POSITIVE-DEFINITE DISTRIBUTIONS AND INTERTWINING OPERATORS

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An example is given of a positive-definite measure $\mu$ on the group $\text{SL}(2, \mathbb{R})$ which is extremal in the cone of positive-definite measures, but the corresponding unitary representation $L^\mu$ is reducible. By considering positive-definite distributions this anomaly disappears, and for an arbitrary Lie group $G$ and positive-definite distribution $\mu$ on $G$ a bijection is established between positive-definite distributions on $G$ bounded by $\mu$ and positive-definite intertwining operators for the representation $L^\mu$. As an application, cyclic vectors for $L^\mu$ are obtained by a simple explicit construction.

Introduction. The use of positive-definiteness as a tool in abstract harmonic analysis has a long history, the most striking early instance being the Gelfand-Raikov proof via positive-definite functions of the completeness of the set of irreducible unitary representations of a locally compact group [5]. More recently, it was observed by R. J. Blattner [1] that the systematic use of positive-definite measures gives very simple proofs of the basic properties of induced representations, and the cone of positive-definite measures on a group was subsequently studied by Effros and Hahn [4].

The purpose of this paper is two-fold. First, we give an example to show that positive-definite measures do not suffice for the study of intertwining operators and irreducibility of induced representations, despite the claim to the contrary in [4]. Specifically, we exhibit a positive-definite measure $\mu$ on $G = \text{SL}(2, \mathbb{R})$ such that $\mu$ lies on an extremal ray in the cone of positive-definite measures on $G$, but the associated unitary representation $L^\mu$ is reducible, contradicting Lemma 4.16 of [4].

Our second aim is to show that when $G$ is any Lie group, then the correspondence between intertwining operators and positive functionals on $G$ asserted by Effros and Hahn does hold, provided one deals throughout with positive-definite distributions instead of just measures. The essential point is the validity of the Schwartz Kernel Theorem for the space $C^\infty_0(G)$, together with a result of Bruhat [3] about distributions on $G \times G$, invariant under the diagonal action of $G$. Using this correspondence, we obtain cyclic vectors for representations defined by positive-definite distributions, using a modification of the construction in [7]. (The proof of cyclicity given in [7] is invalid, since it assumes the existence of a measure on $G$ corresponding to

1. Notation and statement of theorems. Let $G$ be a Lie group, and denote by $\mathcal{D}(G)$ the space $C_c^\infty(G)$ with the usual inductive limit topology [10]. Fix a left Haar measure $dx$ on $G$; then $d(xy) = A_\sigma(y)dx$, where $A_\sigma$ is the modular function for $G$. If $\phi \in \mathcal{D}(G)$, define $\phi^*(x) = \overline{\phi(x^{-1})}A_\sigma(x)^{-1}$. Denote by $\mathcal{D}'(G)$ the space of Schwartz distributions on $G$. A distribution $\alpha$ is positive-definite if $\alpha(\phi^*\phi) \geq 0$ for all $\phi \in \mathcal{D}(G)$, where convolution is defined as usual by

$$(\psi \ast \phi)(x) = \int_G \psi(y)\phi(y^{-1}x)dy.$$ 

If $\alpha$ and $\beta$ are distributions, say that $\alpha \ll \beta$ if $\beta - \alpha$ is positive-definite.

Given a positive-definite distribution $\mu$, one obtains a unitary representation $L_\mu$ of $G$ by a standard construction: Let $L_\phi(x) = \phi(y^{-1}x)$ be the left action of $G$ on $\mathcal{D}(G)$. Then $(L_\phi \ast \phi)(L_\psi \psi) = \phi^*\psi$, so the semi-definite inner product $\mu(\phi^*\phi)$ is invariant under left translations. Define $I_\mu = \{\phi \in \mathcal{D}(G): \mu(\phi^*\phi) = 0\}$. The quotient space $\mathcal{D}_\mu = \mathcal{D}(G)/I_\mu$ is then a pre-Hilbert space with inner product $(\tilde{\psi}, \tilde{\phi})_\mu = \mu(\phi^*\psi)$, where $\phi \mapsto \tilde{\phi}$ is the natural mapping of $\mathcal{D}(G)$ onto $\mathcal{D}_\mu$. Let $\mathcal{H}_\mu$ be the completion of $\mathcal{D}_\mu$. The operators $L_\phi$ pass to the quotient to give a strongly continuous unitary representation $y \mapsto L_\phi^\mu$ of $G$ on $\mathcal{H}_\mu$.

