DIFFERENTIABLE OPEN MAPS OF $(p+1)$-MANIFOLD TO p-MANIFOLD

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Let $f\colon M^{p+1} \to N^p$ be a C^3 open map with $p \geq 1$, let $R_{p-1}(f)$ be the critical set of f, and let

$$\dim (R_{p-1}(f) \cap f^{-1}(y)) \leq 0$$

for each $y \in N^p$. Then (1.1) there is a closed set $X \subset M^{p+1}$ such that $\dim f(X) \leq p-2$ and, for every $x \in M^{p+1} - X$, there is a natural number $d(x)$ with f at x locally topologically equivalent to the map

$$\phi_{d(x)}\colon C \times R^{p-1} \to R \times R^{p-1}$$

defined by

$$\phi_{d(x)}(z, t_1, \cdots, t_{p-1}) = (\mathscr{R}(z^{d(x)}), t_1, \cdots, t_{p-1})$$

($\mathscr{R}(z^{d(x)})$ is the real part of the complex number $z^{d(x)}$).

The hypothesis on the critical set is essential [3, (4.11)], but in [4] we show that any real analytic open map satisfies this hypothesis, and thus this conclusion.

COROLLARY 1.2. *If* $f\colon M^{p+1} \to N^p$ *is a* C^{p+1} *open map with* $\dim (R_{p-1}(f)) \leq 0$, *then at each* $x \in M^{p+1}$, f *is locally topologically equivalent to one of the following maps:*

(a) *the projection map* $\rho\colon R^{p+1} \to R^p$,

(b) $\tau\colon C \times C \to C \times R$ *defined by* $\tau(z, w) = (2z \cdot \bar{w}, |w|^2 - |z|^2)$, *where* $\bar{w}$ *is the complex conjugate of* w.

(c) $\psi_d\colon C \to R$ *defined by* $\psi_d(z) = \mathscr{R}(z^d)$.

In order to read the proofs in this paper, the reader will need to have [3] at hand. In particular, the terms *locally topologically equivalent, branch set* B_f, *layer map, extended embedding,* and *0-regular* are defined in [3; (1.3), (1.5), (2.1), (2.3),* and (4.1), respectively].

2. **Spoke sets.** The definition and lemmas of this section are given in somewhat greater generality than needed in this paper (i.e., for open maps), for use in a subsequent paper.

Let Γ^2 be any 2-manifold (without boundary).

DEFINITION 2.1. Let $\psi_w \times \iota\colon C \times R^{p-1} \to R \times R^{p-1}$ be defined by $\psi_0 \times \iota(z, t) = (|z|, t)$ and $\psi_w \times \iota(z, t) = (\mathscr{R}(z^w), t)$ $(w = 1, 2, \cdots)$. Thus

$B(\psi_1 \times \iota) = \varnothing$ and $B(\psi_w \times \iota) = \{0\} \times R^{p-1}$ otherwise. For $w = 0$ let $L = D^2 \times D^{p-1}$ and let $J = [-1, 1]$; for $w \geqq 1$ and $\eta > 0$ sufficiently small, let

$$L = (D^2 \times D^{p-1}) \cap (\psi_w \times \iota)^{-1} ([-\eta, \eta] \times D^{p-1})$$

and let $J = [-\eta, \eta]$. These examples motivate the following definition.

Let $f: \Gamma^2 \times R^{p-1} \to R \times R^{p-1}$ be a layer map, let $J = [b_0, b_1] \subset R$, and let $W \subset R^{p-1}$ be a closed q-cell $(q = 0, 1, \cdots, p - 1)$. Let $\{\gamma_j\}$ be a (possibly empty) collection of $2w$ disjoint closed arcs in $S^1 (j = 1, 2, \cdots, 2w)$; let $A = \bigcup_j \gamma_j$, and let $\zeta: S^1 \times W \to \Gamma^2 \times W$ be a layer embedding such that $B_f \cap \text{imag } \zeta = \varnothing$, $f \circ \zeta: \gamma_j \times W \approx J \times W$, and for each component $\varPhi$ of $\text{Cl}\,[S^1 - A]$, $f(\zeta(\varPhi \times W)) = \{b_i\} \times W (i = 0$ or 1). A *spoke set of f over $J \times W$* is (i) a compact, connected subspace $L \subset f^{-1}(R \times W)$ such that (ii) $L \cap (\Gamma^2 \times \{t\})$ is a 2-cell for each $t \in W$ and (iii) for some ζ as above, the boundary Ω of L with respect to $f^{-1}(R \times W)$ is $\text{imag } \zeta$. Thus if $A = \varnothing$, $f(\Omega) = \{b_i\} \times W (i = 0$ or 1). (In case $A \neq \varnothing$ and $q = 1$, L is homeomorphic to the hub and spokes of a wagon wheel, where $\zeta(A \times W)$ corresponds to the ends of the spokes.) The index $\xi(L) = 1 - w$.

