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An example is given of a positive-definite measure μ on the group $\mathrm{SL}(2, \boldsymbol{R})$ which is extremal in the cone of positive-definite measures, but the corresponding unitary representation L^{μ} is *reducible*. By considering positive-definite *distributions* this anomaly disappears, and for an arbitrary Lie group G and positive-definite distribution μ on G a bijection is established between positive-definite distributions on G bounded by μ and positive-definite intertwining operators for the representation L^{μ}. As an application, cyclic vectors for L^{μ} are obtained by a simple explicit construction.

Introduction. The use of positive-definiteness as a tool in abstract harmonic analysis has a long history, the most striking early instance being the Gelfand-Raikov proof via positive-definite functions of the completeness of the set of irreducible unitary representations of a locally compact group [5]. More recently, it was observed by R. J. Blattner [1] that the systematic use of positive-definite *measures* gives very simple proofs of the basic properties of induced representations, and the cone of positive-definite measures on a group was subsequently studied by Effros and Hahn [4].

The purpose of this paper is two-fold. First, we give an example to show that positive-definite measures do not suffice for the study of intertwining operators and irreducibility of induced representations, despite the claim to the contrary in [4]. Specifically, we exhibit a positive-definite measure μ on $G = \mathrm{SL}(2, \boldsymbol{R})$ such that μ lies on an extremal ray in the cone of positive-definite measures on G, but the associated unitary representation L^{μ} is *reducible*, contradicting Lemma 4.16 of [4].

Our second aim is to show that when G is any Lie group, then the correspondence between intertwining operators and positive functionals on G asserted by Effros and Hahn does hold, provided one deals throughout with positive-definite *distributions* instead of just measures. The essential point is the validity of the Schwartz Kernel Theorem for the space $C_0^{\infty}(G)$, together with a result of Bruhat [3] about distributions on $G \times G$, invariant under the diagonal action of G. Using this correspondence, we obtain cyclic vectors for representations defined by positive-definite distributions, using a modification of the construction in [7]. (The proof of cyclicity given in [7] is invalid, since it assumes the existence of a measure on G corresponding to an arbitrary intertwining operator. Cf. [6] for a proof of cyclicity using von Neumann algebra techniques.)

1. Notation and statement of theorems. Let G be a Lie group, and denote by $\mathscr{D}(G)$ the space $C_0^\infty(G)$ with the usual inductive limit topology [10]. Fix a left Haar measure dx on G; then $d(xy) = \Delta_G(y)dx$, where Δ_G is the modular function for G. If $\phi \in \mathscr{D}(G)$, define $\phi^*(x) = \overline{\phi(x^{-1})\Delta_G(x)^{-1}}$. Denote by $\mathscr{D}'(G)$ the space of Schwartz distributions on G. A distribution α is *positive-definite* if $\alpha(\phi^* * \phi) \geqq 0$ for all $\phi \in \mathscr{D}(G)$, where convolution is defined as usual by

$$(\psi * \phi)(x) = \int_G \psi(y)\phi(y^{-1}x)dy .$$

If α and β are distributions, say that $\alpha \ll \beta$ if $\beta - \alpha$ is positive-definite.

Given a positive-definite distribution μ, one obtains a unitary representation L^μ of G by a standard construction: Let $L_y\phi(x) = \phi(y^{-1}x)$ be the left action of G on $\mathscr{D}(G)$. Then $(L_y\phi)^* * (L_y\psi) = \phi^* * \psi$, so the semi-definite inner product $\mu(\phi^* * \psi)$ is invariant under left translations. Define $I_\mu = \{\phi \in \mathscr{D}(G) \colon \mu(\phi^* * \phi) = 0\}$. The quotient space $\mathscr{D}_\mu = \mathscr{D}(G)/I_u$ is then a pre-Hilbert space with inner product $(\tilde{\psi}, \tilde{\phi})_\mu = \mu(\phi^* * \psi)$, where $\phi \longrightarrow \tilde{\phi}$ is the natural mapping of $\mathscr{D}(G)$ onto $\mathscr{D}_\mu$. Let $\mathscr{H}_\mu$ be the completion of $\mathscr{D}_\mu$. The operators L_y pass to the quotient to give a strongly continuous unitary representation $y \longrightarrow L_y^\mu$ of G on $\mathscr{H}_\mu$.