Suppose now that $\alpha \in \mathcal{D}'(G)$ satisfies $0 \ll \alpha \ll \mu$, and there exists a unique self-adjoint operator $A$ on $\mathcal{H}_\mu$ such that

$$(1.1) \quad (A\tilde{\phi}, \tilde{\psi})_\mu = \alpha(\psi^*\phi).$$

The operator $A$ obviously satisfies

$$(1.2) \quad 0 \leq A \leq I$$

$$(1.3) \quad L_\phi^\mu A = AL_\phi^\mu,$$

since the Hermitian form $\alpha(\phi^*\phi)$ is nonnegative, bounded by $(\tilde{\phi}, \tilde{\phi})_\mu = \|\tilde{\phi}\|_\mu^2$, and invariant under left translations by $G$. It was asserted (without proof) by Effros and Hahn in [4, §4] that when $\mu$ is a measure, then every operator $A$ satisfying (1.2) and (1.3) is given by formula (1.1), where $\alpha$ is a positive-definite measure. Unfortunately, this is false in general, as shown by the following example:

**Theorem 1.** There is a positive-definite measure $\mu$ on the group $G = \text{SL}(2, \mathbb{R})$ such that:

(i) The only measures $\alpha$ satisfying $0 \ll \alpha \ll \mu$ are the measures $c\mu$, $c \in [0, 1]$. 


(ii) The representation $L^\mu$ of $G$ defined by $\mu$ is reducible.

If we allow positive-definite distributions in formula (1.1), however, then we obtain all intertwining operators, as follows:

**Theorem 2.** Let $G$ be a Lie group, and let $\mu$ be a positive-definite distribution on $G$. Suppose $A$ is an operator on $\mathcal{H}_\mu$ satisfying (1.2) and (1.3). Then there exists a unique positive-definite distribution $\alpha$ on $G$ such that (1.1) holds. Furthermore, the local order of $\alpha$ can be bounded in terms of the local order of $\mu$ and the dimension of $G$.

**Remarks 1.** Theorems 1 and 2 show that the cone of positive-definite measures on $\text{SL}(2, \mathbb{R})$ is not a face of the cone of positive-definite distributions.

2. For a study of unbounded intertwining operators, cf. [9].

3. In case $\mu$ is a positive-definite measure, then the distribution $\alpha$ in Theorem 2 has finite global order at most $2(\dim G + 1)$.

A sequence $\{\phi_n\} \subset \mathcal{D}(G)$ will be called a $\delta$-sequence if $\phi_n(x) \geq 0$, $\lim_n \int_G \phi_n(x) dx = 1$, and $\text{Supp}(\phi_n) \to \{1\}$ as $n \to \infty$. Any $\delta$-sequence is an approximate identity under convolution, of course.

**Corollary.** Let $\{\phi_n\}$ be a delta sequence, and set $w_n = \phi_n^* \phi_n$. Then the vector $\xi = \sum \lambda_n \omega_n$ will be a cyclic vector for the representation $L^\mu$, provided $\lambda_n > 0$ and $\lambda_n \to 0$ sufficiently fast as $n \to \infty$.

2. Proof of Theorem 1. Let $G = \text{SL}(2, \mathbb{R})$ in this section. We distinguish two closed subgroups of $G$: the subgroup $B$ consisting of all matrices $b = \begin{pmatrix} s & t \\ 0 & s^{-1} \end{pmatrix}$, with $s, t$ real, $s \neq 0$, and the subgroup $V$ consisting of all matrices $v = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$, $x$ real. One has $B \cap V = \{1\}$, while $V \cdot B$ consists of all unimodular matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $a \neq 0$. The map $v, b \to v \cdot b$ is a diffeomorphism from $V \times B$ to the open subset $V \cdot B$ of $G$. Let $dv$ and $db$ be left Haar measures on $V$ and $B$, respectively, and let $\Delta_b$ be the modular function of $B$. Left Haar measure $dx$ on $G$ is then given by the formula

