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**DOMAINS OF NEGATIVITY AND APPLICATION TO  
GENERALIZED CONVEXITY ON A REAL TOPOLOGICAL  
VECTOR SPACE**

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# DOMAINS OF NEGATIVITY AND APPLICATION TO GENERALIZED CONVEXITY ON A REAL TOPOLOGICAL VECTOR SPACE

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**The purpose of this paper is to derive conditions for the existence of domains of negativity, and then to determine maximal domains of convexity, quasi-convexity, and pseudo-convexity for a quadratic function defined on a real topological vector space.**

1. Introduction. Martos, in [14] and [15], and Cottle and the author, in [3], [4], [6], and [7], study quasi-convex and pseudo-convex quadratic functions defined on  $E^n$ , the  $n$ -dimensional Euclidean space. Furthermore, in [6] and [7], the author uses the concept of domains of negativity that was introduced, *mutatis mutandis*, by Koecher in [11]. The purpose of this paper is to derive conditions for the existence of domains of negativity, and then to generalize the results found in [6].

In §2, we briefly review definitions needed in the rest of this paper. We also state relations between the classes of convex, quasi-convex, and pseudo-convex quadratic functions on a convex set. Conditions for the existence of domains of negativity and properties of these are given in §3. In §4, convex quadratic functions are studied. Then, domains of quasi-convexity and pseudo-convexity for quadratic forms are specified in §5, and, in §6, we extend this analysis to quadratic functions.

*Note.* Another approach to this theory have been used by Siegfried Schaible in "Quasi-convex Optimization in General Real Linear Spaces", *Zeitschrift für Operations Research*, 1972.

2. DEFINITIONS. Let  $E^1$  denote the field of real numbers with the natural topology and let  $X$  be a vector space over  $E^1$ . We assume that  $X$  admits a *norm*, i.e., there exists a mapping  $x \rightarrow |x|$  from  $X$  into  $E_+^1 = \{\alpha \in E^1 \mid \alpha \geq 0\}$  with the following properties:

- (i)  $|x| = 0$  if and only if  $x = 0$ ,
- (ii)  $|\lambda x| = |\lambda| |x|$  for all  $\lambda \in E^1$  and all  $x \in X$ ,
- (iii)  $|x + y| \leq |x| + |y|$  for all  $x$  and  $y$  in  $X$ .

A topology on  $X$  is determined by this norm, and  $X$ , so endowed, is called a *topological vector space over  $E^1$* .

Let  $X$  and  $Y$  be two real vector spaces. The mapping  $A: X \rightarrow Y$  is a *linear transformation* if and only if for all vectors  $x$  and  $y$  in

$X$  and for all real numbers  $\alpha$  and  $\beta$

$$A(\alpha x + \beta y) = \alpha A(x) + \beta A(y).$$

If  $Y = E^1$ , then  $A$  is said to be a *linear form* from  $X$  into  $E^1$ .

The mapping  $L: X \times X \rightarrow E^1$  is a *bilinear form* on  $X$  if and only if

(i)  $L(x, y) = L(y, x)$  for all  $x$  and  $y$  in  $X$ ,

(ii)  $L(x, y)$  is linear and continuous in  $y$  for each fixed  $x$ .

With each bilinear form  $L$  is associated a unique *quadratic form*  $Q: X \rightarrow E^1$  defined by

$$Q(x) = L(x, x) \text{ for all } x \in X.$$

A *quadratic function* on a real vector space  $X$  is a mapping  $R: X \rightarrow E^1$  defined by

$$R(x) = 1/2Q(x) + P(x) \text{ for all } x \in X,$$

where  $Q$  is a quadratic form and  $P$  is a linear form, both defined on  $X$ .

The *radical* of a bilinear form  $L$  is the set

$$X(L) = \{x \in X \mid L(x, y) = 0 \text{ for all } y \in X\}.$$

$L$  is *nondegenerate* on  $X$  if  $X(L) = 0$ . Otherwise,  $L$  is *degenerate*.

