Pacific Journal of Mathematics

DIFFERENTIABLE OPEN MAPS OF $(p+1)$-MANIFOLD TO p-MANIFOLD

PHILIP THROOP CHURCH AND JAMES TIMOURIAN

Vol. 48, No. 1 March 1973

PACIFIC JOURNAL OF MATHEMATICS
Vol. 48, No. 1, 1973

DIFFERENTIABLE OPEN MAPS OF $(p+1)$-MANIFOLD TO p-MANIFOLD

P. T. CHURCH AND J. G. TIMOURIAN

Let $f: M^{p+1} \to N^p$ be a C^3 open map with $p \geqq 1$, let $R_{p-1}(f)$ be the critical set of f, and let

$$\dim (R_{p-1}(f) \cap f^{-1}(y)) \leqq 0$$

for each $y \in N^p$. Then (1.1) there is a closed set $X \subset M^{p+1}$ such that $\dim f(X) \leqq p-2$ and, for every $x \in M^{p+1} - X$, there is a natural number $d(x)$ with f at x locally topologically equivalent to the map

$$\phi_{d(x)}: C \times R^{p-1} \to R \times R^{p-1}$$

defined by

$$\phi_{d(x)}(z, t_1, \cdots, t_{p-1}) = (\mathscr{R}(z^{d(x)}), t_1, \cdots, t_{p-1})$$

($\mathscr{R}(z^{d(x)})$ is the real part of the complex number $z^{d(x)}$).

The hypothesis on the critical set is essential [3, (4.11)], but in [4] we show that any real analytic open map satisfies this hypothesis, and thus this conclusion.

COROLLARY 1.2. *If $f: M^{p+1} \to N^p$ is a C^{p+1} open map with $\dim (R_{p-1}(f)) \leqq 0$, then at each $x \in M^{p+1}$, f is locally topologically equivalent to one of the following maps*:

(a) *the projection map $\rho: R^{p+1} \to R^p$,*

(b) *$\tau: C \times C \to C \times R$ defined by $\tau(z, w) = (2z \cdot \bar{w}, |w|^2 - |z|^2)$, where $\bar{w}$ is the complex conjugate of w.*

(c) *$\psi_d: C \to R$ defined by $\psi_d(z) = \mathscr{R}(z^d)$.*

In order to read the proofs in this paper, the reader will need to have [3] at hand. In particular, the terms *locally topologically equivalent, branch set B_f, layer map, extended embedding,* and 0-*regular* are defined in [3; (1.3), (1.5), (2.1), (2.3), and (4.1), respectively].

2. Spoke sets.

The definition and lemmas of this section are given in somewhat greater generality than needed in this paper (i.e., for open maps), for use in a subsequent paper.

Let Γ^2 be any 2-manifold (without boundary).

DEFINITION 2.1. Let $\psi_w \times \iota: C \times R^{p-1} \to R \times R^{p-1}$ be defined by $\psi_0 \times \iota(z, t) = (|z|, t)$ and $\psi_w \times \iota(z, t) = (\mathscr{R}(z^w), t)$ $(w = 1, 2, \cdots)$. Thus

$B(\psi_1 \times \iota) = \emptyset$ and $B(\psi_w \times \iota) = \{0\} \times R^{p-1}$ otherwise. For $w = 0$ let $L = D^2 \times D^{p-1}$ and let $J = [-1, 1]$; for $w \geqq 1$ and $\eta > 0$ sufficiently small, let

$$L = (D^2 \times D^{p-1}) \cap (\psi_w \times \iota)^{-1} ([-\eta, \eta] \times D^{p-1})$$

and let $J = [-\eta, \eta]$. These examples motivate the following definition.

Let $f\colon \Gamma^2 \times R^{p-1} \to R \times R^{p-1}$ be a layer map, let $J = [b_0, b_1] \subset R$, and let $W \subset R^{p-1}$ be a closed q-cell ($q = 0, 1, \cdots, p-1$). Let $\{\gamma_j\}$ be a (possibly empty) collection of $2w$ disjoint closed arcs in $S^1 (j = 1, 2, \cdots, 2w)$; let $A = \bigcup_j \gamma_j$, and let $\zeta\colon S^1 \times W \to \Gamma^2 \times W$ be a layer embedding such that $B_f \cap \operatorname{imag} \zeta = \emptyset$, $f \circ \zeta\colon \gamma_j \times W \approx J \times W$, and for each component Φ of $\operatorname{Cl}[S^1 - A]$, $f(\zeta(\Phi \times W)) = \{b_i\} \times W (i = 0$ or $1)$. A *spoke set of f over $J \times W$* is (i) a compact, connected subspace $L \subset f^{-1}(R \times W)$ such that (ii) $L \cap (\Gamma^2 \times \{t\})$ is a 2-cell for each $t \in W$ and (iii) for some ζ as above, the boundary Ω of L with respect to $f^{-1}(R \times W)$ is $\operatorname{imag} \zeta$. Thus if $A = \emptyset$, $f(\Omega) = \{b_i\} \times W$ ($i = 0$ or 1). (In case $A \neq \emptyset$ and $q = 1$, L is homeomorphic to the hub and spokes of a wagon wheel, where $\zeta(A \times W)$ corresponds to the ends of the spokes.) The index $\xi(L) = 1 - w$.