LEMMA 2.2. *Let $f: \Gamma^2 \times R^{p-1} \to R \times R^{p-1}$ be a layer map with* $\dim (B_f \cap (\Gamma^2 \times \{t\})) = \dim (f(B_f) \cap (R \times \{t\})) \leqq 0$ *for each $t \in R^{p-1}$, let* $E \subset B_f$ *be compact, let $a \in R^{p-1}$, and let $\varepsilon > 0$. Then there are a closed* $(p - 1)$-*cell neighborhood W of a, closed intervals $J_j (j = 1, 2, \cdots, m)$, and spoke sets L_j over $J_j \times W$ such that*

(iv) $E \cap L_j \neq \varnothing$ *and* $E \cap (\Gamma^2 \times W) \subset \bigcup_j (L_j - \Omega_j)$,

(v)   *the $L_j - \Omega_j$ are mutually disjoint, and*

(vi)   *each* $\dim L_j < \varepsilon$.

Proof. Let F be a compact neighborhood of E in $\Gamma^2 \times R^{p-1}$, let $\{U_\alpha\}$ be a cover of Γ^2 by interiors of closed 2-cells, and let δ be the Lebesgue number of $\{U_\alpha \times R^{p-1}\}$ as a cover of F. We may suppose that $\varepsilon < \min (\delta, d(E, \text{bdy } F))$. Thus

(1)   for each $\varPsi \subset F$ with $\dim \varPsi < \varepsilon$, there is a closed 2-cell U with $\varPsi \subset (\text{int } U) \times R^{p-1}$.

Given $y \in R$ with $(y, a) \in f(E)$ and $X = E \cap f^{-1}(y, a)$, let Q be the finite set and $\nu: Q \times D \to \Gamma^2 \times R^{p-1}$ be the extended embedding with $\text{imag } \nu \cap B_f = \varnothing$ given by [3, (2.5)] for X and ε. According to that lemma each component K of $f^{-1}(\text{int } D) - \text{imag } \nu$ meeting X has $\dim K < \varepsilon$, and each is open. Since $X = E \cap f^{-1}(y, a)$ and E is compact, one may prove (by contradiction) that it is possible to

select the p-cell neighborhood D of (y, a) in $R \times R^{p-1}$ sufficiently small that each component K of $f^{-1}(\operatorname{int} D) - \operatorname{imag} \nu$ meeting E has diam $K < \varepsilon$. Summarizing.

(2) each component K of $f^{-1}(\operatorname{int} D)$-imag ν with $K \cap E \neq \varnothing$ has diam $K < \varepsilon$, so that $\bar{K} \subset \operatorname{int} F$.

Choose a closed interval $J(y) \subset R$ with $y \in \operatorname{int} J(y)$,

$$J(y) \times \{a\} \subset \operatorname{int} D ,$$

and end points $b_0(y)$, $b_1(y)$ with $(b_0(y), a)$, $(b_1(y), a) \notin f(B_f)$. Since $f(F \cap B_f)$ is closed, there is a closed $(p-1)$-cell neighborhood $W(y)$ of a in R^{p-1} such that $(\partial J(y) \times W(y)) \cap f(F \cap B_f) = \varnothing$ and

$$J(y) \times W(y) \subset D .$$

Let $\nu(y)$ be the corresponding extended embedding (restricted) over $J \times W$.

There are $y_1, y_2, \cdots, y_u \in R$ with $(y_j, a) \in f(E)$ and

$$f(E) \cap (R \times \{a\}) \subset \bigcup_j \operatorname{int} (J(y_j)) \times \{a\} .$$

The points $\{b_i(y_j): i = 0, 1; j = 1, 2, \cdots, u\}$ are the end points of a finite set of closed intervals with mutually disjoint interiors; let $J_h(h = 1, 2, \cdots, r)$ be those intervals with $(J_h \times \{a\}) \cap f(E) \neq \varnothing$. Let W be a closed $(p-1)$-cell neighborhood of $a \in R^{p-1}$ with $W \subset \bigcap_j W(y_j)$. Then $(\partial J_h \times W) \cap f(F \cap B_f) = \varnothing$ and

$$f(E) \cap (R \times W) \subset \bigcup_h ((\operatorname{int} J_h) \times W) \ (h = 1, 2, \cdots, r) .$$

Since each J_h is contained in some $J(y_j)$, restriction of $\nu(y_j)$ yields an extended embedding ν_h over $J_h \times W$.

