{P}$ and $A^1 \approx S^1$ or $[0, 1]$.

In particular, $P \cap (I^2 \times \{s\})$ is a component of $f^{-1}(b_i, s) - \text{imag } \nu$ ($s \in W; i = 0, 1$), and $\text{Cl } [P \cap (I^2 \times \{s\})] \approx A^1$. From the compactness of $F$ and the finiteness of $Q$, the number of such components $P$ is finite.
Let $K$ be a component of $f^{-1}(J \times W)$-imag $\nu$ meeting $E$ (thus by (2) diam $K < \varepsilon$ and $K \subset \text{int } F$) and let $T$ be a component of the boundary of $K$ (i.e., relative to) $I^2 \times W$. Then

$$T \subset f^{-1}([b_0, b_1] \times W) \cup \text{imag } \nu.$$ 

Moreover, from (3) there are a finite union (possibly empty) $A$ of disjoint arcs in $S^1$ and a layer embedding $\zeta: S^1 \times W \to I^2 \times W$ with imag $\zeta = T$, $\zeta(A \times W) = T \cap \text{imag } \nu$, and

$$\zeta(\text{Cl } [S^1 - A] \times W) = T \cap f^{-1}([b_0, b_1] \times W).$$

For each $s \in W$ and component (arc) $\gamma$ of $A$, $f \circ \zeta: \gamma \times s \simeq J \times s$, and for each component $\Delta$ of Cl $[S^1 - A]$, $f(\zeta(\Delta \times \{s\})) = (b_i, s)$ ($i = 0$ or 1). Thus if $A \neq \emptyset$, there are an even number of such components (arcs) $\Delta$, and they alternate in value. Hence there are an even number (possibly zero) of components (arcs) of $A$.

The union of such embeddings $\zeta$ over all $J \in \{J_h: h = 1, 2, \cdots, r\}$ and components $K$ of $f^{-1}(J \times W)$ - imag $\nu$ is finite: call them $\zeta_j (j = 1, 2, \cdots, k)$. Let $\Omega_j = \text{imag } \zeta_j$ and let $K_j$ be the corresponding component $K_j$ by (1) there is a closed 2-cell $U_j \subset I^2$ with $K_j \subset \text{int } U_j \times W$, and thus each $K_j \cap (I^2 \times \{s\})$ is a 2-cell-with-holes contained in int $U_j$. Each $\Omega_j$ separates $U_j \times W$ into two components; let $L_j$ be the closure of the component disjoint from $\partial U_j \times W$. Each $L_j \cap (I^2 \times \{s\})$ is a 2-cell, and since the $K_j$ are mutually disjoint, for $i \neq j$ exactly one of the following is true: $(L_i - \Omega_i) \cap (L_j - \Omega_j) = \emptyset$, $L_i \subseteq L_j$, or $L_i \subseteq L_j$. The desired spoke sets are those $L_j$ with $E \cap L_j \neq \emptyset$ and $L_j \not\subset L_i$, for any $i \neq j$. Since each diam $K_j < \varepsilon$, each diam $\Omega_j < \varepsilon$, so that diam $L_j < \varepsilon$. Since $E \cap (I^2 \times W) \subset \bigcup_j K_j \subset \bigcup_j L_j$, $E \subset B_f$, and $B_f \cap \Omega_j = \emptyset$, $E \cap (I^2 \times W) \subset \bigcup_j (L_j - \Omega_j)$.

**Lemma 2.3.** Let $f: I^2 \times R^{p-1} \to R \times R^{p-1}$ be a layer map, let $L_0$ (resp., $L_j, j = 1, 2, \cdots, q$) be a spoke set over $J \times W$(resp., $J_j \times W'$), and let $s \in W \cap W'$. Suppose that $L_i \cap (I^2 \times \{s\}) \subset L_0$,