LEMMA 2.2. *Let $f\colon \Gamma^2 \times R^{p-1} \to R \times R^{p-1}$ be a layer map with* $\dim (B_f \cap (\Gamma^2 \times \{t\})) = \dim (f(B_f) \cap (R \times \{t\})) \leqq 0$ *for each $t \in R^{p-1}$, let $E \subset B_f$ be compact, let $a \in R^{p-1}$, and let $\varepsilon > 0$. Then there are a closed $(p-1)$-cell neighborhood W of a, closed intervals $J_j (j = 1, 2, \cdots, m)$, and spoke sets L_j over $J_j \times W$ such that*

(iv) *$E \cap L_j \neq \emptyset$ and $E \cap (\Gamma^2 \times W) \subset \bigcup_j (L_j - \Omega_j)$,*

(v) *the $L_j - \Omega_j$ are mutually disjoint, and*

(vi) *each* $\operatorname{diam} L_j < \varepsilon$.

Proof. Let F be a compact neighborhood of E in $\Gamma^2 \times R^{p-1}$, let $\{U_\alpha\}$ be a cover of Γ^2 by interiors of closed 2-cells, and let δ be the Lebesgue number of $\{U_\alpha \times R^{p-1}\}$ as a cover of F. We may suppose that $\varepsilon < \min (\delta, d(E, bdy\ F))$. Thus

(1) for each $\Psi \subset F$ with $\operatorname{diam} \Psi < \varepsilon$, there is a closed 2-cell U with $\Psi \subset (\operatorname{int} U) \times R^{p-1}$.

Given $y \in R$ with $(y, a) \in f(E)$ and $X = E \cap f^{-1}(y, a)$, let Q be the finite set and $\nu\colon Q \times D \to \Gamma^2 \times R^{p-1}$ be the extended embedding with $\operatorname{imag} \nu \cap B_f = \emptyset$ given by [3, (2.5)] for X and ε. According to that lemma each component K of $f^{-1}(\operatorname{int} D)$-$\operatorname{imag} \nu$ meeting X has $\operatorname{diam} K < \varepsilon$, and each is open. Since $X = E \cap f^{-1}(y, a)$ and E is compact, one may prove (by contradiction) that it is possible to select the p-cell neighborhood D of (y, a) in $R \times R^{p-1}$ sufficiently small that each component K of $f^{-1}(\operatorname{int} D) - \operatorname{imag} \nu$ meeting E has diam $K < \varepsilon$. Summarizing.

(2) each component K of $f^{-1}(\operatorname{int} D)$-imag ν with $K \cap E \neq \varnothing$ has diam $K < \varepsilon$, so that $\bar{K} \subset \operatorname{int} F$.

Choose a closed interval $J(y) \subset R$ with $y \in \operatorname{int} J(y)$,

$$J(y) \times \{a\} \subset \operatorname{int} D ,$$

and end points $b_0(y)$, $b_1(y)$ with $(b_0(y), a)$, $(b_1(y), a) \notin f(B_f)$. Since $f(F \cap B_f)$ is closed, there is a closed $(p-1)$-cell neighborhood $W(y)$ of a in R^{p-1} such that $(\partial J(y) \times W(y)) \cap f(F \cap B_f) = \varnothing$ and

$$J(y) \times W(y) \subset D .$$

Let $\nu(y)$ be the corresponding extended embedding (restricted) over $J \times W$.

There are $y_1, y_2, \cdots, y_u \in R$ with $(y_j, a) \in f(E)$ and

$$f(E) \cap (R \times \{a\}) \subset \textstyle\bigcup_j \operatorname{int} (J(y_j)) \times \{a\} .$$

The points $\{b_i(y_j)$: $i = 0, 1$; $j = 1, 2, \cdots, u\}$ are the end points of a finite set of closed intervals with mutually disjoint interiors; let $J_h (h = 1, 2, \cdots, r)$ be those intervals with $(J_h \times \{a\}) \cap f(E) \neq \varnothing$. Let W be a closed $(p-1)$-cell neighborhood of $a \in R^{p-1}$ with $W \subset \bigcap_j W(y_j)$. Then $(\partial J_h \times W) \cap f(F \cap B_f) = \varnothing$ and

$$f(E) \cap (R \times W) \subset \textstyle\bigcup_h ((\operatorname{int} J_h) \times W) \quad (h = 1, 2, \cdots, r) .$$

Since each J_h is contained in some $J(y_j)$, restriction of $\nu(y_j)$ yields an extended embedding ν_h over $J_h \times W$.

Let $J = [b_0, b_1]$ be one of these intervals J_h, let

$$\nu\colon (Q \times J) \times W \longrightarrow \Gamma^2 \times R^{p-1}$$

be the layer embedding ν_h, and let $P \subset F$ be a component of

$$f^{-1}(\{b_i\} \times W) - \operatorname{imag} \nu .$$

Since $(\{b_i\} \times W) \cap f(F \cap B_f) = \varnothing$, $f^{-1}(\{b_i\} \times W) \cap \operatorname{int} F$ is a p-manifold, $\bar{P}$ is a compact connected p-manifold with boundary, and [3, (1.9)] $f \mid \bar{P}$: $\bar{P} \rightarrow \{b_i\} \times W$ is a bundle map. Thus [11; p. 53, (11.4)] it is a product bundle map, and since f is a layer map

(3) there is a layer embedding λ: $\Lambda^1 \times W \rightarrow \Gamma^2 \times W$, where $\lambda(\Lambda^1 \times W) = \bar{P}$ and $\Lambda^1 \approx S^1$ or $[0, 1]$.