Suppose now that $\alpha \in \mathscr{D}'(G)$ satisfies $0 \ll \alpha \ll \mu$. Then $I_\alpha \supseteqq I_\mu$, and there exists a unique self-adjoint operator A on $\mathscr{H}_\mu$ such that

$$(A\tilde{\phi}, \tilde{\psi})_\mu = \alpha(\psi^* * \phi) . \tag{1.1}$$

The operator A obviously satisfies

$$0 \leqq A \leqq I \tag{1.2}$$

$$L_x^\mu A = A L_x^\mu , \tag{1.3}$$

since the Hermitian form $\alpha(\phi^* * \phi)$ is nonnegative, bounded by $(\tilde{\phi}, \tilde{\phi})_\mu = \|\tilde{\phi}\|_\mu^2$, and invariant under left translations by G. It was asserted (without proof) by Effros and Hahn in [4, §4] that when μ is a *measure*, then every operator A satisfying (1.2) and (1.3) is given by formula (1.1), where α is a positive-definite *measure*. Unfortunately, this is false in general, as shown by the following example:

THEOREM 1. *There is a positive-definite measure μ on the group $G = \mathrm{SL}(2, \boldsymbol{R})$ such that:*

(i) *The only measures α satisfying $0 \ll \alpha \ll \mu$ are the measures $c\mu$, $c \in [0, 1]$.*

(ii) *The representation L^μ of G defined by μ is reducible.*

If we allow positive-definite *distributions* in formula (1.1), however, then we obtain all intertwining operators, as follows:

THEOREM 2. *Let G be a Lie group, and let μ be a positive-definite distribution on G. Suppose A is an operator on $\mathscr{H}_\mu$ satisfying* (1.2) *and* (1.3). *Then there exists a unique positive-definite distribution α on G such that* (1.1) *holds. Furthermore, the local order of α can be bounded in terms of the local order of μ and the dimension of G.*

REMARKS 1. Theorems 1 and 2 show that the cone of positive-definite measures on SL(2, $\boldsymbol{R}$) is not a *face* of the cone of positive-definite distributions.

2. For a study of *unbounded* intertwining operators, cf. [9].

3. In case μ is a positive-definite *measure*, then the distribution α in Theorem 2 has finite global order at most $2(\dim G + 1)$.

A sequence $\{\phi_n\} \subset \mathscr{D}(G)$ will be called a δ-*sequence* if $\phi_n(x) \geqq 0$, $\lim_n \int_G \phi_n(x)dx = 1$, and $\operatorname{Supp}(\phi_n) \to \{1\}$ as $n \to \infty$. Any δ-sequence is an approximate identity under convolution, of course.

COROLLARY. *Let $\{\phi_n\}$ be a delta sequence, and set $w_n = \phi_n^* * \phi_n$. Then the vector $\xi = \Sigma\lambda_n \tilde{w}_n$ will be a cyclic vector for the representation L^μ, provided $\lambda_n > 0$ and $\lambda_n \longrightarrow 0$ sufficiently fast as $n \to \infty$.*

2. *Proof of Theorem* 1. Let $G = \mathrm{SL}(2, \boldsymbol{R})$ in this section. We distinguish two closed subgroups of G: the subgroup B consisting of all matrices $b = \begin{pmatrix} s & t \\ 0 & s^{-1} \end{pmatrix}$, with s, t real, $s \neq 0$, and the subgroup V consisting of all matrices $v = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$, x real. One has $B \cap V = \{1\}$, while $V \cdot B$ consists of all unimodular matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $a \neq 0$. The map $v, b \to v \cdot b$ is a diffeomorphism from $V \times B$ to the open subset $V \cdot B$ of G. Let dv and db be left Haar measures on V and B, respectively, and let Δ_B be the modular function of B. Left Haar measure dx on G is then given by the formula

$$(2.1) \qquad \int_G f(x)dx = \int_V \int_B f(vb)\Delta_B(b^{-1})dbdv = \int_B \int_V f(bv)dbdv$$

[2, Chap. VII, §3, Proposition 6].