$$B_f \cap L_0 \cap (I^2 \times \{s\}) \subset \bigcup_{j>0} (L_j - \Omega_j),$$

and the $L_j - \Omega_j$ are mutually disjoint ($j > 0$). Then

$$\zeta(L_0) = \sum_{j>0} \zeta(L_j).$$

**Proof.** Since $B(f_s) \subset B_f \cap (I^2 \times \{s\})$ and $\zeta(L_j) = \zeta(L_j \cap (I^2 \times \{s\}))$, it suffices to prove the lemma for $f = f_s: I^2 \to R$. Thus $L_i \subset L_0$ and $B_f \cap L_0 \subset \bigcup_{j>0} L_j - \Omega_j$. If $A_j$ (see (2.1)) has 2 $w(j)$ components
DIFFERENTIABLE OPEN MAPS OF $(p+1)$-MANIFOLD TO $p$-MANIFOLD

$\omega(j) = 0, 1, \cdots$, define $g_j: L_j \to R$ to agree with $f$ on $\partial L_j = \Omega_j$ and to be topologically equivalent to $\psi_{\omega(j)}$. Let $h: L_0 \to R$ agree with $f$ on $L_0 \cup \bigcup_{j>0} (L_j - \Omega_j)$ and with $g_j$ on $L_j$ $(j = 1, 2, \cdots, q)$. Then $B(h) = \bigcup_{j>0} B(g_j)$, and so is discrete.

Let $D(L_j)$ be the identification space obtained from $(L_j \times \{0\}) \cup (L_j \times \{1\})$ by identifying $(x, 0)$ with $(x, 1)$ for each $x \in A = A(L_j)$, let $D(g_j): D(L_j) \to R$ be defined by $D(g_j) (x, 0) = D(g_j) (x, 1) = g_j(x)$, and let $D(h)$ be defined analogously. Define a vector field $\mathbf{u}_j$ (resp., $\mathbf{v}$) on $D(L_j)$ (resp., $D(L_0)$) which is 0 precisely on the (discrete) branch set $B(D(g_j))$ (resp., $B(D(h))$) and elsewhere is transverse to the level curves of $D(g_j)$ (resp., $D(h)$), i.e., a “gradient vector field” $(j = 0, 1, \cdots, q)$. For any vector field $\alpha$ with isolated zeros, let the sum of the indices of $\alpha$ at its zeros [7, p. 32] be denoted by $\iota(\alpha)$.

Since $L_j \approx D^2$, the Euler characteristic

$$
\chi(D(L_j)) = 2 - 2\omega(j) = 2\xi(L_j).
$$

According to the Poincaré-Hopf Theorem [7, p. 35] (differentiability is not really needed in our case) $\chi(D(L_j)) = \iota(\mathbf{u}_j)$, so that $2\xi(L_j) = \iota(\mathbf{u}_j)$ and $2\xi(L_0) = \iota(\mathbf{u}_0) = \iota(\mathbf{v})$. Thus $2\xi(L_0) = \iota(\mathbf{v}) = 2\sum_{j>0} \iota(\mathbf{v} | L_j)$ (by definition of $\iota = \sum_{j>0} \iota(\mathbf{u}_j) = 2\sum_{j>0} \xi(L_j)$, so that $\xi(L_0) = \sum_{j>0} \xi(L_j)$ (where $j = 1, 2, \cdots, q$).

Alternatively, we could have used [5, p. 370] or [10, p. 35, (4.3.6)]; in this case we would have removed an open 2-cell with boundary a level circle about each local maximum or minimum point of $g_j$ and $h$, in order to have open maps. Or, we could have used a counting argument based on the Euler characteristics of $L_j$, $L_0$, and $L_0 - \bigcup_{j} \text{int } L_j$; the first two spaces are 2-cells, and the last one is disjoint from $B_j$, so that information about it can be obtained from [3, (1.9)].

