

# Pacific Journal of Mathematics

**ON QUASI-COMPLEMENTS**

WILLIAM BUHMANN JOHNSON

## ON QUASI-COMPLEMENTS

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**Results of H. P. Rosenthal and the author on  $w^*$ -basic sequences are combined with known techniques and applied to quasi-complementation problems in Banach spaces.**

1. Introduction. Recall that (closed, linear) subspaces  $Y, Z$  of the Banach space  $X$  are *quasi-complements* (respectively *complements*) provided  $Y \cap Z = \{0\}$  and  $Y + Z$  is dense in  $X$  (respectively,  $Y + Z = X$ ).

Suppose that  $Y, Z$  are quasi-complements, but not complements, for the separable space  $X$ . We show that there exist closed subspaces  $Y_1$  and  $Y_2$  of  $X$  with  $Y_1 \subset Y \subset Y_2$ ,  $\dim Y/Y_1 = \infty = \dim Y_2/Y$ , such that  $Y_1, Z$  are quasi-complements and  $Y_2, Z$  are quasi-complements. This generalizes a theorem of James [5], who proved the existence of  $Y_1$  for the case of general separable  $X$  and the existence of  $Y_2$  for separable, reflexive  $X$ . Our proof uses James' method (and  $w^*$ -basic sequences), but seems simpler than James' construction. Also, our argument provides information for some nonseparable spaces.

We show also the following.

**THEOREM 2.** *Suppose  $Y$  is a subspace of  $X$  and  $Y^*$  is weak\*-separable. If  $X/Y$  has a separable, infinite dimensional quotient space, then  $Y$  is quasi-complemented in  $X$ .*

Theorem 2 was discovered by J. Lindenstrauss and H. P. Rosenthal [unpublished], both of whom apparently use an idea from [3]. Our argument uses  $w^*$ -basic sequences and Rosenthal's proof of Theorem 2 in the case where  $X/Y$  has a reflexive, infinite dimensional quotient (cf. [12]).

The final result of the paper is that every subspace of a separable conjugate space admits a weak\*-closed quasi-complement which is spanned by a boundedly complete  $w^*$ -basic sequence.

The notation and terminology agree with [6]. In particular, subspaces and quotients are assumed to be infinite dimensional and complete. For  $A \subset X$ ,  $A^\perp$  is the annihilator of  $A$  in  $X^*$ , while for  $B \subset X^*$ ,  $B^\perp$  is the annihilator of  $B$  in  $X$  and  $\tilde{B}$  is the weak\*-closure of  $B$  in  $X^*$ .

II. THE THEOREMS. We recall the definition of  $w^*$ -basic sequence

[6]: A sequence  $(y_n) \subset X^*$  is called  $w^*$ -basic provided that there exists  $(x_n) \subset X$  biorthogonal to  $(y_n)$  and, for each  $y$  in the weak\*-closure  $\widetilde{\{y_n\}}$  of the closed linear span  $\{y_n\}$  of  $(y_n)$ ,  $y = w^*\text{-}\lim_n \sum_{i=1}^n y(x_i)y_i$ .

In [6] it was proved that, when  $X$  is separable, if  $(y_n) \subset X^*$ ,  $y_n \xrightarrow{w^*} 0$ , but  $\liminf \|y_n\| > 0$ , then  $(y_n)$  contains a  $w^*$ -basic subsequence. Let us note that the same result is true when  $X$  admits a weakly compact fundamental set. Indeed, in this case there exists by [1] a norm one projection  $P$  on  $X$  with  $PX$  separable and  $(y_n) \subset P^*X^*$ .  $P^*X^*$  is isometric to  $(PX)^*$  and the relative weak\* topology on  $P^*X^*$  from  $X^*$  agrees with the weak\* topology on  $P^*X^*$  considered as the conjugate of  $PX$ . Therefore, the above mentioned result from [6] applies to show that  $(y_n)$  has a  $w^*$ -basic subsequence.

First we prove the extension of James' theorem:

**THEOREM 1.** *Suppose that  $Y, Z$  are quasi-complements, but not complements, for  $X$ .*

(a) *If  $Y$  has a weakly compact fundamental subset, then there exists a subspace  $Y_1$  of  $Y$  with  $\dim Y/Y_1 = \infty$  and  $Y_1, Z$  are quasi-complements.*

(b) *If  $X/Y$  has a weakly compact fundamental subset (in particular, if  $X$  does), then there exists a subspace  $Y_2$  of  $X$  with  $Y_2 \supset Y$ ,  $\dim Y_2/Y = \infty$ , and  $Y_2, Z$  are quasi-complements.*

*Proof.* Pick positive numbers  $(a_n)$  less than 1 so that  $a_1 + a_1a_2 + a_1a_2a_3 + \dots < \infty$ . Let  $p$  be a bijection of  $N \times N$  onto  $N$  ( $N$  is the set of natural numbers) so that for each  $n$  and  $j$ ,  $p(n, j) \geq j$ .

