

Pacific Journal of Mathematics

**ON ADDITIVE FUNCTIONS WHOSE LIMITING
DISTRIBUTIONS POSSESS A FINITE MEAN AND VARIANCE**

P. D. T. A. ELLIOTT

ON ADDITIVE FUNCTIONS WHOSE LIMITING DISTRIBUTIONS POSSESS A FINITE MEAN AND VARIANCE

P. D. T. A. ELLIOTT

In this paper two characterizations are given of those additive arithmetic functions which possess a limiting distribution with a finite mean and variance. It turns out that the study of such functions fits naturally within the framework of the theory of Lambert series.

1. An arithmetic function $f(n)$ is said to be *additive* if for every pair of coprime positive integers a and b the relation

$$f(ab) = f(a) + f(b)$$

is satisfied. If in addition the relations

$$f(p) = f(p^2) = \dots$$

hold for each prime power then we say that $f(n)$ is *strongly additive*. For clarity of exposition only we shall confine ourselves to the study of strongly additive functions in this paper.

For each real number $x \geq 1$ we define the frequency function

$$\nu_x(n; f(n) < z) = x^{-1} \sum_{\substack{n \leq x \\ f(n) < z}} 1.$$

If as $x \rightarrow \infty$ these frequencies converge to a limiting distribution in the usual probabilistic sense then we say that $f(n)$ has a limiting distribution.

2. **THEOREM.** *For any (real valued) additive function $f(n)$ the following three propositions are equivalent:*

(i) $f(n)$ has a limiting distribution with finite mean and variance.

(ii) The series

$$\sum f(p)p^{-1} \text{ and } \sum f^2(p)p^{-1}$$

both converge.

(iii)

$$\limsup_{x \rightarrow \infty} x^{-1} \sum_{n \leq x} f^2(n) < \infty$$

and

$$\lim_{x \rightarrow \infty} x^{-1} \sum_{n \leq x} f(n)$$

exist.

REMARK. The equivalence of Propositions (i) and (ii) is exactly what one should expect from the interpretation of $f(n)$ as the sum of independent random variables which take (respective) values $f(p)$ with probability p^{-1} and zero with probability $1 - p^{-1}$. More surprising, perhaps, is the fact that the hypothesis that $f(n)$ be additive improves the otherwise weak conditions (iii) to equivalence with (i). We shall (perhaps surprisingly) appeal to a result concerning Lambert series.

It will be clear that a form of theorem involving complex-valued additive functions could be proved if we confine our attention to the equivalence of Propositions (ii) and (iii).

3. Proof that (i) implies (ii).

We define the function

$$f^1(p) = \begin{cases} f(p) & \text{if } |f(p)| < 1 \\ 1 & \text{otherwise.} \end{cases}$$

Then the Erdős-Wintner criterion (see for example Kubilius [3] Theorem 4.5 pp. 74-85) asserts that $f(n)$ possesses a limiting frequency (unrestricted) if and only if both of the series

$$\sum f'(p)p^{-1} \text{ and } \sum (f'(p))_{p-1}^2$$

converge. Let $F(z)$ denote the limiting frequency guaranteed by (i). Then for any positive real number B such that $\pm B$ are continuity points of $F(z)$ we see that

$$x^{-1} \sum_{\substack{n \leq x \\ |f(n)| \leq B}} f^2(n) \longrightarrow \int_{|z| \leq B} z^2 dF(z), \quad (x \longrightarrow \infty).$$

Next, for any real $\varepsilon > 0$ there is a number A such that

$$\liminf_{x \rightarrow \infty} \nu_x(n; |f(n)| \leq A) > 1 - \varepsilon.$$

From the Erdős-Wintner criterion we see that those primes q_j for which $|f(q_j)| \geq 1$ are such that the series

$$\sum q_j^{-1}$$

converges. Let us denote the set of these primes by Q .