3. Spoke sets of open maps.

**Lemma 3.1.** Let $f: I^2 \times R^{p-1} \to R \times R^{p-1}$ be an open layer map, and let $L_0$ be a spoke set over $J \times W$, where $W$ is a closed $(p - 1)$-cell. Then

(a) $f^{-1}(y, t) \cap L_0$ does not contain a homeomorph of $S^1$

$((y, t) \in R \times R^{p-1})$

(b) $\xi(L_0) \leq 0$;

(c) $f(L_0) = J \times W$;
(d) \( \xi(L_0) \neq 0 \) implies that \( B_f \cap (L_0 - \Omega_0) \cap (I^\infty \times \{t\}) \neq \emptyset \) for every \( t \in R^{p-1} \);

(e) if \( \dim (f(B_f) \cap (R \times \{t\})) \leq 0 \) for every \( t \in R^{p-1} \),

\[
\dim (B_f \cap f^{-1}(y, t)) \leq 0 \quad \text{for every} \quad (y, t) \in R \times R^{p-1},
\]

and \( \xi(L_0) = 0 \), then \( B_f \cap \text{int} \ L_0 = \emptyset \).

**Proof.** Suppose (a) is false, where \( A \) is the homeomorph of \( S^1 \). Then \( A \) bounds an open 2-cell \( A \) in \( L_0 \cap (I^\infty \times \{t\}) \approx D^2 \). Since \( f_t : I^2 \to R \) is open, \( f_t(A) \) is an open interval, while \( f_t(\Delta) \) is a closed interval with \( f_t(\partial \Delta) \) a single point, and a contradiction results.

If \( \xi(L_0) > 0 \), then \( \Omega_0 \cap (I^\infty \times \{t\}) \) is a component of \( f^{-1}(y, t) \) for some \( y \in R \), and a contradiction of (a) results. Thus (b) is true.

From the definition of \( L_0 \), \( f(L_0) \subset J \times W \), and from that definition and (b), \( f(\Omega_0) = J \times W \), so that (c) \( J \times W = f(L_0) \).

If \( B_f \cap (L_0 - \Omega_0) \cap (I^\infty \times \{t\}) \neq \emptyset \) for some \( t \in W \), then \( \xi(L_0) = 0 \). Conclusion (d) results.

For a spoke set \( L \) over \( I \times U \), let \( *L = L \cap f^{-1}(\text{int}(I \times U)) \); thus \( *L - \Omega = \text{int} \ L \) (interior relative to \( I^\infty \times R^{p-1} \)). Since the restriction map \( \alpha : f^{-1}(\text{int}(J \times W)) \to \text{int}(J \times W) \) is open, \( *L_0 - \Omega_0 \) is open in \( f^{-1}(\text{int}(J \times W)) \), and \( B(f_j | L_0) \cap \Omega_0 = \emptyset \), the restriction map \( \beta_0 : *L_0 \to \text{int}(J \times W) \) is open. Suppose that \( f \) satisfies the hypotheses of (e), i.e., \( \xi(L_0) = 0 \), while \( (x, s) \in B_f \cap \text{int} \ L_0 \). Given \( \varepsilon > 0 \), which we may assume is less than \( d(B_f, \Omega_0) \), let \( W \) and the spoke sets \( L_j(j = 1, 2, \ldots, q) \) be as given by (2.2) for \( f, \varepsilon, a = s \), and \( E = (B_f \cap L_0) \), where \( (x, s) \in \text{int} \ L_i \). From (b) each \( \xi(L_i) \leq 0 \) and from (2.3) \( \xi(L_i) = \sum_{j \geq 0} \xi(L_j) \); thus \( \xi(L_i) = 0 \) for every \( j \), so in particular \( \xi(L_0) = 0 \). Let \( \beta_i : *L_i \to f(*L_i) \) be restriction of \( f \).

For each \( (z, t) \in f(L_i) - f(B_i), (i = 0, 1), (\beta_i)^{-1}(z, t) \) is a 1-manifold with boundary; by (a) each of its components is homeomorphic to \( [0, 1] \), and since \( \xi(L_0) = 0 \), \( (\beta_i)^{-1}(z, t) \approx [0, 1] \). By [3, (4.3)(a)] \( (\beta_i)^{-1}(y, u) \) is arcwise connected for each \( (y, u) \in \text{imag} \beta_0 \). Choose \( \delta > 0 \) such that \( S((x, s), \delta) \subset \text{int} \ L_i \). Then