In particular, $P \cap (\Gamma^2 \times \{s\})$ is a component of $f^{-1}(b_i, s) - \operatorname{imag} \nu$ $(s \in W;\ i = 0, 1)$, and $\operatorname{Cl} [P \cap (\Gamma^2 \times \{s\})] \approx \Lambda^1$. From the compactness of F and the finiteness of Q, the number of such components P is finite.

Let K be a component of $f^{-1}(J \times W)$-imag ν meeting E (thus by (2) $\operatorname{diam} K < \varepsilon$ and $\bar{K} \subset \operatorname{int} F$) and let T be a component of the boundary of K in (i.e., relative to) $\Gamma^2 \times W$. Then

$$T \subset f^{-1}(\{b_0, b_1\} \times W) \cup \operatorname{imag} \nu .$$

Moreover, from (3) there are a finite union (possibly empty) A of disjoint arcs in S^1 and a layer embedding ζ: $S^1 \times W \to \Gamma^2 \times W$ with $\operatorname{imag} \zeta = T$, $\zeta(A \times W) = T \cap \operatorname{imag} \nu$, and

$$\zeta(\operatorname{Cl}[S^1 - A] \times W) = T \cap f^{-1}(\{b_0, b_1\} \times W) .$$

For each $s \in W$ and component (arc) γ of A, $f \circ \zeta$: $\gamma \times s \approx J \times s$, and for each component Δ of $\operatorname{Cl}[S^1 - A]$, $f(\zeta(\Delta \times \{s\})) = (b_i, s)$ ($i = 0$ or 1). Thus if $A \neq \emptyset$, there are an even number of such components (arcs) Δ, and they alternate in value. Hence there are an even number (possibly zero) of components (arcs) of A.

The union of such embeddings ζ over all $J \in \{J_h: h = 1, 2, \cdots, r\}$ and components K of $f^{-1}(J \times W) - \operatorname{imag} \nu$ is finite: call them

$$\zeta_j \ (j = 1, 2, \cdots, k) .$$

Let $\Omega_j = \operatorname{imag} \zeta_j$ and let K_j be the corresponding component K; by (1) there is a closed 2-cell $U_j \subset \Gamma^2$ with $\bar{K}_j \subset (\operatorname{int} U_j) \times W$, and thus each $\bar{K}_j \cap (\Gamma^2 \times \{s\})$ is a 2-cell-with-holes contained in int U_j. Each Ω_j separates $U_j \times W$ into two components; let L_j be the closure of the component disjoint from $\partial U_j \times W$. Each $L_j \cap (\Gamma^2 \times \{s\})$ is a 2-cell, and since the K_j are mutually disjoint, for $i \neq j$ exactly one of the following is true: $(L_i - \Omega_i) \cap (L_j - \Omega_j) = \emptyset$, $L_i \subset L_j$, or $L_j \subset L_i$. The desired spoke sets are those L_j with $E \cap L_j \neq \emptyset$ and $L_j \not\subset L_i$ for any $i \neq j$. Since each $\operatorname{diam} K_j < \varepsilon$, each $\operatorname{diam} \Omega_j < \varepsilon$, so that $\operatorname{diam} L_j < \varepsilon$. Since $E \cap (\Gamma^2 \times W) \subset \bigcup_j K_j \subset \bigcup_j L_j$, $E \subset B_f$, and $B_f \cap \Omega_j = \emptyset$, $E \cap (\Gamma^2 \times W) \subset \bigcup_j (L_j - \Omega_j)$.

LEMMA 2.3. *Let f: $\Gamma^2 \times R^{p-1} \to R \times R^{p-1}$ be a layer map, let L_0 (resp., $L_j, j = 1, 2, \cdots, q$) be a spoke set over $J \times W$ (resp., $J_j \times W'$), and let $s \in W \cap W'$. Suppose that $L_j \cap (\Gamma^2 \times \{s\}) \subset L_0$,*

$$B_f \cap L_0 \cap (\Gamma^2 \times \{s\}) \subset \bigcup_{j>0} (L_j - \Omega_j) ,$$

and the $L_j - \Omega_j$ are mutually disjoint ($j > 0$). Then

$$\xi(L_0) = \sum_{j>0} \xi(L_j) .$$

Proof. Since $B(f_s) \subset B_f \cap (\Gamma^2 \times \{s\})$ and $\xi(L_j) = \xi(L_j \cap (\Gamma^2 \times \{s\}))$, it suffices to prove the lemma for $f = f_s$: $\Gamma^2 \to R$. Thus $L_j \subset L_0$ and $B_f \cap L_0 \subset \bigcup_{j>0} L_j - \Omega_j$. If A_j (see (2.1)) has 2 $w(j)$ components

$(w(j) = 0, 1, \cdots)$, define g_j: $L_j \to R$ to agree with f on $\partial L_j = \Omega_j$ and to be topologically equivalent to $\psi_{w(j)}$. Let h: $L_0 \to R$ agree with f on $L_0 - \bigcup_{j>0}(L_j - \Omega_j)$ and with g_j on L_j $(j = 1, 2, \cdots, q)$. Then $B(h) = \bigcup_{j>0} B(g_j)$, and so is discrete.