A straightforward application of the sieve of Eratosthenes shows that those integers which are prime to every q_j have a natural density. In fact we obtain

$$\nu_x(n; q_j \nmid n \forall j) \longrightarrow \prod_{i=1}^{\infty} \left(1 - \frac{1}{q_i}\right), \quad (x \longrightarrow \infty).$$

Set α for this product, and let A be chosen so that the second of our two assertions above holds with $\varepsilon = \alpha/2$. Let the integers n_i run through all those integers n which satisfy both

$$|f(n)| \leq A \text{ and } q_j \nmid n \forall j.$$

From what we have so far said it is clear that

$$\liminf_{x \rightarrow \infty} \nu_x(n; n = n_i \leq x) \geq \alpha/2,$$

and in particular we have

$$\nu_x(n; n = n_i \leq x) \geq \alpha/4$$

for all $x \geq x_0$, say.

Consider the sum

$$S_x = \sum_{n_i} \sum'_{q_j \leq x} f(n_i q_j)^2$$

where ' denotes that the side condition $2A < |f(q_j)| \leq B - A$ is to be satisfied.

From these restrictions a typical summand satisfies

$$f^2(n_i q_j) \geq (|f(q_j)| - A)^2 \geq \frac{1}{4} f^2(q_j)$$

so that

$$\begin{aligned} S_x &\geq \frac{1}{4} \sum_{q_j^{-1} \leq x} f^2(q_j) \sum_{n_i \leq x q^{-1}} 1 \\ &\geq \frac{1}{4} \sum_{q_j \leq x x_0^{-1}} f^2(q_j) \frac{1}{4} \alpha \frac{x}{q_j} \end{aligned}$$

and therefore

$$\begin{aligned} \limsup_{x \rightarrow \infty} \sum_{q_j \leq x} \frac{f^2(q_j)}{q_j} &\leq \limsup_{x \rightarrow \infty} x^{-1} S_x \\ &\leq \int_{|z| \leq B} z^2 dF(z) \leq \int_{-\infty}^{\infty} z^2 dF(z). \end{aligned}$$

Since these inequalities hold for any sequence of suitable continuity points $\pm B$ which tend (in absolute value) to infinity, we deduce that for any $B > 0, x \geq 0$

$$\sum_{q_j \leq x} f^2(q_j) \leq \int_{-\infty}^{\infty} z^2 dF(z)$$

where

$$2A < |f(q_j)| \leq B - A$$

so that letting $B \rightarrow \infty$ and then x yields

$$\sum_{|f(q_j)| \geq 2A} \frac{f^2(q_j)}{q_j} < \infty .$$

Moreover,

$$\sum_{1 \leq |f(p)| \leq 2A} \frac{|f(p)|}{p} \leq 2A \sum_{j=1}^{\infty} \frac{1}{q_j} < \infty ,$$

and

$$\sum_{|f(p)| < 1} \frac{f^2(p)}{p} < \infty$$

so that altogether the series

$$\sum f^2(p)p^{-1}$$

converges. The convergence of the second series in (ii) follows immediately.

Proof that (ii) implies (iii) and (i).

We begin with the remark that for any additive function, complex valued or otherwise, the Turan-Kubilius inequality (see for example Kubilius [3] pp. 31-35) asserts that for a suitable positive constant c

$$\sum_{n \leq x} |f(n) - \sum_{p \leq x} f(p)p^{-1}|^2 \leq c \sum_{p \leq x} |f^2(p)|p^{-1}, \quad (x \geq 1) .$$

In our present circumstances the sums

$$\sum_{p \leq x} f(p)p^{-1} \quad \text{and} \quad \sum_{p \leq x} f^2(p)p^{-1}$$

are uniformly bounded for all real values of x , so that

$$\begin{aligned} \sum_{n \leq x} f^2(n) &\leq 2 \sum_{n \leq x} \left(f(n) - \sum_{p \leq x} f(p)p^{-1} \right)^2 + 2x \left(\sum_{p \leq x} f(p)p^{-1} \right)^2 \\ &= O(x) , \end{aligned}$$

and

$$\limsup_{x \rightarrow \infty} x^{-1} \sum_{n \leq x} f^2(n) = D < \infty .$$

From the Erdős-Wintner criterion $f(n)$ possesses a limiting distribution $F(z)$, say. For each real number B such that $\pm B$ are continuity points of this limiting distribution, an application of Fatou's lemma yields

$$\int_{|z| \leq B} z^2 dF(z) \leq \liminf_{x \rightarrow \infty} x^{-1} \sum_{n \leq x} f^2(n) \leq D .$$

Since B is otherwise arbitrary $F(z)$ has a finite second moment, and hence a finite mean and variance.