Let $J = [b_0, b_1]$ be one of these intervals J_h, let

$$\nu: (Q \times J) \times W \longrightarrow \varGamma^2 \times R^{p-1}$$

be the layer embedding ν_h, and let $P \subset F$ be a component of

$$f^{-1}(\{b_i\} \times W) - \operatorname{imag} \nu .$$

Since $(\{b_i\} \times W) \cap f(F \cap B_f) = \varnothing$, $f^{-1}(\{b_i\} \times W) \cap \operatorname{int} F$ is a p-manifold, $\bar{P}$ is a compact connected p-manifold with boundary, and [3, (1.9)] $f \mid \bar{P}: \bar{P} \to \{b_i\} \times W$ is a bundle map. Thus [11; p. 53, (11.4)] it is a product bundle map, and since f is a layer map

(3) there is a layer embedding $\lambda: \varLambda^1 \times W \to \varGamma^2 \times W$, where $\lambda(\varLambda^1 \times W) = \bar{P}$ and $\varLambda^1 \approx S^1$ or $[0, 1]$.

In particular, $P \cap (\varGamma^2 \times \{s\})$ is a component of $f^{-1}(b_i, s) - \operatorname{imag} \nu$ $(s \in W; i = 0, 1)$, and $\operatorname{Cl}[P \cap (\varGamma^2 \times \{s\})] \approx \varLambda^1$. From the compactness of F and the finiteness of Q, the number of such components P is finite.

Let K be a component of $f^{-1}(J \times W)$-imag ν meeting E (thus by (2) diam $K < \varepsilon$ and $\bar{K} \subset \operatorname{int} F$) and let T be a component of the boundary of K in (i.e., relative to) $\Gamma^2 \times W$. Then

$$T \subset f^{-1}(\{b_0, b_1\} \times W) \cup \operatorname{imag} \nu .$$

Moreover, from (3) there are a finite union (possibly empty) A of disjoint arcs in S^1 and a layer embedding $\zeta: S^1 \times W \to \Gamma^2 \times W$ with imag $\zeta = T$, $\zeta(A \times W) = T \cap \operatorname{imag} \nu$, and

$$\zeta (\operatorname{Cl} [S^1 - A] \times W) = T \cap f^{-1}(\{b_0, b_1\} \times W) .$$

For each $s \in W$ and component (arc) γ of A, $f \circ \zeta: \gamma \times s \approx J \times s$, and for each component $\varDelta$ of $\operatorname{Cl} [S^1 - A]$, $f(\zeta(\varDelta \times \{s\})) = (b_i, s)$ $(i = 0$ or $1)$. Thus if $A \neq \varnothing$, there are an even number of such components (arcs) $\varDelta$, and they alternate in value. Hence there are an even number (possibly zero) of components (arcs) of A.

The union of such embeddings ζ over all $J \in \{J_h: h = 1, 2, \cdots, r\}$ and components K of $f^{-1}(J \times W) - \operatorname{imag} \nu$ is finite: call them

$$\zeta_j (j = 1, 2, \cdots, k) .$$

Let $\varOmega_j = \operatorname{imag} \zeta_j$ and let K_j be the corresponding component K; by (1) there is a closed 2-cell $U_j \subset \Gamma^2$ with $\bar{K}_j \subset (\operatorname{int} U_j) \times W$, and thus each $\bar{K}_j \cap (\Gamma^2 \times \{s\})$ is a 2-cell-with-holes contained in int U_j. Each $\varOmega_j$ separates $U_j \times W$ into two components; let L_j be the closure of the component disjoint from $\partial U_j \times W$. Each $L_j \cap (\Gamma^2 \times \{s\})$ is a 2-cell, and since the K_j are mutually disjoint, for $i \neq j$ exactly one of the following is true: $(L_i - \varOmega_i) \cap (L_j - \varOmega_j) = \varnothing$, $L_i \subset L_j$, or $L_j \subset L_i$. The desired spoke sets are those L_j with $E \cap L_j \neq \varnothing$ and $L_j \not\subset L_i$ for any $i \neq j$. Since each diam $K_j < \varepsilon$, each diam $\varOmega_j < \varepsilon$, so that diam $L_j < \varepsilon$. Since $E \cap (\Gamma^2 \times W) \subset \bigcup_j K_j \subset \bigcup_j L_j$, $E \subset B_f$, and $B_f \cap \varOmega_j = \varnothing$, $E \cap (\Gamma^2 \times W) \subset \bigcup_j (L_j - \varOmega_j)$.

LEMMA 2.3. *Let* $f: \Gamma^2 \times R^{p-1} \to R \times R^{p-1}$ *be a layer map, let* L_0 *(resp.,* $L_j, j = 1, 2, \cdots, q)$ *be a spoke set over* $J \times W(\text{resp.,} J_j \times W')$, *and let* $s \in W \cap W'$. *Suppose that* $L_j \cap (\Gamma^2 \times \{s\}) \subset L_0$,

$$B_f \cap L_0 \cap (\Gamma^2 \times \{s\}) \subset \bigcup_{j>0} (L_j - \varOmega_j) ,$$

and the $L_j - \varOmega_j$ *are mutually disjoint* $(j > 0)$. *Then*

$$\xi(L_0) = \sum_{j>0} \xi(L_j) .$$

Proof. Since $B(f_s) \subset B_f \cap (\Gamma^2 \times \{s\})$ and $\xi(L_j) = \xi(L_j \cap (\Gamma^2 \cap (\Gamma^2 \times \{s\})))$, it suffices to prove the lemma for $f = f_s: \Gamma^2 \to R$. Thus $L_j \subset L_0$ and $B_f \cap L_0 \subset \bigcup_{j>0} L_j - \varOmega_j$. If A_j (see (2.1)) has 2 $w(j)$ components