$$
\int_G f(x) dx = \int_V \int_B f(vb) \Delta_b(b^{-1}) db dv = \int_B \int_V f(bv) db dv
$$
Suppose that $p$ is a unitary character of $B$. Then $p(b)db$ is a positive-definite measure on $B$, and the measure $\mu$ on $G$ defined by

$$
\int_G f(x) d\mu(x) = \int_B f(b) \Delta_B(b)^{-1/2} p(b)db
$$

is positive-definite [1]. As in §1, we denote by $L^\mu$ the corresponding representation of $G$ on $\mathcal{H}_\mu$. The representation $L^\mu$ is equivalent to the "principal series" representation of $G$ induced from the one-dimensional representation $p$ of $B$. Using the integration formula (2.1), we can identify the representation space $\mathcal{H}_\mu$ with $L_2(V, d\nu)$. (This gives the so-called "non-compact picture" for the principal series [8].) Indeed, if $\phi, \psi \in \mathcal{D}(G)$, then an easy calculation using (2.1) shows that

$$(\tilde{\phi}, \tilde{\psi})_\mu = \int_V \varepsilon(\phi)\overline{\varepsilon(\psi)}d\nu,$$

where

$$
\varepsilon(\phi)(v) = \int_B \phi(vb) \Delta_B(b)^{-1/2} p(b)db .
$$

The restriction of $L^\mu$ to the subgroup $V$ becomes simply the left regular representation of $V$ in this picture.

**Lemma 1.** Let $A$ be a bounded operator on $L_2(V)$ which commutes with left translations by $V$, and suppose that there exists a Radon measure $\alpha$ on $G$ such that

$$(A\varepsilon(\phi), \varepsilon(\psi))_{L_2(V)} = \alpha(\psi^*\phi)$$

for all $\phi, \psi \in \mathcal{D}(G)$. Then there is a Radon measure $\nu$ on $V$ such that $Af = f*\nu$, for $f \in \mathcal{D}(V)$.

**Proof.** Since $A$ is translation invariant, it is enough to establish an estimate

$$(Af)(1) \leq C_K ||f||_\infty ,$$

for all $f \in \mathcal{D}(V)$ supported on an arbitrary compact set $K \subset V$ ($||f||_\infty$ denoting the sup norm). Let $\mathcal{H}_\infty(V)$ be the space of $C^\infty$ vectors for the left regular representation of $V$. By Sobolev's lemma, $\mathcal{H}_\infty(V) \subset C^\infty(V)$, and $A$ leaves the space $\mathcal{H}_\infty(V)$ invariant. Hence, $A\varepsilon(\phi)$ is a $C^\infty$ function for every $\phi \in \mathcal{D}(G)$.

If $f \in \mathcal{D}(V)$ and $g \in \mathcal{D}(B)$, write $f \otimes g$ for the function $f(v)g(b)$. Via the map $v, b \to vb$ we may consider $f \otimes g$ as an element of $\mathcal{D}(G)$. Then $\varepsilon(f \otimes g) = \lambda_g f$, where $\lambda_g = \int_B g(b) \Delta_B(b)^{-1/2} p(b)db$. In particular,
if \( \{f_n\} \) and \( \{g_n\} \) are \( \delta \)-sequence in \( \mathcal{D}(V) \) and \( \mathcal{D}(B) \) respectively, then \( \lambda_{g_n} \rightarrow 1 \) as \( n \rightarrow \infty \) and \( f_n \otimes g_n \) is a \( \delta \)-sequence on \( G \) (by the integration formula (2.1)). Hence, we deduce from (2.2) that

\[
A\varepsilon(\phi)(1) = \alpha(\phi)
\]

for all \( \phi \in \mathcal{D}(G) \). Fix \( g \in \mathcal{D}(B) \) such that \( \lambda_g = 1 \). Then for any \( f \in \mathcal{D}(V) \) we have \( f = \varepsilon(f \otimes g) \), and hence

\[
(2.4) \quad (Af)(1) = \alpha(f \otimes g).
\]

Since \( \alpha \) is a Radon measure, the right side of (2.4) satisfies (2.3), which proves the lemma. (In fact, \( \nu \) is the measure \( f \rightarrow \alpha(f \otimes g) \).)