Suppose that p is a unitary character of B. Then $p(b)db$ is a positive-definite measure on B, and the measure μ on G defined by

$$\int_G f(x)d\mu(x) = \int_B f(b)\Delta_B(b)^{-1/2}p(b)db$$

is positive-definite [1]. As in §1, we denote by L^μ the corresponding representation of G on $\mathscr{H}_\mu$. The representation L^μ is equivalent to the "principal series" representation of G induced from the one-dimensional representation p of B. Using the integration formula (2.1), we can identify the representation space $\mathscr{H}_\mu$ with $L_2(V, dv)$. (This gives the so-called "non-compact picture" for the principal series [8].) Indeed, if $\phi, \psi \in \mathscr{D}(G)$, then an easy calculation using (2.1) shows that

$$(\tilde{\phi}, \tilde{\psi})_\mu = \int_V \varepsilon(\phi)\overline{\varepsilon(\psi)}dv ,$$

where

$$\varepsilon(\phi)(v) = \int_B \phi(vb)\Delta_B(b)^{-1/2}p(b)db .$$

The restriction of L^μ to the subgroup V becomes simply the left regular representation of V in this picture.

LEMMA 1. *Let A be a bounded operator on $L_2(V)$ which commutes with left translations by V, and suppose that there exists a Radon measure α on G such that*

$$(A\varepsilon(\phi), \varepsilon(\psi))_{L_2(V)} = \alpha(\psi^* * \phi) \tag{2.2}$$

for all $\phi, \psi \in \mathscr{D}(G)$. Then there is a Radon measure ν on V such that $Af = f*\nu$, for $f \in \mathscr{D}(V)$.

Proof. Since A is translation invariant, it is enough to establish an estimate

$$|(Af)(1)| \leqq C_K \|f\|_\infty , \tag{2.3}$$

for all $f \in \mathscr{D}(V)$ supported on an arbitrary compact set $K \subset V$ ($\|f\|_\infty$ denoting the sup norm). Let $\mathscr{H}^\infty(V)$ be the space of C^∞ vectors for the left regular representation of V. By Sobolev's lemma, $\mathscr{H}^\infty(V) \subset C^\infty(V)$, and A leaves the space $\mathscr{H}^\infty(V)$ invariant. Hence, $A\varepsilon(\phi)$ is a C^∞ function for every $\phi \in \mathscr{D}(G)$.

If $f \in \mathscr{D}(V)$ and $g \in \mathscr{D}(B)$, write $f \otimes g$ for the function $f(v)g(b)$. Via the map $v, b \to vb$ we may consider $f \otimes g$ as an element of $\mathscr{D}(G)$. Then $\varepsilon(f \otimes g) = \lambda_g f$, where $\lambda_g = \int_B g(b)\Delta_B(b)^{-1/2}p(b)db$. In particular, if $\{f_n\}$ and $\{g_n\}$ are δ-sequence in $\mathscr{D}(V)$ and $\mathscr{D}(B)$ respectively, then $\lambda_{g_n} \to 1$ as $n \to \infty$ and $f_n \otimes g_n$ is a δ-sequence on G (by the integration formula (2.1)). Hence, we deduce from (2.2) that

$$A\varepsilon(\phi)(1) = \alpha(\phi)$$

for all $\phi \in \mathscr{D}(G)$. Fix $g \in \mathscr{D}(B)$ such that $\lambda_g = 1$. Then for any $f \in \mathscr{D}(V)$ we have $f = \varepsilon(f \otimes g)$, and hence

$$(Af)(1) = \alpha(f \otimes g)\ . \tag{2.4}$$

Since α is a Radon measure, the right side of (2.4) satisfies (2.3), which proves the lemma. (In fact, ν is the measure $f \to \alpha(f \otimes g)$.)