Let $D(L_j)$ be the identification space obtained from

$$(L_j \times \{0\}) \cup (L_j \times \{1\})$$

by identifying $(x, 0)$ with $(x, 1)$ for each $x \in A = A(L_j)$, let $D(g_j)$: $D(L_j) \to R$ be defined by $D(g_j)\,(x, 0) = D(g_j)\,(x, 1) = g_j(x)$, and let $D(h)$ be defined analogously. Define a vector field u_j (resp., v) on $D(L_j)$ (resp., $D(L_0)$) which is 0 precisely on the (discrete) branch set $B(D(g_j))$ (resp., $B(D(h))$) and elsewhere is transverse to the level curves of $D(g_j)$ (resp., $D(h)$), i.e., a "gradient vector field" $(j = 0, 1, \cdots, q)$. For any vector field α with isolated zeros, let the sum of the indices of α at its zeros [7, p. 32] be denoted by $\iota(\alpha)$.

Since $L_j \approx D^2$, the Euler characteristic

$$\chi(D(L_j)) = 2 - 2w(j) = 2\xi(L_j)\ .$$

According to the Poincaré-Hopf Theorem [7, p. 35] (differentiability is not really needed in our case) $\chi(D(L_j)) = \iota(u_j)$, so that $2\xi(L_j) = \iota(u_j)$ and $2\xi(L_0) = \iota(u_0) = \iota(v)$. Thus $2\xi(L_0) = \iota(v) = 2\sum_{j>0}\iota(v \mid L_j)$ (by definition of ι) $= \sum_{j>0}\iota(u_j) = 2\sum_{j>0}\xi(L_j)$, so that $\xi(L_0) = \sum_{j>0}\xi(L_j)$ (where $j = 1, 2, \cdots, q$).

Alternatively, we could have used [5, p. 370] or [10, p. 35, (4.3.6)]; in this case we would have removed an open 2-cell with boundary a level circle about each local maximum or minimum point of g_j and h, in order to have open maps. Or, we could have used a counting argument based on the Euler characteristics of L_j, L_0, and $L_0 - \bigcup_j \operatorname{int} L_j$; the first two spaces are 2-cells, and the last one is disjoint from B_f, so that information about it can be obtained from [3, (1.9)].

3. Spoke sets of open maps.

LEMMA 3.1. *Let f: $\Gamma^2 \times R^{p-1} \to R \times R^{p-1}$ be an open layer map, and let L_0 be a spoke set over $J \times W$, where W is a closed $(p-1)$-cell. Then*

(a) $f^{-1}(y, t) \cap L_0$ *does not contain a homeomorph of* S^1

$$((y, t) \in R \times R^{p-1})$$

(b) $\xi(L_0) \leqq 0$;
(c) $f(L_0) = J \times W$;

(d) $\xi(L_0) \neq 0$ *implies that* $B_f \cap (L_0 - \Omega_0) \cap (\Gamma^2 \times \{t\}) \neq \emptyset$ *for every* $t \in R^{p-1}$;

(e) *if* $\dim (f(B_f) \cap (R \times \{t\})) \leqq 0$ *for every* $t \in R^{p-1}$,

$$\dim (B_f \cap f^{-1}(y, t)) \leqq 0 \quad \textit{for every } (y, t) \in R \times R^{p-1} ,$$

and $\xi(L_0) = 0$, *then* $B_f \cap \operatorname{int} L_0 = \emptyset$.

Proof. Suppose (a) is false, where Λ is the homeomorph of S^1. Then Λ bounds an open 2-cell Δ in $L_0 \cap (\Gamma^2 \times \{t\}) \approx D^2$. Since f_t: $\Gamma^2 \to R$ is open, $f_t(\Delta)$ is an open interval, while $f_t(\bar{\Delta})$ is a closed interval with $f_t(\partial\Delta)$ a single point, and a contradiction results.

If $\xi(L_0) > 0$, then $\Omega_0 \cap (\Gamma^2 \times \{t\})$ is a component of $f^{-1}(y, t)$ for some $y \in R$, and a contradiction of (a) results. Thus (b) is true.

From the definition of L_0 (2.1), $f(L_0) \subset J \times W$, and from that definition and (b), $f(\Omega_0) = J \times W$, so that (c) $J \times W = f(L_0)$.

If $B_f \cap (L_0 - \Omega_0) \cap (\Gamma^2 \times \{t\}) = \emptyset$ for some $t \in W$, then

$$g\colon L_0 \cap (\Gamma^2 \times \{t\}) \longrightarrow J \times \{t\}$$

defined by restriction of f has $B_g = \emptyset$ [3, (4.10)], and so is a bundle map [3, (1.9)]. Thus [11, p. 53, (11.4)] $L_0 \cap (\Gamma^2 \times \{t\}) \approx J \times F$, where the fiber F is a 1-manifold with boundary. Since $J \times F \approx D^2$ (2.1) (ii), F is connected and $F \not\approx S^1$. Thus $F \approx [0, 1]$, so that $\xi(L_0) = 0$. Conclusion (d) results.