**Completion of proof of Theorem 1.** Now take for \( p \) the character \( p(b) = \text{sgn}(s) \), when \( b = \left( \begin{array}{cc} s & t \\ 0 & s^{-1} \end{array} \right) \). Then it is known [8] that the induced representation \( L^\nu \) in this case splits into two parts, and when \( \mathcal{H}_\nu \) is realized as \( L_s(V) \), then any nontrivial intertwining operator is a scalar multiple of the classical Hilbert transform

\[
Af(x) = \lim_{\delta \rightarrow 0} \frac{1}{\pi} \int_{|y| > \delta} f(x - y)y^{-1}dy.
\]

(We identify \( V \) with \( R \) via the map \( x \rightarrow (1 \ 0) \).)

The Hilbert transform does not satisfy estimate (2.3). For example, if

\[
f_n(x) = \phi(x) \sum_{k=2}^n \frac{\sin(kx)}{k \log k},
\]

where \( \phi \in \mathcal{D}(R) \) is fixed with \( \phi(x) = 1 \) for \( |x| \leq 1 \), then \( \text{Supp} (f_n) \subseteq \text{Supp} (\phi) \) and \( \sup_n \|f_n\|_\infty < \infty \) [11, p. 182].

On the other hand,

\[
Af_n(0) = \sum_{k=2}^n c_k (k \log k)^{-1} + O(1)
\]

as \( n \rightarrow \infty \), where

\[
c_k = \frac{1}{\pi} \int_{-1}^1 x^{-1} \sin(kx)dx.
\]

Since \( c_k \rightarrow 1 \) as \( k \rightarrow \infty \), and since \( \Sigma(k \log k)^{-1} = +\infty \), it follows that

\[
\sup_n |Af_n(0)| = \infty.
\]

**3. Proof of Theorem 2 and Corollary.** Let \( G \) be an arbitrary Lie group (assumed countable at infinity), and let \( \mu \) be a given positive-
definite distribution on $G$. If we set $||\phi||_\mu = \mu(\phi^*\phi)^{1/2}$, then $\phi \mapsto ||\phi||_\mu$ is a continuous seminorm on $\mathcal{D}(G)$. Suppose now that $A$ is a bounded operator on the representation space $\mathcal{H}_\mu$. We may associate with $A$ a bilinear form $B_A$ on $\mathcal{D}(G)$ by the formula

$$B_A(\psi, \phi) = (A\phi, J\psi)_\mu.$$  

(3.1)

Here $\phi \mapsto \tilde{\phi}$ is the canonical map from $\mathcal{D}(G)$ into $\mathcal{H}_\mu$ as in §1, and $J\phi = \tilde{\phi}$ (complex conjugate). By the Schwarz inequality and the boundedness of $A$ we see that

$$|B_A(\psi, \phi)| \leq ||A|| \cdot ||\phi||_\mu ||J\psi||_\mu.$$  

(3.2)

Clearly, $\psi \mapsto ||J\psi||_\mu$ is also a continuous seminorm on $\mathcal{D}(G)$. Although $||J\psi||_\mu$ need not be bounded in terms of $||\psi||_\mu$, nevertheless, the local order of this seminorm is the same as the local order of $|| \cdot ||_\mu$. (If $K \subset G$ is a compact set and $\rho$ is a continuous seminorm on $\mathcal{D}(G)$, we say that $\rho$ has order $\leq r$ on $K$ if there is a finite set of differential operators $\{D_j\}$ on $G$ each of order $\leq r$, such that $\rho(\phi) \leq \max_j ||D_j\phi||_\infty$ for all $\phi$ with $\text{Supp} (\phi) \subseteq K$.)

The main analytic fact we need is the following version of the "kernel theorem" for continuous bilinear forms:

**Lemma 2.** Suppose $B$ is a bilinear form on $\mathcal{D}(G)$, and $\rho_1, \rho_2$ are continuous seminorms on $\mathcal{D}(G)$ such that

$$|B(\phi, \psi)| \leq \rho_1(\phi) \rho_2(\psi).$$  

(3.3)

Then there is a distribution $T$ on $G \times G$ such that

$$B(\phi, \psi) = T(\phi \otimes \psi).$$

Furthermore, if $K_1$ and $K_2$ are compact subsets of $G$, and if $\rho_j$ has order $\leq r_j$ on $K_j (j = 1, 2)$, then $T$ has order $\leq r_1 + r_2 + 2(\dim G + 1)$ on any compact set $M \subset \text{Interior} (K_1 \times K_2)$.