\[
f^{-1}(y, u) \cap S(x, \delta) \subset (\beta_i)^{-1}(y, u) \subset f^{-1}(y, u) \cap S((x, s), \varepsilon),
\]

so that \( f \) is 0-regular at \( (x, s) [3, (4.1)] \). Since \( (x, s) \in B_f \cap L_0 \) is arbitrary, by [3, (4.2)] \( f \) is 0-regular at each point of \( L_0 \). Thus \( \beta_0 \) is
a bundle map [3, (4.3) (b)], so that $B_f \cap \text{int } L_0 = \emptyset$.

**Lemma 3.2.** Let $g: I^a \times R^{p-1} \to R \times R^{p-1}$ be an open layer map, let $L$ be a spoke set over $J \times W$ where $W$ is a $(p - 1)$-cell and let $\alpha; W \approx B_s \cap L$ with $\pi \circ \alpha$ the identity map. Then $g \mid \text{int } L$ is topologically equivalent to $\varphi_w \times \iota$ ($w = 2, 3, \ldots$; see (2.1)).

**Proof.** We may as well replace $g$ by its restriction to $g\mid \text{int } J \times \text{int } W$, and $L$ by $L \cap g^{-1}(\text{int } J \times \text{int } W)$, i.e., we may as well suppose that $\text{int } J = R$ and $\text{int } W = R^{p-1}$. Let $h: R \times R^{p-1} \to R \times R^{p-1}$ be the layer homeomorphism defined by $h(y, t) = (y, t) - g(\alpha(t))$, and let $\lambda = h \circ g \mid L$. Then $B_f = B_s \cap L$ and $\lambda(B_f) = [0] \times R^{p-1}$.

Let $J_i$ be $(-\infty, 0]$ or $[0, \infty)$ according as $i$ is odd or even. (1) Let $K$ be a component of $\lambda^{-1}((\text{int } J_i) \times R^{p-1})$, and let $\beta: K \to \text{int } J_i \times R^{p-1}$ and $\gamma: K \to J_i \times R^{p-1}$ be the restriction of $\lambda$. Since $B_s = \emptyset$, $\beta$ is a bundle map with fiber a 1-manifold $F$ [3, (1.9)], and so $K \approx F \times \text{int } J_i \times R^{p-1}$ [11, p. 53, (11.4)]. Since $K$ is connected, $F$ is also, and by (3.1(a)) $F \approx [0, 1]$. By [3, (4.3)(a)], $\gamma^{-1}(0, t)$ is arcwise connected for each $t \in R^{p-1}$.

Given $(x, s) \in B_s \cap \gamma^{-1}([0] \times R^{p-1})$ and $\varepsilon > 0$ with $S((x, s), \varepsilon) \subseteq \text{int } L$, let $L'$ be a spoke set over $J' \times W'$ given by (2.2) for $\lambda$, $E = \{(x, s), \}$, $a = s$, and $\varepsilon$. Then $L'$ satisfies the original hypotheses, so that $(\gamma')^{-1}(y, t)$ is arcwise connected for every $(y, t)$. Choose $\delta > 0$ with $S((x, s), \delta) \subseteq \text{int } L'$. Then

$$S((x, s), \delta) \cap \gamma^{-1}(y, t) \subset (\gamma')^{-1}(y, t) \subset S((x, s), \varepsilon) \cap \gamma^{-1}(y, t)$$

for each $(y, t) \in J' \times W'$, so that $\gamma'$ is 0-regular at $(x, s)$. By [3, (4.2)] $\gamma$ is 0-regular, and (by [3, (4.3)(b)]) (2) $\gamma$ is a (product) bundle map with fiber $[0, 1]$.