To prove (a), we use the fact that  $Y + Z$  is not closed to select unit vectors  $(y_n)$  in  $Y$  with  $d(y_n, Z) \equiv \inf \{\|y_n + z\| : z \in Z\} \rightarrow 0$ . Since  $Y \cap Z = \{0\}$ , 0 is the only possible weak cluster point of  $(y_n)$ , and hence either  $y_n \xrightarrow{w} 0$  or the weak closure of  $(y_n)$  is not weakly compact. Thus, by either [2] or [11],  $(y_n)$  has a basic subsequence, which we also denote by  $(y_n)$ .

Let  $(y_n^*)$  be a bounded sequence of functionals in  $Y^*$  biorthogonal to  $(y_n)$ . Since  $Y$  admits a weakly compact fundamental set, the unit ball of  $Y^*$  is weak\* sequentially compact (cf. [1]), so we may assume, by passing to a subsequence, that  $y_n^* \xrightarrow{w^*} y^*$ .  $(y_n^* - y^*)$  converges  $w^*$  to 0 and is bounded away from zero, so it has a  $w^*$ -basic subsequence. Thus by passing to a subsequence of  $(y_n, y_n^* - y^*)$ , we have that there exists a biorthogonal sequence  $(x_n, x_n^*)$  in  $Y$  with  $\|x_n\| = 1$ ,  $(\|x_n^*\|)$  bounded,  $d(x_n, Z) \leq n^{-1}a_1a_2a_3 \dots a_n$ ,  $(x_n)$  is basic, and  $(x_n^*)$  is  $w^*$ -basic.

Let  $Y_1 = [(x_i^*)^\tau \cup (a_i x_{p(n,i)} - x_{p(n,i+1)})_{i=n=1}^\infty]$ . (The annihilator of  $(x_i^*)$  is of course taken in  $Y$ .) We claim that  $Y_1 \cap [x_{p(n,1)}] = \{0\}$ . To see this, first note that  $w_n^* = x_{p(n,1)}^* + a_1 x_{p(n,2)}^* + a_1 a_2 x_{p(n,3)}^* + \dots$  is absolutely convergent,  $w_n^*(x_{p(n,1)}) = 1$ , while  $w_n^*(x_{p(m,1)}) = 0$  when  $n \neq m$ . By construction,  $Y_1 \subset (w_n^*)^\tau$ , and  $(w_n^*)^\tau \cap [(x_{p(n,1)})] = \{0\}$  because  $(x_{p(n,1)})$  is basic under some ordering and  $(x_{p(n,1)}, w_n^*)$  is biorthogonal. Hence,  $Y_1 \cap [x_{p(n,1)}] = \{0\}$ , whence  $\dim Y/Y_1 = \infty$ .

We complete the proof by showing that  $Y_1 + Z$  is dense in  $X$ . Now  $(x_n^*)^\tau + [x_n]$  is dense in  $Y$  because  $(x_n^*)$  is  $w^*$ -basic, so we need show only that  $(x_{p(n,1)}) \subset \overline{Y_1 + Z}$ . But

$$\begin{aligned} x_{p(n,1)} &- a_1^{-1}(a_1 x_{p(n,1)} - x_{p(n,2)}) - (a_1 a_2)^{-1}(a_2 x_{p(n,2)} - x_{p(n,3)}) \\ &- \dots - (a_1 a_2 \dots a_j)^{-1}(a_j x_{p(n,j)} - x_{p(n,j+1)}) \\ &= (a_1 a_2 \dots a_j)^{-1} x_{p(n,j+1)}. \end{aligned}$$

Since  $d(x_{p(n,j+1)}, Z) \leq p(n, j + 1)^{-1} a_1 a_2 \dots a_{p(n,j+1)} \leq (j + 1)^{-1} a_1 a_2 \dots a_j$ , it follows that  $d(x_{p(n,1)}, Y_1 + Z) \leq (j + 1)^{-1}$ . Since  $j$  is arbitrary, this completes the proof of (a).

The proof of (b) is very similar to the above: Since  $Y, Z$  are not complements,  $Y^\perp + Z^\perp$  is not closed in  $X^*$ . Thus there exists a sequence  $(y_n^*)$  of unit vectors in  $Y^\perp$  with  $d(y_n^*, Z^\perp) \rightarrow 0$ . Of necessity,  $y_n^* \xrightarrow{w^*} 0$ . Now  $Y^\perp = (X/Y)^*$  in the canonical way, so  $(y_n^*)$  has a  $w^*$ -basic subsequence. Hence for an appropriate subsequence  $(x_n^*)$  of  $(y_n^*)$ , we have that there exists a biorthogonal sequence  $(x_n, x_n^*)$  in  $X$  with  $(\|x_n\|)$  bounded,  $\|x_n^*\| = 1$ ,  $(x_n^*) \subset Y^\perp$ ,  $(x_n^*)$   $w^*$ -basic, and  $d(x_n^*, Z^\perp) \leq n^{-1} a_1 a_2 \dots a_n$ .