For a spoke set L of f over $I \times U$, let $^*L$ be $L \cap f^{-1}(\operatorname{int}(I \times U))$; thus $^*L - \Omega = \operatorname{int} L$ (interior relative to $\Gamma^2 \times R^{p-1}$). Since the restriction map α: $f^{-1}(\operatorname{int}(J \times W)) \to \operatorname{int}(J \times W)$ is open, $^*L_0 - \Omega_0$ is open in $f^{-1}(\operatorname{int}(J \times W))$, and $B(f \mid L_0) \cap \Omega_0 = \emptyset$, the restriction map β_0: $^*L_0 \to \operatorname{int}(J \times W)$ is open. Suppose that f satisfies the hypotheses of (e), i.e., $\xi(L_0) = 0$, while $(x, s) \in B_f \cap \operatorname{int} L_0$. Given $\varepsilon > 0$, which we may assume is less than $d(B_f, \Omega_0)$, let W' and the spoke sets $L_j (j = 1, 2, \cdots, q)$ be as given by (2.2) for f, ε, $a = s$, and $E = (B_f \cap L_0)$, where $(x, s) \in \operatorname{int} L_1$. From (b) each $\xi(L_j) \leqq 0$ and from (2.3) $\xi(L_0) = \sum_{j>0} \xi(L_j)$; thus $\xi(L_j) = 0$ for every j, so in particular $\xi(L_1) = 0$. Let β_1: $^*L_1 \to f(^*L_1)$ be restriction of f.

For each $(z, t) \in f(L_i) - f(B_f)$, $(i = 0, 1)$, $(\beta_i)^{-1}(z, t)$ is a 1-manifold with boundary; by (a) each of its components is homeomorphic to $[0, 1]$, and since $\xi(L_i) = 0$, $(\beta_i)^{-1}(z, t) \approx [0, 1]$. By [3, (4.3)(a)] $(\beta_i)^{-1}(y, u)$ is arcwise connected for each $(y, u) \in \operatorname{imag} \beta_i$. Choose $\delta > 0$ such that $S((x, s), \delta) \subset \operatorname{int} L_1$. Then

$$f^{-1}(y, u) \cap S(x, \delta) \subset (\beta_1)^{-1}(y, u) \subset f^{-1}(y, u) \cap S((x, s), \varepsilon) ,$$

so that f is 0-regular at (x, s) [3, (4.1)]. Since $(x, s) \in B_f \cap L_0$ is arbitrary, by [3, (4.2)] f is 0-regular at each point of L_0. Thus β_0 is a bundle map [3, (4.3) (b)], so that $B_f \cap \operatorname{int} L_0 = \varnothing$.

LEMMA 3.2. *Let* $g\colon \Gamma^2 \times R^{p-1} \to R \times R^{p-1}$ *be an open layer map, let* L *be a spoke set over* $J \times W$ *where* W *is a* $(p-1)$*-cell and let* α; $W \approx B_g \cap L$ *with* $\pi \circ \alpha$ *the identity map. Then* $g \mid int\ L$ *is topologically equivalent to* $\psi_w \times \iota$ $(w = 2, 3, \cdots$; *see* (2.1)).

Proof. We may as well replace g by its restriction to $g^{-1}(\operatorname{int} J \times \operatorname{int} W)$, and L by $L \cap g^{-1}(\operatorname{int} J \times \operatorname{int} W)$, i.e., we may as well suppose that $\operatorname{int} J = R$ and $\operatorname{int} W = R^{p-1}$. Let $h\colon R \times R^{p-1} \to R \times R^{p-1}$ be the layer homeomorphism defined by $h(y, t) = (y, t) - g(\alpha(t))$, and let $\lambda = h \circ g \mid L$. Then $B_\lambda = B_g \cap L$ and $\lambda(B_\lambda) = \{0\} \times R^{p-1}$.

Let J_i be $(-\infty, 0]$ or $[0, \infty)$ according as i is odd or even. (1) Let K be a component of $\lambda^{-1}((\operatorname{int} J_i) \times R^{p-1})$, and let $\beta\colon K \to \operatorname{int} J_i \times R^{p-1}$ and $\gamma\colon \bar{K} \to J_i \times R^{p-1}$ be the restriction of λ. Since $B_\beta = \varnothing$, β is a bundle map with fiber a 1-manifold F [3, (1.9)], and so $K \approx F \times \operatorname{int} J_i \times R^{p-1}$ [11, p. 53, (11.4)]. Since K is connected, F is also, and by (3.1(a)) $F \approx [0, 1]$. By [3, (4.3)(a)], $\gamma^{-1}(0, t)$ is arcwise connected for each $t \in R^{p-1}$.