$(w(j) = 0, 1, \cdots)$, define $g_j \colon L_j \to R$ to agree with f on $\partial L_j = \Omega_j$ and to be topologically equivalent to $\psi_{w(j)}$. Let $h \colon L_0 \to R$ agree with f on $L_0 - \bigcup_{j>0} (L_j - \Omega_j)$ and with g_j on L_j $(j = 1, 2, \cdots, q)$. Then $B(h) = \bigcup_{j>0} B(g_j)$, and so is discrete.

Let $D(L_j)$ be the identification space obtained from

$$(L_j \times \{0\}) \cup (L_j \times \{1\})$$

by identifying $(x, 0)$ with $(x, 1)$ for each $x \in A = A(L_j)$, let $D(g_j) \colon D(L_j) \to R$ be defined by $D(g_j)(x, 0) = D(g_j)(x, 1) = g_j(x)$, and let $D(h)$ be defined analogously. Define a vector field u_j (resp., v) on $D(L_j)$ (resp., $D(L_0)$) which is 0 precisely on the (discrete) branch set $B(D(g_j))$ (resp., $B(D(h))$) and elsewhere is transverse to the level curves of $D(g_j)$ (resp., $D(h)$), i.e., a "gradient vector field" $(j = 0, 1, \cdots, q)$. For any vector field α with isolated zeros, let the sum of the indices of α at its zeros [7, p. 32] be denoted by $\iota(\alpha)$.

Since $L_j \approx D^2$, the Euler characteristic

$$\chi(D(L_j)) = 2 - 2w(j) = 2\xi(L_j) .$$

According to the Poincaré-Hopf Theorem [7, p. 35] (differentiability is not really needed in our case) $\chi(D(L_j)) = \iota(u_j)$, so that $2\xi(L_j) = \iota(u_j)$ and $2\xi(L_0) = \iota(u_0) = \iota(v)$. Thus $2\xi(L_0) = \iota(v) = 2 \sum_{j>0} \iota(v \mid L_j)$ (by definition of ι) $= \sum_{j>0} \iota(u_j) = 2 \sum_{j>0} \xi(L_j)$, so that $\xi(L_0) = \sum_{j>0} \xi(L_j)$ (where $j = 1, 2, \cdots, q$).

Alternatively, we could have used [5, p. 370] or [10, p. 35, (4.3.6)]; in this case we would have removed an open 2-cell with boundary a level circle about each local maximum or minimum point of g_j and h, in order to have open maps. Or, we could have used a counting argument based on the Euler characteristics of L_j, L_0, and $L_0 - \bigcup_j \text{int } L_j$; the first two spaces are 2-cells, and the last one is disjoint from B_f, so that information about it can be obtained from [3, (1.9)].

3. Spoke sets of open maps.

LEMMA 3.1. *Let* $f \colon \Gamma^2 \times R^{p-1} \to R \times R^{p-1}$ *be an open layer map, and let* L_0 *be a spoke set over* $J \times W$, *where* W *is a closed* $(p-1)$-*cell. Then*

(a) $f^{-1}(y, t) \cap L_0$ *does not contain a homeomorph of* S^1

$$((y, t) \in R \times R^{p-1})$$

(b) $\xi(L_0) \leq 0$;

(c) $f(L_0) = J \times W$;

(d) $\xi(L_0) \neq 0$ *implies that* $B_f \cap (L_0 - \Omega_0) \cap (\Gamma^2 \times \{t\}) \neq \varnothing$ *for every* $t \in R^{p-1}$;

(e) *if* $\dim (f(B_f) \cap (R \times \{t\})) \leqq 0$ *for every* $t \in R^{p-1}$,

$$\dim (B_f \cap f^{-1}(y, t)) \leqq 0 \quad \text{for every} \quad (y, t) \in R \times R^{p-1},$$

and $\xi(L_0) = 0$, *then* $B_f \cap \operatorname{int} L_0 = \varnothing$.

Proof. Suppose (a) is false, where Λ is the homeomorph of S^1. Then Λ bounds an open 2-cell Δ in $L_0 \cap (\Gamma^2 \times \{t\}) \approx D^2$. Since f_t: $\Gamma^2 \to R$ is open, $f_t(\Delta)$ is an open interval, while $f_t(\bar{\Delta})$ is a closed interval with $f_t(\partial \Delta)$ a single point, and a contradiction results.