If  $X_1$  and  $X_2$  are subsets of  $X$ , then the *complement of  $X_2$  relative to  $X_1$*  is the set

$$X_1 \setminus X_2 = \{x \in X_1 \mid x \notin X_2\}.$$

Also, the *sum of  $X_1$  and  $X_2$*  is the set

$$X_1 + X_2 = \{x \in X \mid x = u + v, u \in X_1, \text{ and } v \in X_2\}.$$

If  $E_1$  and  $E_2$  are subspaces of  $X$ , then  $X = E_1 \oplus E_2$ , the *direct sum* of  $E_1$  and  $E_2$ , if and only if for each  $x \in X$  there exists a unique pair  $u \in E_1$  and  $v \in E_2$  such that  $x = u + v$ .

In [11], Koecher introduces the notion of domains of positivity in a real topological vector space, and *mutatis mutandis*, we define a *domain of negativity* in  $X$  determined by  $L$  as a subset  $Y$  of  $X$  having the following properties:

(i)  $Y$  is open and nonempty,

(ii)  $L(x, y) < 0$  for all  $x$  and  $y \in Y$ ,

(iii) for all  $x \notin Y$  there exists a vector  $y \in \bar{Y} \setminus X(L)$  such that  $L(x, y) \geq 0$ . (Note that  $\bar{Y}$  is the closure of  $Y$ .)

A subset  $S$  of  $X$  is said to be *convex* if and only if for all  $x, y$  in  $S$  and for all  $\theta \in [0, 1]$

$$x(\theta) = (1 - \theta)x + \theta y \in S.$$

Furthermore,  $S$  is *solid* if and only if it has a nonempty interior,  $S^\circ$ .

The quadratic function  $R(x) = 1/2Q(x) + P(x)$  is *convex* on a convex set  $S$  in  $X$  if and only if for all  $x$  and  $y$  in  $S$  and for all  $\theta \in [0, 1]$ ,

$$(1) \quad R((1 - \theta)x + \theta y) \leq (1 - \theta)R(x) + \theta R(y) .$$

The quadratic function  $R(x) = 1/2Q(x) + P(x)$  is *quasi-convex* on a set  $S$  in  $X$  if and only if for all  $x$  and  $y$  in  $S$

$$(2) \quad R(y) \leq R(x) \text{ implies } L(x, y - x) + P(y - x) \leq 0 .$$

The quadratic function  $R(x) = 1/2Q(x) + P(x)$  is *pseudo-convex* on a set  $S$  in  $X$  if and only if for all  $x$  and  $y$  in  $S$

$$(3) \quad L(x, y - x) + P(y - x) \geq 0 \text{ implies } R(y) \geq R(x) .$$

Observe that if we take  $P(x) = 0$  for all  $x \in X$ , then (1), (2), and (3) are the conditions for the quadratic form  $Q$  to be convex, quasi-convex, and pseudo-convex, respectively.

If  $S$  is a convex set, then denote by  $C(S)$ ,  $QC(S)$ , and  $PC(S)$  the classes of all quadratic functions  $R$  that are convex on  $S$ , quasi-convex on  $S$ , and pseudo-convex on  $S$ , respectively.

Notice that Mangasarian's results in Chapters 6 and 9 of [13] hold for a quadratic function  $R(x) = 1/2Q(x) + P(x)$  defined on an arbitrary real topological vector space if we replace the expression  $(\nabla R(x), y - x)$  by  $L(x, y - x) + P(y - x)$ . (Recall that in  $E^n$  the gradient of  $R$  evaluated at  $x$ ,  $\nabla R(x)$ , is the column vector of the partial derivatives of  $R$  at  $x$ .) Thus, from [13, Theorem 9.1.4], we have this equivalent definition: a quadratic function  $R(x)$  is *quasi-convex* on a *convex* set  $S$  in  $X$  if and only if for all  $x, y \in S$  and for all  $\theta \in [0, 1]$

$$(4) \quad R((1 - \theta)x + \theta y) \leq \text{Max} \{R(x), R(y)\} .$$

Furthermore the results in [13], [Chapters 6 and 9] imply that if  $S$  is a convex set in  $X$ , then

$$(5) \quad C(S) \subset PC(S) \subset QC(S) .$$

In [3], Cottle and the author have shown the following.

(6) **PROPOSITION.** *If the real valued function  $h$  is quasi-convex on a nonempty convex set  $S$  in  $E^n$  and continuous on  $\bar{S}$ , then  $h$  is quasi-convex on  $\bar{S}$ , the closure of  $S$ .*

Since this result holds for a quadratic function  $R$  defined on an arbitrary real topological vector space, if  $S$  is convex, then

$$(7) \quad QC(S) \subset QC(\bar{S}) .$$

It follows from (5) and (7) that for a convex set  $S \subset X$

$$(8) \quad C(S) \subset PC(S) \subset QC(S) \subset QC(\bar{S}) .$$

Observe the similarity with Ponstein's results for  $X = E^n$ . See [16].