We define  $Y_2^\perp$  to be the weak\*-closure of  $[Y^\perp \cap (x_n)^\perp \cup (a_i x_{p(n,i)}^* - x_{p(n,i+1)}^*)_{n,i=1}^\infty]$ . Since  $Y_2^\perp \subset Y^\perp$ , we have  $Y_2 \supset Y$ . To show that  $\dim Y_2/Y = \infty$ , it clearly suffices to prove that  $Y_2^\perp \cap \widehat{[x_{p(n,1)}^*]} = \{0\}$ . But note that  $y_n \equiv x_{p(n,1)} + a_1 a_2 x_{p(n,2)} + a_1 a_2 a_3 x_{p(n,3)} + \dots$  is absolutely convergent,  $x_{p(n,1)}^*(y_n) = 1$ , while  $x_{p(m,1)}^*(y_n) = 0$  when  $m \neq n$ . By construction,  $(y_n)^\perp \supset (a_i x_{p(n,i)}^* - x_{p(n,i+1)}^*)_{n,i=1}^\infty$  and  $(y_n)^\perp \supset (x_n)^\perp$ , hence  $(y_n)^\perp \supset Y_2^\perp$ . But  $(y_n)^\perp \cap \widehat{[x_{p(n,1)}^*]} = \{0\}$  because  $(x_{p(n,1)}^*)$  is  $w^*$ -basic in some ordering and  $(y_n, x_{p(n,1)}^*)$  is biorthogonal.

Since  $Y_2^\perp \cap Z^\perp \subset Y^\perp \cap Z^\perp = \{0\}$ , we have that  $Y_2 + Z$  is dense in  $X$ . To show that  $Y_2 \cap Z = \{0\}$ , we prove the equivalent fact that  $Y_2^\perp + Z^\perp$  is  $w^*$  dense in  $X^*$ . But  $Y^\perp \cap (x_n)^\perp + [x_n^*]$  is  $w^*$  dense in  $Y^\perp$  because  $(x_n^*)$  is  $w^*$ -basic, so we need only show that each  $x_{p(n,1)}^*$  is in the closure of  $Y_2^\perp + Z$ . To see that this last statement is true, write

$$\begin{aligned} x_{p(n,1)}^* &- a_1^{-1}[a_1 x_{p(n,1)}^* - x_{p(n,2)}^*] - (a_1 a_2)^{-1}[a_2 x_{p(n,2)}^* - x_{p(n,3)}^*] - \dots \\ &- (a_1 a_2 \dots a_j)^{-1}[a_j x_{p(n,j)}^* - x_{p(n,j+1)}^*] \\ &= (a_1 a_2 \dots a_j)^{-1} x_{p(n,j+1)}^* . \end{aligned}$$

Since  $d(x_{p(n,j+1)}^*, Z) \leq p(n, j+1)^{-1} a_1 a_2 \dots a_{p(n,j+1)} \leq (j+1)^{-1} a_1 \dots a_j$ , we have  $d(x_{p(n,1)}^*, Y_2^\perp + Z) \leq (j+1)^{-1}$  for arbitrary  $j$ .

Next we prove the result of Lindenstrauss and Rosenthal.

*Proof of Theorem 2.* Since  $X/Y$  has a separable quotient, there exists a biorthogonal sequence  $(x_n, x_n^*)$  in  $X$  with  $(x_n^*) \subset Y^\perp$ ,  $(x_n^*)$   $w^*$ -basic, and normalized so that  $\|x_n\| = 1$ . Since  $Y^*$  is  $w^*$ -separable, a biorthogonalization argument (cf., e.g., [8] or [7]) shows that there exists a biorthogonal sequence  $(y_n, y_n^*)$  for  $Y$  with  $(y_n^*) \subset X^*$ ,  $Y \cap (y_n^*)^\tau = \{0\}$ , and normalized so that  $\|y_n^*\| = 1$ .

Define  $T: X \rightarrow X$  by  $Tx = \sum_{n=1}^\infty 2^{-n-1} y_n^*(x) x_n$ . Then  $\|T\| \leq 1/2$ , so  $I + T$  is an isomorphism. Hence  $(I + T)^*$  is a weak\*-isomorphism on  $X^*$ , whence  $(x_n^* + T^* x_n^*)$  is a  $w^*$ -basic sequence  $w^*$ -equivalent to  $(x_n^*)$ .