For each $t \in R^{p-1}$ and component $K$ (see (1)), $\gamma \mid (\bar{K} \cap (I^a \times \{t\}))$ is a product bundle map over $J_i \times \{t\}$ with fiber $[0, 1]$, so that $\lambda^{-1}(0, t)$ is a deformation retract of $L \cap (I^a \times \{t\}) \approx D^1$. Thus $\lambda^{-1}(0, t)$ is connected. Since $\lambda^{-1}(0, t)$ contains no homeomorph of $S^1$ (3.1(a)), and $\lambda^{-1}(0, t) - \{\alpha(t)\}$ is a 1-manifold with boundary points the $2w$ ($\xi(L) = 1 - w$) points of $\lambda^{-1}(0, t) \cap \emptyset$ (2.1), it follows that $\lambda^{-1}(0, t)$ is homeomorphic to the union of $2w$ arcs disjoint except for their common endpoint $\alpha(t)$. As a result $\alpha(t) \in \bar{K} \cap (I^a \times \{t\})$, so that each $\bar{K}$ contains imag $\alpha$, i.e., $B_s$.

Let $K_i$ ($i = 1, 2, \ldots, 2w$) be the components $K$ enumerated so that for any $t \in R^{p-1}$, $(\text{int } K_i) \cap (I^a \times \{t\})$ are the components of

$$(\text{int } L) \cap ((I^a \times \{t\}) - \lambda^{-1}(0, t))$$

in counterclockwise order around $\alpha(t)$ with $\lambda(\bar{K}_i) = J_i \times R^{p-1}$. Let
$A_i = \bar{K}_i \cap \text{int } L$, let $\psi = \psi_w \times \iota$ (see (2.1)), and let $A_i$ be the closures of the components of $\psi^{-1} (\text{int } J_i \times R^{p-1})$ enumerated in analogous fashion.

By (2) there is an orientation-preserving homeomorphism $\mu_i$ of $A_i$ onto $R \times J_i \times R^{p-1}$ with $\pi \circ \mu_i = \lambda | A_i$. Let $\nu_i$ be the homeomorphism of $R \times J_i \times R^{p-1}$ onto itself defined by

$$\nu_i(x, y, t) = (x, y, t) - \mu_i(\alpha(t)) + (0, 0, t),$$

and let $\zeta_i = \nu_i \circ \mu_i$. Then $\zeta_i(\alpha(t)) = (0, 0, t)$, so that

$$\zeta_i(B_i) = \{0\} \times \{0\} \times R^{p-1}.$$ 

There is an analogous orientation-preserving homeomorphism $\xi_i$ of $A_i$ onto $R \times J_i \times R^{p-1}$ with $\pi \circ \xi_i = \lambda | A_i$ and $\xi_i(B_i) = \{0\} \times \{0\} \times R^{p-1}$.

Let $\Phi = (\text{int } L) \cap \lambda^{-1} (\{0\} \times R^{p-1})$, and let $Y_i$ (resp., $\Psi_i$) be the closure in $\Phi$ (resp., $\psi^{-1}(\{0\} \times R^{p-1})$) of the component in $\Phi - B_i$ (resp., $\psi^{-1}(\{0\} \times R^{p-1}) - B_i$) meeting both $A_i$ and $A_{i+1}$ (resp., $A_i$ and $A_{i+1}$), where $i$ and $i + 1$ are interpreted mod $2w$. In case $w = 1$ there are two such components, and $Y_i$ is so chosen that, for each $t \in R^{p-1}$, a counter-clockwise path around $\alpha(t)$ from $A_i$ to $A_{i+1}$ passes through $Y_i$. Then $(\xi_i)^{-1} \circ \zeta_i$ (also $(\xi_{i+1})^{-1} \circ \xi_i$) defines a homeomorphism of $Y_i$ onto $\psi_i$ with $(\xi_i)^{-1} \circ \zeta_i(B_i) = B_i$. Let $\rho: \Phi \approx \psi^{-1}(\{0\} \times R^{p-1})$ agree with $(\xi_i)^{-1} \circ \zeta_i$ on $Y_i$.