**3. Domains of negativity.** In this section we give necessary and sufficient conditions for a bilinear form to determine a pair of domains of negativity in a real topological vector space. The importance of domains of negativity in the study of quasi-convexity and pseudo-convexity will become apparent in §§5 and 6.

First we introduce the following notation. For each  $x \in X$  we denote by  $E(x)$  the *subspace generated by  $x$* , i.e.,

$$E(x) = \{z \in X \mid z = \alpha x, \alpha \in E^1\} .$$

Given a certain bilinear form  $L$  and an arbitrary subspace  $E$  of  $X$ , we denote

$$E_L = \{z \in X \mid L(x, z) = 0 \text{ for all } x \in E\} .$$

Referring to [10, p. 6], the following is true.

$$(9) \quad \text{PROPOSITION.} \quad \text{If } x \in X \text{ and } Q(x) \neq 0, \text{ then } X = E(x) \oplus E_L(x).$$

Relative to a bilinear form  $L$ , we say that a nonzero vector  $z \in X$  is

$$\begin{aligned} &\textit{positive-valued} \text{ if and only if } Q(z) > 0 , \\ &\textit{negative-valued} \text{ if and only if } Q(z) < 0 , \\ &\textit{zero-valued} \text{ if and only if } Q(z) = 0 . \end{aligned}$$

Suppose that  $L$  is a nondegenerate bilinear form, i.e.,  $X(L) = 0$ . Furthermore, suppose there exists a vector  $x \in X$  such that  $Q(x) = -1$  and  $E_L(x)$  is an *inner product space* where  $L(u, v)$  is the inner product, i.e.,

$$\begin{aligned} L(u, v) &= L(v, u) \text{ for all } u, v \in E_L(x) \\ Q(u) &\geq 0 \text{ for all } u \in E_L(x) \\ Q(u) = 0 &\text{ implies } u = 0. \end{aligned}$$

For details see Schaefer [17, p. 44] or Greub [9, p. 160]. From (9),

$$X = E(x) \oplus E_L(x) .$$

Using the same type of argument as in [9, p. 268], the following can be shown.

(10) **PROPOSITION.** *If  $z$  is a negative-valued vector or if  $z$  is a nonzero but zero-valued vector, then  $L(x, z) \neq 0$ .*

Define the sets

$$\begin{aligned} Y^+ &= \{z \in X \mid Q(z) < 0 \text{ and } L(x, z) < 0\}, \\ Y^- &= \{z \in X \mid Q(z) < 0 \text{ and } L(x, z) > 0\}, \end{aligned}$$

Notice that  $Y^+$  and  $Y^-$  are nonempty since  $x \in Y^+$  and  $-x \in Y^-$ . It is easy to verify that

$$\begin{aligned} \bar{Y}^+ &= \{z \in X \mid Q(z) \leq 0 \text{ and } L(x, z) < 0\} \cup \{0\} \\ \bar{Y}^- &= \{z \in X \mid Q(z) \leq 0 \text{ and } L(x, z) > 0\} \cup \{0\}, \end{aligned}$$

and that  $Y^+ \cup \{0\}$ ,  $Y^- \cup \{0\}$ ,  $\bar{Y}^+$ , and  $\bar{Y}^-$  are solid convex cones. Furthermore, a modified version of arguments [6, (3.22) and (3.32)] shows that  $Y^+$  and  $Y^-$  are *domains of negativity*.

The definitions of  $Y^+$  and  $Y^-$  and (10) imply the following result.

(11) **THEOREM.** *Given the pair of domains of negativity  $Y^+$  and  $Y^-$  in  $X$  determined by  $L$ , then*

- (a)  $z \in X^- = Y^+ \cup Y^-$  if and only if  $Q(z) < 0$ ,
- (b)  $z \in X^0 = (\bar{Y}^+ \setminus Y^+) \cup (\bar{Y}^- \setminus Y^-)$  if and only if  $Q(z) = 0$ ,
- (c)  $z \in X^+ = X \setminus (\bar{Y}^+ \cup \bar{Y}^-)$  if and only if  $Q(z) > 0$ .