If $\xi(L_0) > 0$, then $\Omega_0 \cap (\Gamma^2 \times \{t\})$ is a component of $f^{-1}(y, t)$ for some $y \in R$, and a contradiction of (a) results. Thus (b) is true.

From the definition of L_0 (2.1), $f(L_0) \subset J \times W$, and from that definition and (b), $f(\Omega_0) = J \times W$, so that (c) $J \times W = f(L_0)$.

If $B_f \cap (L_0 - \Omega_0) \cap (\Gamma^2 \times \{t\}) = \varnothing$ for some $t \in W$, then

$$g: L_0 \cap (\Gamma^2 \times \{t\}) \longrightarrow J \times \{t\}$$

defined by restriction of f has $B_g = \varnothing$ [3, (4.10)], and so is a bundle map [3, (1.9)]. Thus [11, p. 53, (11.4)] $L_0 \cap (\Gamma^2 \times \{t\}) \approx J \times F$, where the fiber F is a 1-manifold with boundary. Since $J \times F \approx D^2$ (2.1) (ii), F is connected and $F \not\approx S^1$. Thus $F \approx [0, 1]$, so that $\xi(L_0) = 0$. Conclusion (d) results.

For a spoke set L of f over $I \times U$, let $*L$ be $L \cap f^{-1}(\operatorname{int}(I \times U))$; thus $*L - \Omega = \operatorname{int} L$ (interior relative to $\Gamma^2 \times R^{p-1}$). Since the restriction map $\alpha: f^{-1}(\operatorname{int}(J \times W)) \to \operatorname{int}(J \times W)$ is open, $*L_0 - \Omega_0$ is open in $f^{-1}(\operatorname{int}(J \times W))$, and $B(f \mid L_0) \cap \Omega_0 = \varnothing$, the restriction map $\beta_0: *L_0 \to \operatorname{int}(J \times W)$ is open. Suppose that f satisfies the hypotheses of (e), i.e., $\xi(L_0) = 0$, while $(x, s) \in B_f \cap \operatorname{int} L_0$. Given $\varepsilon > 0$, which we may assume is less than $d(B_f, \Omega_0)$, let W' and the spoke sets $L_j(j = 1, 2, \cdots, q)$ be as given by (2.2) for f, ε, $a = s$, and $E = (B_f \cap L_0)$, where $(x, s) \in \operatorname{int} L_1$. From (b) each $\xi(L_j) \leqq 0$ and from (2.3) $\xi(L_0) = \sum_{j>0} \xi(L_j)$; thus $\xi(L_j) = 0$ for every j, so in particular $\xi(L_1) = 0$. Let $\beta_1: *L_1 \to f(*L_1)$ be restriction of f.

For each $(z, t) \in f(L_i) - f(B_f)$, $(i = 0, 1)$, $(\beta_i)^{-1}(z, t)$ is a 1-manifold with boundary; by (a) each of its components is homeomorphic to $[0, 1]$, and since $\xi(L_i) = 0$, $(\beta_i)^{-1}(z, t) \approx [0, 1]$. By [3, (4.3)(a)] $(\beta_i)^{-1}(y, u)$ is arcwise connected for each $(y, u) \in \operatorname{imag} \beta_i$. Choose $\delta > 0$ such that $S((x, s), \delta) \subset \operatorname{int} L_1$. Then

$$f^{-1}(y, u) \cap S(x, \delta) \subset (\beta_1)^{-1}(y, u) \subset f^{-1}(y, u) \cap S((x, s), \varepsilon),$$

so that f is 0-regular at (x, s) [3, (4.1)]. Since $(x, s) \in B_f \cap L_0$ is arbitrary, by [3, (4.2)] f is 0-regular at each point of L_0. Thus β_0 is

a bundle map [3, (4.3) (b)], so that $B_f \cap \text{int } L_0 = \varnothing$.

LEMMA 3.2. *Let* $g: \Gamma^2 \times R^{p-1} \to R \times R^{p-1}$ *be an open layer map,
let* L *be a spoke set over* $J \times W$ *where* W *is a* $(p-1)$-*cell and let*
$\alpha; W \approx B_g \cap L$ *with* $\pi \circ \alpha$ *the identity map. Then* $g \mid \text{int } L$ *is
topologically equivalent to* $\psi_w \times \iota$ $(w = 2, 3, \cdots; \text{see } (2.1))$.

Proof. We may as well replace g by its restriction to g^{-1} (int $J \times$
int W), and L by $L \cap g^{-1}$ (int $J \times$ int W), i.e., we may as well suppose
that int $J = R$ and int $W = R^{p-1}$. Let $h: R \times R^{p-1} \to R \times R^{p-1}$ be the
layer homeomorphism defined by $h(y, t) = (y, t) - g(\alpha(t))$, and let
$\lambda = h \circ g \mid L$. Then $B_\lambda = B_g \cap L$ and $\lambda(B_\lambda) = \{0\} \times R^{p-1}$.