Let $\sigma_i$ be the layer homeomorphism of $R \times \{0\} \times R^{p-1}$ onto itself which is the restriction of $\xi_i \circ \rho \circ \zeta_i^{-1}$, (on $\sigma_i(\gamma_{i-1})$, $\sigma_i$ agrees with the identity map) and let $\tau_i$ be its first coordinate map. Let $\phi_i$ be the homeomorphism of $R \times J_i \times R^{p-1}$ onto itself defined by $\phi_i(x, y, t) = (\tau_i(x, t), y, t)$, and let $\chi_i = (\xi_i)^{-1} \circ \phi_i \circ \zeta_i$. Then $\chi_i: A_i \approx A_i$, they agree with $\rho$, and they thus define $\chi$: int $L \approx C \times R^{p-1}$; since $\pi \circ \zeta_i = \lambda | A_i$ and $\pi \circ \zeta_i = \psi | A_i$, where $\pi: R \times J_i \times R^{p-1} \to J_i \times R^{p-1}$ is projection, $\psi \circ \chi = \lambda | \text{int } L$. This is the desired conclusion.

4. The Proof of the theorem.

Remark 4.1. According to the Rank Theorem [3, (1.6)] $B_f \subset R_{p-1}(f)$, and we prove (1.1) under the weaker hypothesis that $\dim (B_f \cap f^{-1}(y)) \leq 0$ for each $y \in N^p$.

Proof. Let $X$ be the complement of the set on which $f$ has the desired structure; then $X \subset B_f$ is closed. We suppose that

$$\dim f(X) \geq p - 1,$$

and will obtain a contradiction.

Since $f$ is $C^1$, $\dim (f(R_{p-1}(f))) \leq p - 2$ [2, p. 1037]. If, for every
$\alpha \in M^{p+1} - f^{-1}(f(R_{p-2}(f)))$, there is an open neighborhood

$$U_\alpha \subset M^{p+1} - f^{-1}(f(R_{p-2}(f)))$$

of $x$ with $\bar{U}_x$ compact and $\dim(f(U_\alpha \cap X)) \leq p - 2$, it follows from the fact that $\{U_\alpha\}$ has a countable subcover that $\dim(f(X)) \leq p - 2$. Thus, there is an $\bar{x} \in M^{p+1} - f^{-1}(f(R_{p-2}(f)))$ such that, (1) for every open neighborhood $U \subset M^{p+1} - f^{-1}(f(R_{p-2}(f)))$ of $\bar{x}$, $\dim(f(U \cap X)) \geq p - 1$.

By [1, p. 87, (1.1)] there are open neighborhoods $U$ of $\bar{x}$ and $V$ of $f(\bar{x})$ and $C^r$ diffeomorphisms $\sigma: \mathbb{R}^2 \times \mathbb{R}^{p-1} \approx U$ and $\rho: V \approx R \times \mathbb{R}^{p-1}$ such that $\rho \circ f \circ \sigma = g$ is a $C^r$ layer map and $\sigma(0, 0) = \bar{x}$. By hypothesis $\dim(B_g \cap \Lambda^t) \leq 0$ for each $(y, t) \in R \times \mathbb{R}^{p-1}$.

Since $\sigma^{-1}(X) \subset B_g$, $B_\alpha \subset R_{p-1}(g)$ (by the Rank Theorem [3, (1.6)]), $R_{p-1}(g) \cap (R^2 \times \{t\}) = R_{p-1}(g)$, and $\dim(g_i(R_{p-1}(g))) \leq 0$ by Sard's Theorem (e.g. [2, p. 1037]), (2) $\dim(g(B_\alpha) \cap (R^2 \times \{t\})) \leq 0$ and

$$\dim(g(\sigma^{-1}(X)) \cap (R \times \{t\})) \leq 0.$$ On the other hand, by (1) $\dim(g(\sigma^{-1}(X))) \geq p - 1$, so there is an $r > 0$ such that

$$A = (C \cap [S(0, r)] \times \mathbb{R}^{p-1}) \cap \sigma^{-1}(X)$$

has $\dim(g(A)) \geq p - 1$. If $\pi: R \times \mathbb{R}^{p-1} \rightarrow \mathbb{R}^{p-1}$ is projection, then $\dim(\pi(g(A))) \geq p - 1$ (by (2) and [6, p. 91]), and there is an open $(p - 1)$-cell $T \subset \pi(g(A))$ [6, p. 44] with $\bar{T}$ compact. Thus (3)

$$A \cap (R^2 \times \{t\}) \neq \emptyset$$

for each $t \in T$.