Since  $Y^+$  and  $Y^-$  are maximal ([11, p. 5]), then it follows from (11) that the pair  $Y^+$  and  $Y^-$  in  $X$  determined by  $L$  is unique.

In summary, if the vector  $x \in X$  is such that  $Q(x) = -1$  and  $E_L(x)$  is an inner product space, then there exists a pair of domains of negativity in  $X$  determined by  $L$ . This sufficient condition can be expressed into another form. To see this, we need the following result.

(12) **PROPOSITION.** *If there exists a vector  $x \in X$  such that  $Q(x) = -1$  and  $E_L(x)$  is an inner product space, then for all  $z \in X$  such that  $Q(z) < 0$  the subspace  $E_L(z)$  is an inner product space.*

*Proof.* For contradiction, suppose that  $Q(z) < 0$  for some  $z \in X$  and  $E_L(z)$  is not an inner product space. Hence, there exists a nonzero vector  $y \in E_L(z)$  such that  $Q(y) \leq 0$ . On the other hand, by definition of  $x$  there exists a pair  $Y^+$  and  $Y^-$  of domains of negativity in  $X$  determined by  $L$ .

Suppose  $z \in Y^+$ . If  $Q(y) < 0$ , then via (11), either the pair  $y$  and  $z$  belongs to  $Y^+$  or the pair  $-y$  and  $z$  belongs to  $Y^+$ . Since  $L(y, z) =$

$L(-y, z) = 0$ , in either case we have a contradiction to the definition of domains of negativity.

If  $Q(y) = 0$ , then, via (11), either  $y \in \bar{Y}^+ \setminus Y^+$  or  $-y \in \bar{Y}^+ \setminus Y^+$ . Since  $y \neq 0$ , either the pair  $z$  and  $y$  or the pair  $z$  and  $-y$  contradicts the property that if  $u \in Y^+$  and  $v \in \bar{Y}^+ \setminus X(L)$ , then  $L(u, v) < 0$  ([11, Theorem 1 a.]). The proof is complete.

Relying on (12), if the set  $\{x \in X \mid Q(x) < 0\}$  is nonempty and for each  $x$  in this set the subspace  $E_L(x)$  is an inner product space, then there exists a pair of domains of negativity. Other trivial sufficient conditions for the existence of such a pair are  $Q(x) < 0$  and  $E_L(x)$  empty (i.e.,  $\dim X = 1$ ). Now we turn to the necessity of these conditions.

(13) **THEOREM.** *If there exists a pair  $Y^+$  and  $Y^-$  of domains of negativity in  $X$  determined by  $L$ , then the set  $\{x \in X \mid Q(x) < 0\}$  is nonempty and for all  $x \in X$  such that  $Q(x) < 0$  the subspace  $E_L(x)$  is an inner product space or is empty.*

*Proof.* Since  $Y^+$  is nonempty, it follows that  $\{x \in X \mid Q(x) < 0\}$  is nonempty. The second condition is shown by a similar argument as in (12), and this completes the proof.

We are left with the problem of studying conditions for the existence of domains of negativity when the bilinear form  $L$  is degenerate in  $X$ , i.e., when  $X(L) \neq 0$ . Referring to Schaefer [17, p. 20], the vector space  $X$  can always be expressed as

$$X = (X/X(L)) \oplus X(L)$$

where  $X/X(L)$  is called the *quotient space* of  $X$  over  $X(L)$ . It is well-known that the bilinear form  $L$  is nondegenerate on  $X/X(L)$ .

If there exists a pair  $Y_L^+$  and  $Y_L^-$  of domains of negativity in  $X/X(L)$  determined by  $L$ , then denote

$$\begin{aligned} Y^+ &= Y_L^+ \oplus X(L) \\ Y^- &= Y_L^- \oplus X(L) . \end{aligned}$$

First, since  $Y_L^+$  and  $Y_L^-$  are nonempty and open, so are  $Y^+$  and  $Y^-$ . The other conditions for  $Y^+$  and  $Y^-$  to be domains of negativity in  $X$  follow from the fact that if  $x, y \in X$ , then

$$\begin{aligned} x &= u + t, & u \in X/X(L) \text{ and } t \in X(L), \\ y &= v + z, & v \in X/X(L) \text{ and } z \in X(L), \end{aligned}$$

and

$$L(x, y) = L(u, v) + L(t, z) = L(u, v) .$$

Hence a pair  $Y^+$  and  $Y^-$  of domains of negativity in  $X$  determined by  $L$  exists if and only if such a pair exists when  $L$  is restricted to  $X/X(L)$ .