Completion of proof of Theorem 1. Now take for p the character $p(b) = \operatorname{sgn}(s)$, when $b = \begin{pmatrix} s & t \\ 0 & s^{-1} \end{pmatrix}$. Then it is known [8] that the induced representation L^μ in this case splits into two parts, and when $\mathscr{H}_\mu$ is realized as $L_2(V)$, then any nontrivial intertwining operator is a scalar multiple of the classical Hilbert transform

$$Af(x) = \lim_{\delta \to 0} \frac{1}{\pi} \int_{|y| > \delta} f(x - y) y^{-1} dy\ .$$

(We identify V with $\boldsymbol{R}$ via the map $x \to \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$.)

The Hilbert transform does not satisfy estimate (2.3). For example, if

$$f_n(x) = \phi(x) \sum_{k=2}^{n} \frac{\sin(kx)}{k \log k}\ ,$$

where $\phi \in \mathscr{D}(\boldsymbol{R})$ is fixed with $\phi(x) = 1$ for $|x| \leqq 1$, then $\operatorname{Supp}(f_n) \subseteqq \operatorname{Supp}(\phi)$ and $\sup_n \|f_n\|_\infty < \infty$ [11, p. 182].
On the other hand,

$$Af_n(0) = \sum_{k=2}^{n} c_k (k \log k)^{-1} + O(1)$$

as $n \to \infty$, where

$$c_k = \frac{1}{\pi} \int_{-1}^{1} x^{-1} \sin(kx) dx\ .$$

Since $c_k \to 1$ as $k \to \infty$, and since $\Sigma(k \log k)^{-1} = +\infty$, it follows that

$$\sup_n |Af_n(0)| = \infty\ .$$

3. *Proof of Theorem* 2 *and Corollary.* Let G be an arbitrary Lie group (assumed countable at infinity), and let μ be a given positive-

definite distribution on G. If we set $\|\phi\|_\mu = \mu(\phi^* * \phi)^{1/2}$, then $\phi \to \|\phi\|_\mu$ is a continuous seminorm on $\mathscr{D}(G)$. Suppose now that A is a bounded operator on the representation space $\mathscr{H}_\mu$. We may associate with A a bilinear form B_A on $\mathscr{D}(G)$ by the formula

$$B_A(\psi, \phi) = (A\tilde{\phi}, \widetilde{J\psi})_\mu . \tag{3.1}$$

Here $\phi \to \tilde{\phi}$ is the canonical map from $\mathscr{D}(G)$ into $\mathscr{H}_\mu$ as in §1, and $J\phi = \bar{\phi}$ (complex conjugate). By the Schwarz inequality and the boundedness of A we see that

$$|B_A(\psi, \phi)| \leqq \|A\| \, \|\phi\|_\mu \|J\psi\|_\mu . \tag{3.2}$$

Clearly, $\psi \to \|J\psi\|_\mu$ is also a continuous seminorm on $\mathscr{D}(G)$. Although $\|J\psi\|_\mu$ need not be bounded in terms of $\|\psi\|_\mu$, nevertheless, the local order of this seminorm is the same as the local order of $\|\cdot\|_\mu$. (If $K \subset G$ is a compact set and ρ is a continuous seminorm on $\mathscr{D}(G)$, we say that ρ has order $\leqq r$ on K if there is a finite set of differential operators $\{D_j\}$ on G each of order $\leqq r$, such that $\rho(\phi) \leqq \max_j \|D_j\phi\|_\infty$ for all ϕ with $\operatorname{Supp}(\phi) \subseteqq K$.)

The main analytic fact we need is the following version of the "kernel theorem" for continuous bilinear forms:

LEMMA 2. *Suppose B is a bilinear form on $\mathscr{D}(G)$, and ρ_1, ρ_2 are continuous seminorms on $\mathscr{D}(G)$ such that*

$$|B(\phi, \psi)| \leqq \rho_1(\phi)\rho_2(\psi) . \tag{3.3}$$

Then there is a distribution T on $G \times G$ such that

$$B(\phi, \psi) = T(\phi \otimes \psi) .$$

Furthermore, if K_1 and K_2 are compact subsets of G, and if ρ_j has order $\leqq r_j$ on $K_j (j = 1, 2)$, then T has order $\leqq r_1 + r_2 + 2(\dim G + 1)$ on any compact set $M \subset \operatorname{Interior}(K_1 \times K_2)$.