Let $W \subset T$ and the spoke sets $L_j (j = 1, 2, \ldots, q)$ be as given by (2.2) for $g$, any $a \in T$, $E = A \cap (R^2 \times \bar{T})$, and (say) $\varepsilon = 1$. If (4) (i) the cardinality $\varrho(t) \geq 1$ of $B_\varepsilon \cap (R^2 \times \{t\}) \cap (\bigcup_j L_j)$ ($t \in \text{int } W$) is bounded above by $|\sum_j \xi(L_j)|$, choose $s \in \text{int } W$ such that $\varrho(s)$ is maximal and let $(x_i, s)$ ($i = 1, 2, \ldots, \varrho(s)$) be these points. Otherwise, (4) (ii) there are $s \in \text{int } W$ and distinct points $(x_i, s)$ ($i = 1, 2, \ldots, |\sum_j \xi(L_j)| + 1$) of $B_\varepsilon \cap (R^2 \times \{t\}) \cap (\bigcup_j L_j)$. Let $\varrho'$ be $\varrho(s)$ in case (4) (i) and $|\sum_j \xi(L_j)| + 1$ in case (4) (ii). Then $\varepsilon > 0$ be less than $d(x_i, x_k)$ for $h \neq i$ and $d(B_\varepsilon, \bigcup_j \Omega_j)$, and let $W' \subset \text{int } W$ and $\{L'_k\}$ be as given by (2.2) for $g$, $a = s$, $E = \bigcup_j L_j \cap B_\varepsilon$, and this $\varepsilon$. Thus (5) the $(x_i, s)$, are in distinct spoke sets $L'_k$.

By hypothesis and by (2), the hypothesis of (3.1) (e) is satisfied, so that by (3.1) (d) and (e) $\xi(L_j) = 0$ if and only if $L_j \cap B_\varepsilon = \emptyset$. We may thus omit those $L_j$ and $L'_k$ with $\xi(L_j) = 0 = \xi(L'_k)$. From (3.1) (b) each $\xi(L_j) < 0$ and $\xi(L'_k) < 0$, and from (5) and (3.1) (d) the cardinality $c$ of $\{L'_k\}$ satisfies $w' \leq c \leq |\sum_k \xi(L'_k)|$. Since each $L'_k$ is
contained in some \( L_j \), \( \sum_j \xi(L_j) = \sum_h \xi(L_h) \) by (2.3), and so \( w' \leq \sum_j \xi(L_j) \); this contradicts (4) (ii), and hence (4) (i) must be true.

For \( t \in W' \), \( w(t) \geq c \) by (3.1) (d), while \( c \geq w(s) \) by (4) (i), so that \( w(t) = w(s) \). Thus (by (3.1) (d)) each \( B_x \cap (R^t \times \{t\}) \cap L_x \) is a single point for \( t \in W' \), and since \( B_x \) is closed, there is a homeomorphism \( \alpha_x : W' \approx L'_x \cap B_x \) with \( \pi \circ \alpha \) the identity map on \( W' \). By (3.2) \( \bigcup_k (\sigma^{-1}(X) \cap L'_x) = \emptyset \). But this set contains \( A \cap (R^s \times W') \), contradicting (3).

**Remark 4.2.** In case \( p = 1 \), \( C^2 \) may be replaced by \( C^1 \) and the argument can be shortened considerably. In that case (4.1) results from [12, p. 103, Theorem 1] (cf. [18, pp. 7–8]), and (4.1) in case \( B_f \) is discrete is [10, p. 28, (4.3.1)] and [9]. Considerable information relating to open maps \( f : M^2 \to N^1 \) is given in [5], [8], and [10].