4. Domains of convexity for a quadratic function. In this section, we want to determine the convex sets in  $X$  over which a quadratic function is convex. In [2], Cottle has studied this problem for quadratic functions defined on  $E^n$ , and, as we shall see, these results hold on an arbitrary real topological vector space.

Using definition (1), this result follows immediately.

(14) PROPOSITION. *The quadratic function  $R$  is convex on a convex set  $S$  in  $X$  if and only if the quadratic form  $Q$  is convex on  $S$ .*

The same kind of argument, as when the quadratic form is defined on  $E^n$ , can be used to show the following result.

(15) PROPOSITION. *The quadratic form  $Q$  is convex on a convex set  $S$  in  $X$  if and only if for all  $x$  and  $y$  in  $S$*

$$Q(x - y) \geq 0.$$

Notice this generalization of Cottle's result [2, (2)].

Recall that a set  $K$  in  $X$  is said to be a linear manifold if it is of the form

$$K = E + x$$

where  $x \in X$  and  $E$  is a vector subspace of  $X$ . ([1]).

With each convex set  $S$  in  $X$  is associated a *carrying plane*  $K(S)$  defined as the linear manifold of least dimension which contains  $S$ . The same argument as in [2] shows the following property.

(16) PROPOSITION. *If the quadratic form  $Q$  is convex on a convex set  $S$  in  $X$ , then  $Q$  is convex on  $K(S)$ .*

It follows that if the quadratic form  $Q$  is convex on a solid convex set  $S$  in  $X$ , then  $Q$  is convex on  $X$ .

5. Domains of quasi-convexity and pseudo-convexity for quadratic forms. The results found in Chapter 3 of [6] hold even for quadratic forms defined on a real topological vector space. Since only slight modifications of these arguments are needed for the generalization, we will restrict ourselves to the statements of the results.



Suppose that  $Y$  is a domain of negativity in  $X$  determined by  $L$ .

(17) **THEOREM.** *The quadratic form  $Q$  is quasi-convex on  $\bar{Y}$  and pseudo-convex on  $\bar{Y} \setminus X(L)$ .*

(18) **THEOREM.** *If the quadratic form  $Q$  is quasi-convex, but not convex, on a solid convex set  $S$ , then there exists a unique pair of domains of negativity,  $Y^+$  and  $Y^-$ , in  $X$  determined by  $L$ , and  $S \subset \bar{Y}^+$  or  $S \subset \bar{Y}^-$ .*

(19) **THEOREM.** *If the quadratic form  $Q$  is pseudo-convex, but not convex, on a solid convex set  $S$ , then there exists a unique pair of domains of negativity,  $Y^+$  and  $Y^-$ , in  $X$  determined by  $L$ , and  $S \subset \bar{Y}^+ \setminus X(L)$  or  $S \subset \bar{Y}^- \setminus X(L)$ .*

Therefore, if  $Y^+$  and  $Y^-$  is a pair of domains of negativity in  $X$  determined by  $L$ , then  $\bar{Y}^+$  and  $\bar{Y}^-$  are *maximal domains of quasi-convexity*, and  $\bar{Y}^+ \setminus X(L)$  and  $\bar{Y}^- \setminus X(L)$  are *maximal domains of pseudo-convexity* for a quadratic form  $Q$ .

## 6. Domains of quasi-convexity and pseudo-convexity for quadratic functions.

We wish to extend the analysis of Section 5 to quadratic functions.

With each quadratic function  $R(x) = 1/2Q(x) + P(x)$ , associate the set

$$M = \{a \in X \mid L(a, x) + P(x) = 0 \text{ for all } x \in X\}.$$

A direct generalization of results in Chapter 4 of [6] gives this sufficient condition.

(20) **THEOREM.** *If  $Y \subset X$  is a domain of negativity determined by  $L$  and  $M$  is nonempty, then the quadratic function  $R(x)$  is quasi-convex on  $\bar{Y} + M$  and pseudo-convex on  $\bar{Y} \setminus X(L) + M$ .*

Before we proceed to determine necessary conditions for the quasi-convexity of a quadratic function on a solid convex set, we have to specify under what conditions the set  $M$  is nonempty.