Let J_i be $(-\infty, 0]$ or $[0, \infty)$ according as i is odd or even. (1)
Let K be a component of $\lambda^{-1}((\text{int } J_i) \times R^{p-1})$, and let $\beta: K \to \text{int } J_i \times
R^{p-1}$ and $\gamma: \bar{K} \to J_i \times R^{p-1}$ be the restriction of λ. Since $B_\beta = \varnothing$,
β is a bundle map with fiber a 1-manifold F [3, (1.9)], and so
$K \approx F \times \text{int } J_i \times R^{p-1}$ [11, p. 53, (11.4)]. Since K is connected, F is
also, and by (3.1(a)) $F \approx [0, 1]$. By [3, (4.3)(a)], $\gamma^{-1}(0, t)$ is arcwise
connected for each $t \in R^{p-1}$.

Given $(x, s) \in B_r \cap \gamma^{-1}(\{0\} \times R^{p-1})$ and $\varepsilon > 0$ with $S((x, s), \varepsilon) \subset \text{int } L$,
let L' be a spoke set over $J' \times W'$ given by (2.2) for $\lambda, E = \{(x, s)\}$,
$a = s$, and ε. Then L' satisfies the original hypotheses, so that
$(r')^{-1}(y, t)$ is arcwise connected for every (y, t). Choose $\delta > 0$ with
$S((x, s), \delta) \subset \text{int } L'$. Then

$$S((x, s), \delta) \cap \gamma^{-1}(y, t) \subset (\gamma')^{-1}(y, t) \subset S((x, s), \varepsilon) \cap \gamma^{-1}(y, t)$$

for each $(y, t) \in J' \times W'$, so that γ' is 0-regular at (x, s). By [3, (4.2)]
γ is 0-regular, and (by [3, (4.3)(b)]) (2) γ is a (product) bundle map
with fiber $[0, 1]$.

For each $t \in R^{p-1}$ and component K (see (1)), $\gamma \mid (\bar{K} \cap (\Gamma^2 \times \{t\}))$ is
a product bundle map over $J_i \times (t)$ with fiber $[0, 1]$, so that $\lambda^{-1}(0, t)$
is a deformation retract of $L \cap (\Gamma^2 \times \{t\}) \approx D^2$. Thus $\lambda^{-1}(0, t)$ is con-
nected. Since $\lambda^{-1}(0, t)$ contains no homeomorph of S^1 (3.1(a)), and
$\lambda^{-1}(0, t) - \{\alpha(t)\}$ is a 1-manifold with boundary points the $2w$
$(\xi(L) = 1 - w)$ points of $\lambda^{-1}(0, t) \cap \Omega$ (2.1), it follows that $\lambda^{-1}(0, t)$ is
homeomorphic to the union of $2w$ arcs disjoint except for their com-
mon endpoint $\alpha(t)$. As a result $\alpha(t) \in \bar{K} \cap (\Gamma^2 \times \{t\})$, so that each $\bar{K}$
contains imag α, i.e., B_λ.

Let K_i $(i = 1, 2, \cdots, 2w)$ be the components K enumerated so
that for any $t \in R^{p-1}$, (int K_i) $\cap (\Gamma^2 \times \{t\})$ are the components of

$$(\text{int } L) \cap ((\Gamma^2 \times \{t\}) - \lambda^{-1}(0, t))$$

in counterclockwise order around $\alpha(t)$ with $\lambda(\bar{K}_i) = J_i \times R^{p-1}$. Let

$\Lambda_i = \bar{K}_i \cap \operatorname{int} L$, let $\psi = \psi_w \times \iota$ (see (2.1)), and let Δ_i be the closures of the components of $\psi^{-1} (\operatorname{int} J_i \times R^{p-1})$ enumerated in analogous fashion.

By (2) there is an orientation-preserving homeomorphism μ_i of Λ_i onto $R \times J_i \times R^{p-1}$ with $\pi \circ \mu_i = \lambda \mid \Lambda_i$. Let ν_i be the homeomorphism of $R \times J_i \times R^{p-1}$ onto itself defined by

$$\nu_i(x, y, t) = (x, y, t) - \mu_i(\alpha(t)) + (0, 0, t) ,$$

and let $\zeta_i = \nu_i \circ \mu_i$. Then $\zeta_i(\alpha(t)) = (0, 0, t)$, so that

$$\zeta_i(B_\lambda) = \{0\} \times \{0\} \times R^{p-1} .$$

There is an analogous orientation-preserving homeomorphism ξ_i of Δ_i onto $R \times J_i \times R^{p-1}$ with $\pi \circ \xi_i = \psi \mid \Delta_i$ and $\xi_i(B_\psi) = \{0\} \times \{0\} \times R^{p-1}$.