Proof. Since multiplication by a C^∞ function is an operator of order zero, we may use a partition of unity and local coordinates to reduce the problem to a local one in $\boldsymbol{R}^d$, $d = \dim G$, such that $K_j = \{|x| \leqq 2\} \subseteqq \boldsymbol{R}^d$ and $M = \{(x, y); |x| \leqq 1, |y| \leqq 1\} \subseteqq \boldsymbol{R}^d \times \boldsymbol{R}^d$.

Let $\phi_0 \in \mathscr{D}(\boldsymbol{R}^d)$ satisfy $\phi_0 = 1$ on $\{|x| \leqq 1\}$ and $\operatorname{Supp}(\phi_0) \subseteqq K_1$. Set $e_n(x) = \phi_0(x)e^{in\cdot x}$, where $n \in \boldsymbol{N}^d$ and $n\cdot x = n_1x_1 + \cdots + n_dx_d$. Then if D is a differential operator of order r, one has $\|De_n\|_\infty \leqq C(1 + |n|)^r$. Hence, the a priori estimate (3.3) implies that for some constant $C > 0$,

$$|B(e_m, e_n)| \leqq C(1 + |m|)^{r_1}(1 + |n|)^{r_2} \tag{3.4}$$

for all $m, n \in \boldsymbol{N}^d$.

Suppose now that f is a C^∞ function on $\boldsymbol{R}^d \times \boldsymbol{R}^d$ with $\operatorname{Supp}(f) \subseteqq M$. Then the Fourier series of f can be written as

$$f(x, y) = \sum_{m,n} \hat{f}(m, n) e_m(x) e_n(y) ,$$

where $\{\hat{f}(m, n)\}$ are the Fourier coefficients of f. Define

$$T(f) = \sum_{m,n} \hat{f}(m, n) B(e_m, e_n) . \tag{3.5}$$

The series (3.5) is absolutely convergent, and by (3.4) we have the estimate

$$| T(f) | \leqq C_1 \sup_{m,n} \{|\hat{f}(m, n)| (1 + |m|)^{r_1+d+1} (1 + |n|)^{r_2+d+1}\} , \tag{3.6}$$

where $C_1 = C \sum_{m,n} (1 + |m|)^{-d-1} (1 + |n|)^{-d-1} < \infty$. Since the right side of (3.6) is a seminorm of order $r_1 + r_2 + 2d + 2$ on M, this proves the lemma.

Completion of proof of Theorem 2. Suppose now that the operator A in formula (3.1) commutes with the representation L^μ. Then the distribution T on $G \times G$ such that $B_A(\phi, \psi) = T(\phi \otimes \psi)$, which was constructed in Lemma 2, satisfies for all $z \in G$,

$$T(\delta_z f) = T(f) , \qquad f \in \mathscr{D}(G \times G) , \tag{3.7}$$

where $\delta_z f(x, y) = f(z^{-1}x, z^{-1}y)$.

The structure of distributions satisfying (3.7) was determined by Bruhat [3, Prop. 3.3]. Let ι denote the distribution on G determined by left Haar measure, and let $\Phi: G \times G \to G \times G$ be the map $\Phi(x, y) = (x, xy)$. Then (3.7) forces T to have the form

$$T(f) = (\iota \otimes \alpha)(f \circ \Phi) ,$$

where α is a distribution on G. Symbolically,

$$T(f) = \iint f(x, xy) dx d\alpha(y) .$$

In particular, if $\phi, \psi \in \mathscr{D}(G)$, then

$$\begin{aligned} (A\tilde{\phi}, \tilde{\psi})_\mu &= T(J\psi \otimes \phi) \\ &= \iint \overline{\psi(x)} \phi(xy) dx d\alpha(y) \\ &= \alpha(\psi^* * \phi) . \end{aligned}$$

Hence, α serves to represent the intertwining operator A, and is obviously positive-definite if $A \geqq 0$. Since Φ is a diffeomorphism, the order of $\iota \otimes \alpha$ on a compact set $M \subset G \times G$ is the same as the order of T on $\Phi^{-1}(M)$. By Lemma 2 and inequality (3.2), the local order of $\iota \otimes \alpha$ (and, hence, the local order of α) can, therefore, be bounded in terms of the local order of μ and the dimension of G, as claimed.