**4.3. Proof of (1.2).** The hypotheses of (1.1) are satisfied (with \( C^2 \) if \( p = 1 \)). In case \( p = 1 \), \( X = \emptyset \), so that at each \( x \in M^{p+1}, f \) at \( x \) is locally topologically equivalent to \( \psi_{d(x)} \). In case \( p \geq 2 \), for each \( x \in M^{p+1} - X \) with \( d(x) \neq 1 \) (i.e., \( x \in B_f \)), \( \dim B_f = p - 1 \geq 1 \) in a neighborhood of \( x \); the assumption that \( \dim R_{p-1}(f) \leq 0 \) contradicts the Rank Theorem [3, (1.6)]. Thus \( B_f \subset X \), so that

\[
\dim f(B_f) \leq p - 2.
\]

That \( f \) is locally topological equivalent to \( \rho \) or to \( \tau \) is now a consequence of [3, (4.7)].

**References**


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SYRACUSE UNIVERSITY
UNIVERSITY OF TENNESSEE
AND
UNIVERSITY OF ALBERTA
Jan Aarts and David John Lutzer, *Pseudo-completeness and the product of Baire spaces* ........................................ 1
Gordon Owen Berg, *Metric characterizations of Euclidean spaces* ................................................................. 11
Ajit Kaur Chilana, *The space of bounded sequences with the mixed topology* ............................................. 29
Philip Throop Church and James Timourian, *Differentiable open maps of \((p+1)\)-manifold to \(p\)-manifold* ................................................................. 35
P. D. T. A. Elliott, *On additive functions whose limiting distributions possess a finite mean and variance* ................................................................. 47
M. Solveig Espelie, *Multiplicative and extreme positive operators* ................................................................. 57
Jacques A. Ferland, *Domains of negativity and application to generalized convexity on a real topological vector space* ................................................................. 67
Michael Benton Freeman and Reese Harvey, *A compact set that is locally holomorphically convex but not holomorphically convex* ................................................................. 77
Roe William Goodman, *Positive-definite distributions and intertwining operators* ................................................................. 83
Elliot Charles Gootman, *The type of some \(C^*\) and \(W^*\)-algebras associated with transformation groups* ................................................................. 93
David Charles Haddad, *Angular limits of locally finitely valent holomorphic functions* ................................................................. 107
William Buhmann Johnson, *On quasi-complements* ................................................................. 113
William M. Kantor, *On 2-transitive collineation groups of finite projective spaces* ................................................................. 119
Joachim Lambek and Gerhard O. Michler, *Completions and classical localizations of right Noetherian rings* ................................................................. 133
Kenneth Lamar Lange, *Borel sets of probability measures* ................................................................. 141
David Lowell Lovelady, *Product integrals for an ordinary differential equation in a Banach space* ................................................................. 163
Jorge Martinez, *A hom-functor for lattice-ordered groups* ................................................................. 169
W. K. Mason, *Weakly almost periodic homeomorphisms of the two sphere* ................................................................. 185
Anthony G. Mucci, *Limits for martingale-like sequences* ................................................................. 197
Eugene Michael Norris, *Relationally induced semigroups* ................................................................. 203
Arthur E. Olson, *A comparison of \(c\)-density and \(k\)-density* ................................................................. 209
Donald Steven Passman, *On the semisimplicity of group rings of linear groups. II* ................................................................. 215
Charles Radin, *Ergodicity in von Neumann algebras* ................................................................. 235
P. Rosenthal, *On the singularities of the function generated by the Bergman operator of the second kind* ................................................................. 241
Arthur Argyle Sagle and J. R. Schumi, *Multiplications on homogeneous spaces, nonassociative algebras and connections* ................................................................. 247
Leo Sario and Cecilia Wang, *Existence of Dirichlet finite biharmonic functions on the Poincaré 3-ball* ................................................................. 267
Ramachandran Subramanian, *On a generalization of martingales due to Blake* ................................................................. 275
Bui An Ton, *On strongly nonlinear elliptic variational inequalities* ................................................................. 279
Seth Warner, *A topological characterization of complete, discretely valued fields* ................................................................. 293
Chi Song Wong, *Common fixed points of two mappings* ................................................................. 299