Given $(x, s) \in B_\gamma \cap \gamma^{-1}(\{0\} \times R^{p-1})$ and $\varepsilon > 0$ with $S((x, s), \varepsilon) \subset \operatorname{int} L$, let L' be a spoke set over $J' \times W'$ given by (2.2) for λ, $E = \{(x, s)\}$, $a = s$, and ε. Then L' satisfies the original hypotheses, so that $(r')^{-1}(y, t)$ is arcwise connected for every (y, t). Choose $\delta > 0$ with $S((x, s), \delta) \subset \operatorname{int} L'$. Then

$$S((x, s), \delta) \cap \gamma^{-1}(y, t) \subset (\gamma')^{-1}(y, t) \subset S((x, s), \varepsilon) \cap \gamma^{-1}(y, t)$$

for each $(y, t) \in J' \times W'$, so that γ' is 0-regular at (x, s). By [3, (4.2)] γ is 0-regular, and (by [3, (4.3)(b)]) (2) γ is a (product) bundle map with fiber $[0, 1]$.

For each $t \in R^{p-1}$ and component K (see (1)), $\gamma \mid (\bar{K} \cap (\Gamma^2 \times \{t\}))$ is a product bundle map over $J_i \times (t)$ with fiber $[0, 1]$, so that $\lambda^{-1}(0, t)$ is a deformation retract of $L \cap (\Gamma^2 \times \{t\}) \approx D^2$. Thus $\lambda^{-1}(0, t)$ is connected. Since $\lambda^{-1}(0, t)$ contains no homeomorph of S^1 (3.1(a)), and $\lambda^{-1}(0, t) - \{\alpha(t)\}$ is a 1-manifold with boundary points the $2w$ $(\xi(L) = 1 - w)$ points of $\lambda^{-1}(0, t) \cap \Omega$ (2.1), it follows that $\lambda^{-1}(0, t)$ is homeomorphic to the union of $2w$ arcs disjoint except for their common endpoint $\alpha(t)$. As a result $\alpha(t) \in \bar{K} \cap (\Gamma^2 \times \{t\})$, so that each $\bar{K}$ contains imag α, i.e., B_λ.

Let K_i $(i = 1, 2, \cdots, 2w)$ be the components K enumerated so that for any $t \in R^{p-1}$, $(\operatorname{int} K_i) \cap (\Gamma^2 \times \{t\})$ are the components of

$$(\operatorname{int} L) \cap ((\Gamma^2 \times \{t\}) - \lambda^{-1}(0, t))$$

in counterclockwise order around $\alpha(t)$ with $\lambda(\bar{K}_i) = J_i \times R^{p-1}$. Let

$\Lambda_i = \bar{K}_i \cap \operatorname{int} L$, let $\psi = \psi_w \times \iota$ (see (2.1)), and let Δ_i be the closures of the components of $\psi^{-1}(\operatorname{int} J_i \times R^{p-1})$ enumerated in analogous fashion.

By (2) there is an orientation-preserving homeomorphism μ_i of Λ_i onto $R \times J_i \times R^{p-1}$ with $\pi \circ \mu_i = \lambda \mid \Lambda_i$. Let ν_i be the homeomorphism of $R \times J_i \times R^{p-1}$ onto itself defined by

$$\nu_i(x, y, t) = (x, y, t) - \mu_i(\alpha(t)) + (0, 0, t) ,$$

and let $\zeta_i = \nu_i \circ \mu_i$. Then $\zeta_i(\alpha(t)) = (0, 0, t)$, so that

$$\zeta_i(B_\lambda) = \{0\} \times \{0\} \times R^{p-1} .$$

There is an analogous orientation-preserving homeomorphism ξ_i of Δ_i onto $R \times J_i \times R^{p-1}$ with $\pi \circ \xi_i = \psi \mid \Delta_i$ and $\xi_i(B_\psi) = \{0\} \times \{0\} \times R^{p-1}$.

Let $\Phi = (\operatorname{int} L) \cap \lambda^{-1}(\{0\} \times R^{p-1})$, and let Υ_i (resp., Ψ_i) be the closure in Φ (resp., $\psi^{-1}(\{0\} \times R^{p-1})$) of the component in $\Phi - \beta_\lambda$(resp., $\Psi^{-1}(\{0\} \times R^{p-1}) - B_\psi$) meeting both Λ_i and Λ_{i+1} (resp., Δ_i and Δ_{i+1}), where i and $i+1$ are interpreted mod $2w$. In case $w = 1$ there are two such components, and Υ_i is so chosen that, for each $t \in R^{p-1}$, a counter-clockwise path around $\alpha(t)$ from Λ_i to Λ_{i+1} passes through Υ_i. Then $(\xi_i)^{-1} \circ \zeta_i$ (also $(\xi_{i+1})^{-1} \circ \zeta_{i+1}$) defines a homeomorphism of Υ_i onto ψ_i with $(\xi_i)^{-1} \circ \zeta_i(B_\lambda) = B_\psi$. Let ρ: $\Phi \approx \psi^{-1}(\{0\} \times R^{p-1})$ agree with $(\xi_i)^{-1} \circ \zeta_i$ on Υ_i.