This completes the proof of (i).

Furthermore,

$$x^{-1} \sum_{\substack{n \leq x \\ |f(n)| \leq B}} f(n) \longrightarrow \int_{|z| \leq B} z dF(z), \quad (x \rightarrow \infty),$$

whilst

$$\limsup_{x \rightarrow \infty} x^{-1} \sum_{\substack{n \leq x \\ |f(n)| > B}} |f(n)| \leq B^{-1} \limsup_{x \rightarrow \infty} x^{-1} \sum_{n \leq x} f^2(n) \leq B^{-1} D$$

from which it follows trivially that as $x \rightarrow \infty$

$$x^{-1} \sum_{n \leq x} f(n)$$

converges to the mean of $F(z)$.

This completes the proof of (iii).

Proof that (iii) implies (ii) (which will complete the proof of the theorem).

As one would expect this part of the proof takes a little more effort since we have to start, so to speak, from scratch. We recall that an additive function $f(n)$ is said to be *finitely distributed* if and only if there are two positive real numbers c_1 and c_2 so that for an unbound sequence of real numbers $x \geq 1$ we can find at least $k \geq c_2 x$ integers $1 \leq a_1 < a_2 < \dots < a_k \leq x$ so that

$$|f(a_i) - f(a_j)| \leq c_1$$

holds for every pair (a_i, a_j) , $1 \leq i, j \leq k$. This concept was introduced by Erdős [1] who proved

LEMMA 1. *A function $f(n)$ is finitely distributed if and only if there is a constant c_3 and an additive function $g(n)$ so that*

$$f(n) = c_3 \log n + g(n),$$

where

$$\sum (g'(p))^2 p^{-1} < \infty.$$

There is an alternative proof, on somewhat different lines, given by Ryavec [4].

In our present circumstances we have

$$x^{-1} \sum_{n \leq x} f^2(n) \leq E$$

for all $x \geq 2$ (say). Thus for any positive real number $A > E^{1/2}$,

$$\nu_x(n; |f(n)| \geq A) \leq EA^{-2} < 1, \quad (x \geq 2).$$

It follows from Lemma 1 that $f(n)$ is finitely distributed, and has the form

$$c_3 \log n + g(n).$$

Let π denote the set of primes q_j on which $|g(q_j)| > A$. Let n_i run through those squarefree integers which are prime to each q_j . Since

$$\sum_{q \in \pi} q^{-1}$$

converges, a straightforward application of the sieve of Eratosthenes shows that

$$\nu_x(n; n = n_i \leq x) \longrightarrow \prod_{i=1}^{\infty} \left(1 - \frac{1}{q_i + 1}\right) = \beta > 0, \quad (x \longrightarrow \infty),$$

say. For each integer n let $\nu(n)$ denote the number of distinct prime divisors of n . We next assume that $c_3 \neq 0$ and obtain a contradiction.

Let c_4 be sufficiently large that the inequality $A\nu(n) \leq c_4 \log n$ holds for all integers $n \geq 2$. Then for every real number $x \geq 2$ we have

$$\begin{aligned} Ex &\geq \sum_{n_i \leq x} f^2(n_i) \geq \sum_{n_i \leq x} (c_3 \log n_i - A\nu(n_i))^2 \\ &= c_3^2 \sum_{n_i \leq x} \log^2 n_i + O\left(\log x \sum_{n \leq x} \nu(n)\right). \end{aligned}$$

For all sufficiently large values of x the first of these two terms is

$$(1 + o(1))\beta c_3^2 x \log^2 x$$

whilst the second is at most $O(x \log x \log \log x)$. This clearly yields a contradiction. Hence $c_3 = 0$ and the additive function $f(n)$ satisfies

$$\sum_p (f'(p))^2 p^{-1} < \infty.$$

We now argue exactly as in the proof that the existence of a limiting distribution for $f(n)$ which has a finite variance implies that the series

$$\sum f^2(p)p^{-1}$$

converges, and deduce the same result.