**Proof.** Since multiplication by a $C^\infty$ function is an operator of order zero, we may use a partition of unity and local coordinates to reduce the problem to a local one in $R^d$, $d = \dim G$, such that $K_j = \{|x| \leq 2\} \subseteq R^d$ and $M = \{(x, y); |x| \leq 1, |y| \leq 1\} \subseteq R^d \times R^d$.

Let $\phi_0 \in \mathcal{D}(R^d)$ satisfy $\phi_0 = 1$ on $\{|x| \leq 1\}$ and $\text{Supp} (\phi_0) \subseteq K_1$. Set $e_n(x) = \phi_0(x)e^{in\cdot x}$, where $n \in N^d$ and $n \cdot x = n_1x_1 + \cdots + n_dx_d$. Then if $D$ is a differential operator of order $r$, one has $||De_n||_\infty \leq C(1 + |n|)^r$. Hence, the a priori estimate (3.3) implies that for some constant $C > 0$,

$$|B(e_m, e_n)| \leq C(1 + |m|)^r(1 + |n|)^r$$  

(3.4)

for all $m, n \in N^d$. 

Suppose now that $f$ is a $C^\infty$ function on $\mathbb{R}^d \times \mathbb{R}^d$ with $\text{Supp}(f) \subseteq M$. Then the Fourier series of $f$ can be written as

$$f(x, y) = \sum_{m,n} \hat{f}(m,n)e_m(x)e_n(y),$$

where $\{\hat{f}(m,n)\}$ are the Fourier coefficients of $f$. Define

$$T(f) = \sum_{m,n} \hat{f}(m,n)B(e_m, e_n).$$

The series (3.5) is absolutely convergent, and by (3.4) we have the estimate

$$|T(f)| \leq C_i \sup_{m,n} \{|\hat{f}(m,n)| (1 + |m|)^{r_1 + d + 1}(1 + |n|)^{r_2 + d + 1}\},$$

where $C_i = C \sum_{m,n} (1 + |m|)^{-d-1}(1 + |n|)^{-d-1} < \infty$. Since the right side of (3.6) is a seminorm of order $r_1 + r_2 + 2d + 2$ on $M$, this proves the lemma.

**Completion of proof of Theorem 2.** Suppose now that the operator $A$ in formula (3.1) commutes with the representation $L^\nu$. Then the distribution $T$ on $G \times G$ such that $B_A(\phi, \psi) = T(\phi \otimes \psi)$, which was constructed in Lemma 2, satisfies for all $z \in G$,

$$T(\delta_z f) = T(f), \quad f \in \mathcal{D}(G \times G),$$

where $\delta_z f(x, y) = f(z^{-1}x, z^{-1}y)$.

The structure of distributions satisfying (3.7) was determined by Bruhat [3, Prop. 3.3]. Let $\iota$ denote the distribution on $G$ determined by left Haar measure, and let $\Phi: G \times G \to G \times G$ be the map $\Phi(x, y) = (x, xy)$. Then (3.7) forces $T$ to have the form

$$T(f) = (\iota \otimes \alpha)(f \circ \Phi),$$

where $\alpha$ is a distribution on $G$. Symbolically,

$$T(f) = \iint f(x, xy) dxd\alpha(y).$$

In particular, if $\phi, \psi \in \mathcal{D}(G)$, then

$$(A\phi, \psi)_\mu = T(J\psi \otimes \phi) = \iint \psi(x)\phi(xy) dxd\alpha(y) = \alpha(\psi^*\phi).$$