Let σ_i be the layer homeomorphism of $R \times \{0\} \times R^{p-1}$ onto itself which is the restriction of $\xi_i \circ \rho \circ \zeta_i^{-1}$, (on $\zeta_i(\gamma_{i-1})$, σ_i agrees with the identity map) and let τ_i be its first coordinate map. Let ϕ_i be the homeomorphism of $R \times J_i \times R^{p-1}$ onto itself defined by $\phi_i(x, y, t) = (\tau_i(x, t), y, t)$, and let $\chi_i = (\xi_i)^{-1} \circ \phi_i \circ \zeta_i$. Then χ_i: $\Lambda_i \approx \Delta_i$, they agree with ρ, and they thus define χ: $\operatorname{int} L \approx C \times R^{p-1}$; since $\pi \circ \zeta_i = \lambda \mid \Lambda_i$ and $\pi \circ \xi_i = \psi \mid \Delta_i$, where π: $R \times J_i \times R^{p-1} \to J_i \times R^{p-1}$ is projection, $\psi \circ \chi = \lambda \mid \operatorname{int} L$. This is the desired conclusion.

4. The Proof of the theorem.

REMARK 4.1. According to the Rank Theorem [3, (1.6)] $B_f \subset R_{p-1}(f)$, and we prove (1.1) under the weaker hypothesis that $\dim (B_f \cap f^{-1}(y)) \leqq 0$ for each $y \in N^p$.

Proof. Let X be the complement of the set on which f has the desired structure; then $X \subset B_f$ is closed. We suppose that

$$\dim f(X) \geqq p - 1 ,$$

and will obtain a contradiction.

Since f is C^3, $\dim (f(R_{p-2}(f))) \leqq p - 2$ [2, p. 1037]. If, for every

$x \in M^{p+1} - f^{-1}(f(R_{p-2}(f)))$, there is an open neighborhood

$$U_x \subset M^{p+1} - f^{-1}(f(R_{p-2}(f)))$$

of x with $\bar{U}_x$ compact and $\dim(f(U_x \cap X)) \leqq p-2$, it follows from the fact that $\{U_x\}$ has a countable subcover that $\dim(f(X)) \leqq p-2$. Thus, there is an $\bar{x} \in M^{p+1} - f^{-1}(f(R_{p-2}(f)))$ such that, (1) for every open neighborhood $U \subset M^{p+1} - f^{-1}(f(R_{p-2}(f)))$ of $\bar{x}$, $\dim(f(U \cap X)) \geqq p-1$.

By [1, p. 87, (1.1)] there are open neighborhoods U of $\bar{x}$ and V of $f(\bar{x})$ and C^r diffeomorphisms $\sigma\colon R^2 \times R^{p-1} \approx U$ and $\rho\colon V \approx R \times R^{p-1}$ such that $\rho \circ f \circ \sigma = g$ is a C^r layer map and $\sigma(0,0) = \bar{x}$. By hypothesis $\dim(B_g \cap g^{-1}(y,t)) \leqq 0$ for each $(y,t) \in R \times R^{p-1}$.

Since $\sigma^{-1}(X) \subset B_g$, $B_g \subset R_{p-1}(g)$ (by the Rank Theorem [3, (1.6)]), $R_{p-1}(g) \cap (R^2 \times (t)) = R_0(g_t)$, and $\dim(g_t(R_0(g_t))) \leqq 0$ by Sard's Theorem (e.g. [2, p. 1037]), (2) $\dim(g(B_g) \cap (R^2 \times \{t\})) \leqq 0$ and

$$\dim(g(\sigma^{-1}(X)) \cap (R \times \{t\})) \leqq 0 .$$

On the other hand, (by (1)) $\dim(g(\sigma^{-1}(X)) \geqq p-1$, so there is an $r > 0$ such that

$$A = (\mathrm{Cl}\,[S(0,r)] \times R^{p-1}) \cap \sigma^{-1}(X)$$

has $\dim g(A) \geqq p-1$. If $\pi\colon R \times R^{p-1} \to R^{p-1}$ is projection, then $\dim(\pi(g(A))) \geqq p-1$ (by (2) and [6, p. 91]), and there is an open $(p-1)$-cell $T \subset \pi(g(A))$ [6, p. 44] with $\bar{T}$ compact. Thus (3)

$$A \cap (R^2 \times \{t\}) \neq \varnothing \qquad \text{for each } t \in T .$$

Let $W \subset T$ and the spoke sets L_j $(j = 1, 2, \cdots, q)$ be as given by (2.2) for g, any $a \in T$, $E = A \cap (R^2 \times \bar{T})$, and (say) $\varepsilon = 1$. If (4) (i) the cardinality $w(t) \geqq 1$ of $B_g \cap (R^2 \times \{t\}) \cap (\bigcup_j L_j)$ $(t \in \mathrm{int}\, W)$ is bounded above by $|\sum_j \xi(L_j)|$, choose $s \in \mathrm{int}\, W$ such that $w(s)$ is maximal and let (x_i, s) $(i = 1, 2, \cdots, w(s))$ be these points. Otherwise, (4) (ii) there are $s \in \mathrm{int}\, W$ and distinct points (x_i, s) $(i = 1, 2, \cdots, |\sum_j \xi(L_j)| + 1)$ of $B_g \cap (R^2 \times \{t\}) \cap (\bigcup_j L_j)$. Let w' be $w(s)$ in case (4) (i) and $|\sum_j \xi(L_j)| + 1$ in case (4) (ii). Let $\varepsilon > 0$ be less than $d(x_h, x_i)$ for $h \neq i$ and $d(B_g, \bigcup_j \Omega_j)$, and let $W' \subset \mathrm{int}\, W$ and $\{L'_k\}$ be as given by (2.2) for g, $a = s$, $E = \bigcup_j L_j \cap B_g$, and this ε. Thus (5) the (x_i, s), are in distinct spoke sets L'_k.