It remains to secure the convergence of the series

$$\sum f(p)p^{-1}.$$

(We do not as yet know that a limiting distribution for $f(n)$ exists, although if we set $\alpha_n = \sum_{p \leq n} f'(p)p^{-1}$ then we do know that $f(n) - \alpha_n$ has a limiting distribution. See, for example, Kubilius [3] Theorem 4.4 pp. 72-74.)

Consider the generating function

$$G(z) = \sum_{n=1}^{\infty} f(n)z^n.$$

If N is any positive integer and z is any complex number then by the Cauchy-Schwarz inequality

$$\begin{aligned} \left| \sum_{N < n \leq 2N} f(n)z^n \right|^2 &\leq \sum_{n \leq 2N} f^2(n) \sum_{N < n \leq 2N} |z|^{2n} \\ &\leq EN^2 |z|^{2N}. \end{aligned}$$

It is easily seen that $G(z)$ is defined by an absolutely convergent series if z satisfies $|z| < 1$. By means of the representation

$$f(n) = \sum_{p|n} f(p)$$

we invert the order of summation to obtain:

$$G(z) = \sum_p f(p) \frac{z^p}{1 - z^p}.$$

Since

$$x^{-1} \sum_{n \leq x} f(n) \longrightarrow A, \quad (x \longrightarrow \infty), \quad \text{say,}$$

it is readily established that for real values of z

$$G(z) \sim \frac{A}{1 - z} \quad \text{as } z \longrightarrow 1 -.$$

We now appeal to a Tauberian theorem concerning Lambert series.

LEMMA 2. *Let a_n $n = 1, 2, \dots$ be a series of real numbers, and define*

$$H(y) = \sum_{n=1}^{\infty} a_n \frac{ny e^{-ny}}{1 - e^{-ny}}$$

for positive real values of y . Let $H(y) \rightarrow A$ as $y \rightarrow 0 +$. Let the sum of the a_n be a slowly decreasing function in the sense of Hardy [2] §6.2 pp. 124-125, that is if $x < y$ are real numbers, so that as $x \rightarrow \infty$ and $y \rightarrow \infty$ in such a manner that $y/x \rightarrow 1$, then

$$\liminf_{x \rightarrow \infty} \sum_{x < n \leq y} a_n \geq 0 .$$

Then

$$\sum_{n \leq x} a_n \longrightarrow A , \quad (x \longrightarrow \infty) .$$

REMARK. If the a_n are allowed to be complex then provided that we replace the condition of slowly decreasing by a condition of slow oscillation viz:

$$\lim_{x \rightarrow \infty} \sum_{x < n \leq y} a_n = 0 ,$$

the same conclusion may be drawn. A proof of this lemma can be found in Hardy [2], Theorem 261, pp. 373-374.

In our present circumstances we set

$$a_n = \begin{cases} f(p)p^{-1} & \text{if } n = p , \\ 0 & \text{otherwise} \end{cases}$$

and have established that

$$H(y) = yG(e^{-y}) \longrightarrow A , \quad (y \longrightarrow 0+) .$$

Moreover,

$$\left(\sum_{x < n \leq 2x} |a_n| \right)^2 \leq \sum_{x < n \leq 2x} f^2(p)p^{-1} \sum_{x < n \leq 2x} p^{-1} ,$$

so that since the series $\sum f^2(p)p^{-1}$ converges and

$$\sum_{x < n \leq 2x} \frac{1}{p} = \log \left(\frac{\log 2x}{\log x} \right) + O((\log x)^{-1}) \leq c_4 < \infty ,$$

we see that the condition of slow decreasing required for an application of Lemma 2 is satisfied.

We deduce that

$$\lim_{x \rightarrow \infty} \sum_{p \leq x} \frac{f(p)}{p} = A = \lim_{x \rightarrow \infty} x^{-1} \sum_{n \leq x} f(n) .$$

Moreover, by (ii) a limiting distribution exists for $f(n)$, which has the finite mean of value A .