Hence, $\alpha$ serves to represent the intertwining operator $A$, and is obviously positive-definite if $A \succeq 0$. Since $\Phi$ is a diffeomorphism, the order of $\iota \otimes \alpha$ on a compact set $M \subset G \times G$ is the same as the order of $T$ on $\Phi^{-1}(M)$. By Lemma 2 and inequality (3.2), the local order
Proof of Corollary. Using Theorem 2, we are able to rehabilitate
the attempted proof of cyclicity in [7]. Given a \( \delta \)-sequence \( \{ \psi_n \} \) on
\( G \), let \( K \subset G \) be a compact set such that \( K = K^{-1} \) and \( \text{Supp} (\psi_n) \subseteq K \)
for all \( n \). Since \( \| \psi \|_\mu \) is a continuous seminorm on \( \mathcal{D}(G) \), there are
right-invariant differential operators \( D_i, \ldots, D_r \) on \( G \) such that
\[
\| \psi \|_\mu \leq \max_j \| D_j \psi \|_\infty
\]
for all \( \psi \) supported on the set \( K^2 \).

Now set \( w_n = \psi_n^{*} \psi_n \), and let \( \{ \lambda_n \} \) be any sequence such that
\( \lambda_n > 0 \) and
\[
\sum_n \lambda_n \max_j \| D_j \psi_n \|_\infty < \infty .
\]
The series \( \xi = \sum \lambda_n \tilde{w}_n \) then converges absolutely in \( \mathcal{H}_\mu \) (since \( \| w_n \|_\mu \leq \| \psi_n \|_\mu \)). Let \( \mathcal{N} \) be the \( G \)-cyclic subspace generated by \( \xi \), and let \( A \) be the projection onto \( \mathcal{N}^{-1} \). Since \( A \xi = 0 \), we have \( \sum \lambda_n (A \tilde{w}_n, \tilde{\phi})_\mu = 0 \) for all \( \phi \in \mathcal{D}(G) \). But \( \phi * \psi = L_\mu (\phi) \tilde{\psi} \), where \( L_\mu (f) = \int f(x) L_\mu (x) dx \)
is the integrated form of the representation. Since \( A \) commutes with \( L_\mu \), this gives \( (A \tilde{w}_n, \tilde{\phi})_\mu = (A \tilde{\psi}_n, \tilde{\psi}_n * \tilde{\phi})_\mu \). Thus taking \( \phi = \psi_k \) and
letting \( k \to \infty \), we see that
\[
\lim_{k \to \infty} (A \tilde{w}_n, \tilde{\psi}_k)_\mu = (A \tilde{\psi}_n, \tilde{\psi}_n)_\mu
\]
(note that \( \phi \to \tilde{\phi} \) is continuous from \( \mathcal{D}(G) \) to \( \mathcal{H}_\mu \)). Furthermore, by
the Schwartz inequality, the boundedness of \( A \), and the calculation
just made, we have the estimate
\[
\| (A \tilde{w}_n, \tilde{\psi}_k)_\mu \|^{1/2} \leq \| \psi_n \|_\mu \| \psi_n * \psi_k \|_\mu
\leq C \max_j \| D_j \psi_n \|_\infty
\]
(Here we have used estimate (3.8), the right-invariance of \( D_j \), and
the inequality \( \| f * g \|_\infty \leq \| f \|_\infty \| g \|_{L_1} \).) Thus we may apply the domi-
nated convergence theorem to conclude from (3.9) and (3.10) that
\( \sum \lambda_n (A \tilde{\psi}_n, \tilde{\psi}_n)_\mu = 0 \). But \( \lambda_n > 0 \) and \( A = 0 \), so in fact \( (A \tilde{\psi}_n, \tilde{\psi}_n)_\mu = 0 \) for all \( n \). (So far we have simply followed the line of proof of
[7], replacing uniform convergence of the series \( \sum \lambda_n w_n \) by the stronger
condition (3.9), in return for allowing \( \mu \) which are distributions rather
than measures.) Finally let \( \alpha \) be the positive-definite distribution on
\( G \) representing \( A \), which exists by Theorem 2. Then \( \alpha (\psi_n^{*} \psi_n) = 0 \) for
all \( n \). By the Schwarz inequality, this implies that \( \alpha (\phi^{*} \psi_n) = 0 \) for
all \( \phi \in \mathcal{D}(G) \) and all \( n \). Letting \( n \to \infty \), we conclude that \( \alpha = 0 \).
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