By hypothesis and by (2), the hypothesis of (3.1) (e) is satisfied, so that by (3.1) (d) and (e) $\xi(L_j) = 0$ if and only if $L_j \cap B_g = \varnothing$. We may thus omit those L_j and L'_k with $\xi(L_j) = 0 = \xi(L'_k)$. From (3.1) (b) each $\xi(L_j) < 0$ and $\xi(L'_k) < 0$, and from (5) and (3.1) (d) the cardinality c of $\{L'_k\}$ satisfies $w' \leqq c \leqq |\sum_k \xi(L'_k)|$. Since each L'_k is contained in some L_j, $\sum_j \xi(L_j) = \sum_k \xi(L'_k)$ by (2.3), and so $w' \leqq |\sum_j \xi(L_j)|$; this contradicts (4) (ii), and hence (4) (i) must be true.

For $t \in W'$, $w(t) \geqq c$ by (3.1) (d), while $c \geqq w(s)$ by (4) (i), so that $w(t) = w(s)$. Thus (by (3.1) (d)) each $B_g \cap (R^1 \times \{t\}) \cap L'_k$ is a single point for $t \in W'$, and since B_g is closed, there is a homeomorphism α_i: $W' \approx L'_k \cap B_g$ with $\pi \circ \alpha_i$ the identity map on W'. By (3.2) $\bigcup_k (\sigma^{-1}(X) \cap L'_k) = \varnothing$. But this set contains $\Lambda \cap (R^2 \times W')$, contradicting (3).

REMARK 4.2. In case $p = 1$, C^3 may be replaced by C^2 and the argument can be shortened considerably. In that case (4.1) results from [12, p. 103, Theorem 1] (cf. [18, pp. 7-8]), and (4.1) in case B_f is discrete is [10, p. 28, (4.3.1)] and [9]. Considerable information relating to open maps f: $M^2 \to N^1$ is given in [5], [8], and [10].

4.3. *Proof of* (1.2). The hypotheses of (1.1) are satisfied (with C^2 if $p = 1$). In case $p = 1$, $X = \varnothing$, so that at each $x \in M^{p+1}$, f at x is locally topologically equivalent to $\psi_{d(x)}$. In case $p \geqq 2$, for each $x \in M^{p+1} - X$ with $d(x) \neq 1$ (i.e., $x \in B_f$), $\dim B_f = p - 1 \geqq 1$ in a neighborhood of x; the assumption that $\dim R_{p-1}(f) \leqq 0$ contradicts the Rank Theorem [3, (1.6)]. Thus $B_f \subset X$, so that

$$\dim f(B_f) \leqq p - 2 .$$

That f is locally topological equivalent to ρ or to τ is now a consequence of [3, (4.7)].

REFERENCES

1. P. T. Church, *Differentiable open maps on manifolds*, Trans. Amer. Math. Soc., **109** (1963), 87-100.

2. ———, *On points of Jacobian rank k. II*, Proc. Amer. Math. Soc., **16** (1965), 1035-1038.

3. P. T. Church and J. G. Timourian, *Differentiable maps with O-dimensional critical set, I*, Pacific J. Math., (to appear).

4. ———, *Real analytic open maps*, Pacific J. Math., **41** (1972), 615-630.

5. W. C. Fox, *The critical points of peano interior functions defined on 2-manifolds*, Trans. Amer. Math. Soc., **83** (1956), 338-370.

6. W. Hurewicz and H. Wallman, *Dimension Theory*, 2nd edition, Princeton University Press, Princeton, N. J., 1948.

7. J. Milnor, *Topology from the Differentiable Viewpoint*, The University Press of Virginia, Charlottesville, Virginia, 1965.

8. M. Morse, *Topological Methods in the Theory of Functions of a Complex Variable*, Princeton University Press, Princeton, N. J. 1947.

9. W. D. Nathan, *Open mappings on 2-manifolds*, Pacific J. Math., **41** (1972), 495-501.

10. ———, *Open Mappings on Manifolds*, Ph. D. dissertation, Syracuse University, Syracuse, 1968.

11. N. Steenrod, *The Topology of Fiber Bundles*, Princeton University Press, Princeton, N. J., 1951.

12. Y. K. Toki, *A topological characterization of pseudo-harmonic functions,* Osaka Math. J., **3** (1951), 101-122.

Received June 21, 1972. Work of the first author supported in part by NSF Grant GP-6871, and that of the second author by NSF Grant GP-8888 and NRC Grant A7357.

SYRACUSE UNIVERSITY
UNIVERSITY OF TENNESSEE
AND
UNIVERSITY OF ALBERTA

PACIFIC JOURNAL OF MATHEMATICS

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics
Vol. 48, No. 1 March, 1973