Proof of Corollary. Using Theorem 2, we are able to rehabilitate the attempted proof of cyclicity in [7]. Given a δ-sequence $\{\psi_n\}$ on G, let $K \subset G$ be a compact set such that $K = K^{-1}$ and $\operatorname{Supp}(\psi_n) \subseteqq K$ for all n. Since $\|\psi\|_\mu$ is a continuous seminorm on $\mathscr{D}(G)$, there are right-invariant differential operators $D_1, \cdots, D_r$ on G such that

$$\|\psi\|_\mu \leqq \max_j \|D_j\psi\|_\infty \tag{3.8}$$

for all ψ supported on the set K^2.

Now set $w_n = \psi_n^* * \psi_n$, and let $\{\lambda_n\}$ be any sequence such that $\lambda_n > 0$ and

$$\sum_n \lambda_n \max_j \|D_j\psi_n\|_\infty^2 < \infty . \tag{3.9}$$

The series $\xi = \sum \lambda_n \tilde{w}_n$ then converges absolutely in $\mathscr{H}_\mu$ (since $\|w_n\|_\mu \leqq \|\psi_n\|_\mu^2$). Let $\mathscr{N}$ be the G-cyclic subspace generated by ξ, and let A be the projection onto $\mathscr{N}^\perp$. Since $A\xi = 0$, we have $\sum \lambda_n (A\tilde{w}_n, \tilde{\phi})_\mu = 0$ for all $\phi \in \mathscr{D}(G)$. But $\widetilde{\phi * \psi} = L_\mu(\phi)\tilde{\psi}$, where $L_\mu(f) = \int f(x) L_\mu(x)dx$ is the integrated form of the representation. Since A commutes with L_μ, this gives $(A\tilde{w}_n, \tilde{\phi})_\mu = (A\tilde{\psi}_n, \widetilde{\psi_n * \phi})_\mu$. Thus taking $\phi = \psi_k$ and letting $k \to \infty$, we see that

$$\lim_{k\to\infty} (A\tilde{w}_n, \tilde{\psi}_k)_\mu = (A\tilde{\psi}_n, \tilde{\psi}_n)_\mu \tag{3.10}$$

(note that $\phi \to \tilde{\phi}$ is continuous from $\mathscr{D}(G)$ to $\mathscr{H}_\mu$). Furthermore, by the Schwartz inequality, the boundedness of A, and the calculation just made, we have the estimate

$$\begin{aligned} |(A\tilde{w}_n, \tilde{\psi}_k)_\mu| &\leqq \|\psi_n\|_\mu \, \|\psi_n * \psi_k\|_\mu \\ &\leqq C \max_j \|D_j\psi_n\|_\infty^2 . \end{aligned}$$

(Here we have used estimate (3.8), the right-invariance of D_j, and the inequality $\|f * g\|_\infty \leqq \|f\|_\infty \|g\|_{L_1}$.) Thus we may apply the dominated convergence theorem to conclude from (3.9) and (3.10) that $\sum \lambda_n (A\tilde{\psi}_n, \tilde{\psi}_n)_\mu = 0$. But $\lambda_n > 0$ and $A \geqq 0$, so in fact $(A\tilde{\psi}_n, \tilde{\psi}_n)_\mu = 0$ for all n. (So far we have simply followed the line of proof of [7], replacing uniform convergence of the series $\sum \lambda_n w_n$ by the stronger condition (3.9), in return for allowing μ which are distributions rather than measures.) Finally let α be the positive-definite distribution on G representing A, which exists by Theorem 2. Then $\alpha(\psi_n^* * \psi_n) = 0$ for all n. By the Schwarz inequality, this implies that $\alpha(\phi * \psi_n) = 0$ for all $\phi \in \mathscr{D}(G)$ and all n. Letting $n \to \infty$, we conclude that $\alpha = 0$.

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