Let $\Phi = (\operatorname{int} L) \cap \lambda^{-1} (\{0\} \times R^{p-1})$, and let Υ_i (resp., Ψ_i) be the closure in Φ (resp., $\psi^{-1}(\{0\} \times R^{p-1})$) of the component in $\Phi - \beta_\lambda$(resp., $\Psi^{-1} (\{0\} \times R^{p-1}) - B_\psi$) meeting both Λ_i and Λ_{i+1} (resp., Δ_i and Δ_{i+1}), where i and $i + 1$ are interpreted mod $2w$. In case $w = 1$ there are two such components, and Υ_i is so chosen that, for each $t \in R^{p-1}$, a counter-clockwise path around $\alpha(t)$ from Λ_i to Λ_{i+1} passes through Υ_i. Then $(\xi_i)^{-1} \circ \zeta_i$ (also $(\xi_{i+1})^{-1} \circ \zeta_{i+1}$) defines a homeomorphism of Υ_i onto ψ_i with $(\xi_i)^{-1} \circ \zeta_i(B_\lambda) = B_\psi$. Let $\rho \colon \Phi \approx \psi^{-1} (\{0\} \times R^{p-1})$ agree with $(\xi_i)^{-1} \circ \zeta_i$ on Υ_i.

Let σ_i be the layer homeomorphism of $R \times \{0\} \times R^{p-1}$ onto itself which is the restriction of $\xi_i \circ \rho \circ \zeta_i^{-1}$, (on $\zeta_i(\gamma_{i-1})$, σ_i agrees with the identity map) and let τ_i be its first coordinate map. Let ϕ_i be the homeomorphism of $R \times J_i \times R^{p-1}$ onto itself defined by $\phi_i (x, y, t) = (\tau_i(x, t), y, t)$, and let $\chi_i = (\xi_i)^{-1} \circ \phi_i \circ \zeta_i$. Then $\chi_i \colon \Lambda_i \approx \Delta_i$, they agree with ρ, and they thus define $\chi \colon \operatorname{int} L \approx C \times R^{p-1}$; since $\pi \circ \zeta_i = \lambda \mid \Lambda_i$ and $\pi \circ \xi_i = \psi \mid \Delta_i$, where $\pi \colon R \times J_i \times R^{p-1} \to J_i \times R^{p-1}$ is projection, $\psi \circ \chi = \lambda \mid \operatorname{int} L$. This is the desired conclusion.

4. The Proof of the theorem.

REMARK 4.1. According to the Rank Theorem [3, (1.6)] $B_f \subset R_{p-1}(f)$, and we prove (1.1) under the weaker hypothesis that $\dim (B_f \cap f^{-1}(y)) \leq 0$ for each $y \in N^p$.

Proof. Let X be the complement of the set on which f has the desired structure; then $X \subset B_f$ is closed. We suppose that

$$\dim f(X) \geq p - 1 ,$$

and will obtain a contradiction.

Since f is C^3, $\dim (f(R_{p-2}(f))) \leq p - 2$ [2, p. 1037]. If, for every

$x \in M^{p+1} - f^{-1}(f(R_{p-2}(f)))$, there is an open neighborhood

$$U_x \subset M^{p+1} - f^{-1}(f(R_{p-2}(f)))$$

of x with $\bar{U}_x$ compact and dim $(f(U_x \cap X)) \leqq p - 2$, it follows from the fact that $\{U_x\}$ has a countable subcover that $\dim(f(X)) \leqq p - 2$. Thus, there is an $\bar{x} \in M^{p+1} - f^{-1}(f(R_{p-2}(f)))$ such that, (1) for every open neighborhood $U \subset M^{p+1} - f^{-1}(f(R_{p-2}(f)))$ of $\bar{x}$, dim $(f(U \cap X)) \geqq p - 1$.

By [1, p. 87, (1.1)] there are open neighborhoods U of $\bar{x}$ and V of $f(\bar{x})$ and C^r diffeomorphisms $\sigma \colon R^2 \times R^{p-1} \approx U$ and $\rho \colon V \approx R \times R^{p-1}$ such that $\rho \circ f \circ \sigma = g$ is a C^r layer map and $\sigma(0, 0) = \bar{x}$. By hypothesis dim $(B_g \cap g^{-1}(y, t)) \leqq 0$ for each $(y, t) \in R \times R^{p-1}$.

Since $\sigma^{-1}(X) \subset B_g$, $B_g \subset R_{p-1}(g)$ (by the Rank Theorem [3, (1.6)]), $R_{p-1}(g) \cap (R^2 \times (t)) = R_0(g_t)$, and $\dim(g_t(R_0(g_t))) \leqq 0$ by Sard's Theorem (e.g. [2, p. 1037]), (2) dim $(g(B_g) \cap (R^2 \times \{t\})) \leqq 0$ and

$$\dim (g(\sigma^{-1}(X)) \cap (R \times \{t\})) \leqq 0 \ .$$

On the other hand, (by (1)) dim $(g(\sigma^{-1}(X)) \geqq p - 1$, so there is an $r > 0$ such that

$$\varLambda = (\mathrm{Cl} \, [S(0, r)] \times R^{p-1}) \cap \sigma^{-1}(X)$$

has dim $g(\varLambda) \geqq p - 1$. If $\pi \colon R \times R^{p-1} \to R^{p-1}$ is projection, then dim $(\pi(g(\varLambda))) \geqq p - 1$ (by (2) and [6, p. 91]), and there is an open $(p - 1)$-cell $T \subset \pi(g(\varLambda))$ [6, p. 44] with $\bar{T}$ compact. Thus (3)

$$\varLambda \cap (R^2 \times \{t\}) \neq \varnothing \qquad\qquad \text{for each } t \in T \ .$$