It is obvious that the real topological vector space  $X$  can be expressed as

$$X = E^+ \oplus E^- \oplus E^0$$

where  $E^+$ ,  $E^-$  and  $E^0$  are subspaces of  $X$  such that

$$\begin{aligned} Q(x) &> 0 \text{ for all } x \in E^+ \setminus 0, \\ Q(x) &< 0 \text{ for all } x \in E^- \setminus 0, \\ Q(x) &= 0 \text{ for all } x \in E^0, \end{aligned}$$

This decomposition may not be unique, but for the rest of this section we make the following *assumption*:

(21) There exists at least one decomposition

$$X = E^+ \oplus E^- \oplus E^0$$

where  $E^+$  and  $E^-$  are *complete* (i.e., each Cauchy sequence in  $E^+$  or  $E^-$  is convergent).

Under this assumption the following is true:

(22) **PROPOSITION.** *If  $R(x) = 1/2Q(x) + P(x)$ , then either the set  $M = \{a \in X \mid L(a, x) + P(x) = 0 \text{ for all } x \in X\}$  is nonempty or there exists a vector  $t \in X$  such that  $P(t) \neq 0$  and  $L(x, t) = 0$  for all  $x \in X$ .*

*Proof.* First we show that both conditions cannot hold simultaneously. Indeed, suppose there is an  $a \in M$ ; i.e.,  $L(a, x) + P(x) = 0$  for all  $x \in X$ . On the other hand, if  $t$  is such that  $L(x, t) = 0$  for all  $x \in X$  and  $P(t) \neq 0$ , then  $x = a$  gives a contradiction.

Next, suppose that if  $L(x, t) = 0$  for all  $x \in X$ , then  $P(t) = 0$ . Hence  $X = E^+ \oplus E^- \oplus E^0$  implies that for all  $x \in X$

$$L(a, x) + P(x) = (L(a^+, x^+) + P(x^+)) + (L(a^-, x^-) + P(x^-))$$

where  $a^+, x^+ \in E^+$  and  $a^-, x^- \in E^-$ . Relying on [17, p. 44] it follows that there exist at least one  $a^+ \in E^+$  and one  $a^- \in E^-$  such that for all  $x^+ \in E^+$

$$L(a^+, x^+) + P(x^+) = 0$$

and for all  $x^- \in E^-$

$$L(a^-, x^-) + P(x^-) = 0.$$

This shows that  $M$  is nonempty and the proof is complete.

Notice this proposition generalizes to an arbitrary real topological vector space  $X$ , satisfying assumption (21), a well-known result proved in Gale's book [8, Theorem 2.5] for the case  $X = E^n$ .

This proposition and similar arguments as in [6, (4.4), (4.13), and (4.15)] are combined to show these results.

(23) **THEOREM.** *If the quadratic function  $R(x) = 1/2Q(x) + P(x)$  is quasi-convex, but not convex, on a solid convex set  $S$ , then*

(i)  $M$  is not empty,

- (ii) *there exists a unique pair of domains of negativity,  $Y^+$  and  $Y^-$ , in  $X$  determined by  $L$ ,*
- (iii)  $S \subset \bar{Y}^+ + M$  or  $S \subset \bar{Y}^- + M$ .

(24) **THEOREM.** *If the quadratic function  $R(x) = 1/2Q(x) + P(x)$  is pseudo-convex, but not convex, on a solid convex set  $S$  in  $X$ , then*

- (i)  *$M$  is not empty,*
- (ii) *there exists a unique pair of domains of negativity,  $Y^+$  and  $Y^-$ , in  $X$  determined by  $L$ ,*
- (iii)  $S \subset (\bar{Y}^+ \setminus X(L) + M)$  or  $S \subset (\bar{Y}^- \setminus X(L) + M)$ .

Therefore, if  $M$  is nonempty and  $Y^+$  and  $Y^-$  are a pair of domains of negativity in  $X$  determined by  $L$ , then  $\bar{Y}^+ + M$  and  $\bar{Y}^- + M$  are maximal domains of quasi-convexity, and  $\bar{Y}^+ \setminus X(L) + M$  and  $\bar{Y}^- \setminus X(L) + M$  are maximal domains of pseudo-convexity for a quadratic function  $R$ .

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