This completes the proof of the theorem.

REMARK. The use of the Tauberian theorem in Lemma 2 is very convenient for the study of additive functions. If $f(p)$ assumes complex values the side condition $f(p) = O(\log p)$ will suffice in order for Lemma 2 to be applicable. This is a condition which is satisfied in nearly every case of number theoretical interest.

REFERENCES

1. P. Erdős, *On the distribution of additive functions*, Ann. Math., **47** (1946), 1-20.
2. G. H. Hardy, *Divergent Series*, Oxford, 1949.
3. J. P. Kubilius, *Probabilistic Methods in the Theory of Numbers*, Amer. Math. Soc. Transl., Vol. 11, 1964.
4. C. A. Ryavec, *A characterization of finitely distributed additive functions*. J. Number Theory, **2** (4) (1970), 393-403.

Received May 22, 1972.

UNIVERSITY OF COLORADO

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)
University of California
Los Angeles, California 90024

J. DUGUNDJI*
Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT
University of Washington
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM
Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

* C. R. DePrima California Institute of Technology, Pasadena, CA 91109, will replace J. Dugundji until August 1974.

Jan Aarts and David John Lutzer, <i>Pseudo-completeness and the product of Baire spaces</i>	1
Gordon Owen Berg, <i>Metric characterizations of Euclidean spaces</i>	11
Ajit Kaur Chilana, <i>The space of bounded sequences with the mixed topology</i>	29
Philip Throop Church and James Timourian, <i>Differentiable open maps of $(p + 1)$-manifold to p-manifold</i>	35
P. D. T. A. Elliott, <i>On additive functions whose limiting distributions possess a finite mean and variance</i>	47
M. Solveig Espelie, <i>Multiplicative and extreme positive operators</i>	57
Jacques A. Ferland, <i>Domains of negativity and application to generalized convexity on a real topological vector space</i>	67
Michael Benton Freeman and Reese Harvey, <i>A compact set that is locally holomorphically convex but not holomorphically convex</i>	77
Roe William Goodman, <i>Positive-definite distributions and intertwining operators</i>	83
Elliot Charles Gootman, <i>The type of some C^* and W^*-algebras associated with transformation groups</i>	93
David Charles Haddad, <i>Angular limits of locally finitely valent holomorphic functions</i>	107
William Buhmann Johnson, <i>On quasi-complements</i>	113
William M. Kantor, <i>On 2-transitive collineation groups of finite projective spaces</i>	119
Joachim Lambek and Gerhard O. Michler, <i>Completions and classical localizations of right Noetherian rings</i>	133
Kenneth Lamar Lange, <i>Borel sets of probability measures</i>	141
David Lowell Lovelady, <i>Product integrals for an ordinary differential equation in a Banach space</i>	163
Jorge Martinez, <i>A hom-functor for lattice-ordered groups</i>	169
W. K. Mason, <i>Weakly almost periodic homeomorphisms of the two sphere</i>	185
Anthony G. Mucci, <i>Limits for martingale-like sequences</i>	197
Eugene Michael Norris, <i>Relationally induced semigroups</i>	203
Arthur E. Olson, <i>A comparison of c-density and k-density</i>	209
Donald Steven Passman, <i>On the semisimplicity of group rings of linear groups. II</i>	215
Charles Radin, <i>Ergodicity in von Neumann algebras</i>	235
P. Rosenthal, <i>On the singularities of the function generated by the Bergman operator of the second kind</i>	241
Arthur Argyle Sagle and J. R. Schumi, <i>Multiplications on homogeneous spaces, nonassociative algebras and connections</i>	247
Leo Sario and Cecilia Wang, <i>Existence of Dirichlet finite biharmonic functions on the Poincaré 3-ball</i>	267
Ramachandran Subramanian, <i>On a generalization of martingales due to Blake</i>	275
Bui An Ton, <i>On strongly nonlinear elliptic variational inequalities</i>	279
Seth Warner, <i>A topological characterization of complete, discretely valued fields</i>	293
Chi Song Wong, <i>Common fixed points of two mappings</i>	299