Let $W \subset T$ and the spoke sets $L_j (j = 1, 2, \cdots, q)$ be as given by (2.2) for g, any $a \in T$, $E = \varLambda \cap (R^2 \times \bar{T})$, and (say) $\varepsilon = 1$. If (4) (i) the cardinality $w(t) \geqq 1$ of $B_g \cap (R^2 \times \{t\}) \cap (\bigcup_j L_j)$ $(t \in \mathrm{int}\, W)$ is bounded above by $|\sum_j \xi(L_j)|$, choose $s \in \mathrm{int}\, W$ such that $w(s)$ is maximal and let (x_i, s) $(i = 1, 2, \cdots, w(s))$ be these points. Otherwise, (4) (ii) there are $s \in \mathrm{int}\, W$ and distinct points (x_i, s) $(i = 1, 2, \cdots, |\sum_j \xi(L_j)| + 1)$ of $B_g \cap (R^2 \times \{t\}) \cap (\bigcup_j L_j)$. Let w' be $w(s)$ in case (4) (i) and $|\sum_j \xi(L_j)| + 1$ in case (4) (ii). Let $\varepsilon > 0$ be less than $d(x_h, x_i)$ for $h \neq i$ and $d(B_g, \bigcup_j \varOmega_j)$, and let $W' \subset \mathrm{int}\, W$ and $\{L_k'\}$ be as given by (2.2) for g, $a = s$, $E = \bigcup_j L_j \cap B_g$, and this ε. Thus (5) the (x_i, s), are in distinct spoke sets L_k'.

By hypothesis and by (2), the hypothesis of (3.1) (e) is satisfied, so that by (3.1) (d) and (e) $\xi(L_j) = 0$ if and only if $L_j \cap B_g = \varnothing$. We may thus omit those L_j and L_k' with $\xi(L_j) = 0 = \xi(L_k')$. From (3.1) (b) each $\xi(L_j) < 0$ and $\xi(L_k') < 0$, and from (5) and (3.1) (d) the cardinality c of $\{L_k'\}$ satisfies $w' \leqq c \leqq |\sum_k \xi(L_k')|$. Since each L_k' is

contained in some L_j, $\sum_j \xi(L_j) = \sum_k \xi(L'_k)$ by (2.3), and so $w' \leqq |\sum_j \xi(L_j)|$; this contradicts (4) (ii), and hence (4) (i) must be true.

For $t \in W'$, $w(t) \geqq c$ by (3.1) (d), while $c \geqq w(s)$ by (4) (i), so that $w(t) = w(s)$. Thus (by (3.1) (d)) each $B_g \cap (R^1 \times \{t\}) \cap L'_k$ is a single point for $t \in W'$, and since B_g is closed, there is a homeomorphism $\alpha_i \colon W' \approx L'_k \cap B_g$ with $\pi \circ \alpha_i$ the identity map on W'. By (3.2) $\bigcup_k (\sigma^{-1}(X) \cap L'_k) = \varnothing$. But this set contains $\varLambda \cap (R^2 \times W')$, contradicting (3).

REMARK 4.2. In case $p = 1$, C^3 may be replaced by C^2 and the argument can be shortened considerably. In that case (4.1) results from [12, p. 103, Theorem 1] (cf. [18, pp. 7–8]), and (4.1) in case B_f is discrete is [10, p. 28, (4.3.1)] and [9]. Considerable information relating to open maps $f \colon M^2 \to N^1$ is given in [5], [8], and [10].

4.3. *Proof of* (1.2). The hypotheses of (1.1) are satisfied (with C^2 if $p = 1$). In case $p = 1$, $X = \varnothing$, so that at each $x \in M^{p+1}$, f at x is locally topologically equivalent to $\psi_{d(x)}$. In case $p \geqq 2$, for each $x \in M^{p+1} - X$ with $d(x) \neq 1$ (i.e., $x \in B_f$), $\dim B_f = p - 1 \geqq 1$ in a neighborhood of x; the assumption that $\dim R_{p-1}(f) \leqq 0$ contradicts the Rank Theorem [3, (1.6)]. Thus $B_f \subset X$, so that

$$\dim f(B_f) \leqq p - 2 \,.$$

That f is locally topological equivalent to ρ or to τ is now a consequence of [3, (4.7)].

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