Computing  $T^* x_n^*$ , we have  $T^* x_n^*(x) = x_n^* Tx = x_n^* \sum_{m=1}^\infty 2^{-m-1} y_m^*(x) x_m = 2^{-n-1} y_n^*(x)$ ; i.e.,  $T^* x_n^* = 2^{-n-1} y_n^*$ .

We claim that  $(x_n^* + 2^{-n-1} y_n^*)^\tau$  is a quasi-complement to  $Y$ . First we show that  $Y^\perp \cap [x_n^* + 2^{-n-1} y_n^*] = \{0\}$  (so that  $Y + (x_n^* + 2^{-n-1} y_n^*)^\tau$  is dense). But if  $x^* \in [x_n^* + 2^{-n-1} y_n^*]$ , then, since  $(x_n^* + 2^{-n-1} y_n^*)$  is  $w^*$ -equivalent to  $(x_n^*)$ , we can write  $x^* = w^*\text{-}\lim_{n \rightarrow \infty} \sum_{i=1}^n \alpha_i x_i^* + \sum_{i=1}^\infty 2^{-i-1} \alpha_i y_i^*$  for some sequence  $(\alpha_i)$  of scalars. Thus for each  $n$ ,  $x^*(y_n) = 2^{-n-1} \alpha_n$ , hence, since  $x^* \in Y^\perp$ ,  $\alpha_n = 0$ .

We complete the proof by showing that  $Y \cap (x_n^* + 2^{-n-1} y_n^*)^\tau = \{0\}$ . For suppose  $y$  is in this intersection. Since  $y \in Y$ ,  $x_n^*(y) = 0$  for each  $n$ . Hence  $y_n^*(y) = 0$  for each  $n$ , whence  $y \in (y_n^*)^\tau \cap Y = \{0\}$ .

**THEOREM 3.** *Suppose  $X^*$  is separable and  $Y$  is a subspace of  $X^*$  with  $\dim X^*/Y = \infty$ . Then there exists a weak\*-closed subspace  $Z$  of  $X^*$  with  $Y, Z$  quasi-complements and  $Z = [z_n]$  for some boundedly complete,  $w^*$ -basic sequence  $(z_n)$ .*

*Proof.* Mackey [8] showed that  $Y$  has a quasi-complement, say,  $W$ . Let  $(w_n, w_n^*)$  be a biorthogonal sequence in  $W$  with  $\|w_n\| = 1$  and  $[w_n] = W$  (cf. [9]). By Theorem III. 2 of [6], there exists a

biorthogonal sequence  $(x_n, x_n^*)$  in  $X$  with  $(x_n^*) \subset Y$ ,  $(x_n^*)$  boundedly complete and  $w^*$ -basic, normalized so that  $\|x_n\| = 1$ .

Define  $T: X \rightarrow X$  by  $Tx = \sum_{n=1}^{\infty} 2^{-n-1} w_n(x_n)x_n$ . Then  $\|T\| \leq 1/2$ , so  $I + T$  is an isomorphism and hence  $(I + T)^*$  is a weak\*-isomorphism. One checks that  $T^*x_n^* = 2^{-n-1}w_n$ , so that  $(x_n^* + 2^{-n-1}w_n)$  is a  $w^*$ -basic sequence  $w^*$ -equivalent to  $(x_n^*)$ . Letting  $Z = [x_n^* + 2^{-n-1}w_n]$ , we have by Proposition 1 of [6] that  $Z$  is weak\*-closed.

Certainly  $Z + Y \supset (w_n)$ , so  $Z + Y \supset Y + W$  and thus is dense. Suppose that  $z \in Z \cap Y$ . Then  $z = \sum_{n=1}^{\infty} \alpha_n(x_n^* + 2^{-n-1}w_n)$  for some scalars  $(\alpha_n)$  because  $(x_n^* + 2^{-n-1}w_n)$  is basic. Hence also  $\sum_{n=1}^{\infty} \alpha_n x_n^*$  converges, whence  $z - \sum_{n=1}^{\infty} \alpha_n w_n^* = \sum_{n=1}^{\infty} \alpha_n 2^{-n-1}w_n$  is again in  $Y$ . Certainly  $\sum_{n=1}^{\infty} \alpha_n 2^{-n-1}w_n$  is also in  $W$  so that  $\sum_{n=1}^{\infty} \alpha_n 2^{-n-1}w_n = 0$ . Thus  $\alpha_n 2^{-n-1} = w_n^*(\sum_{m=1}^{\infty} \alpha_m 2^{-m-1}w_m) = 0$ , so that  $z = 0$ .

REMARK. Separability of  $X^*$  in Theorem 3 is essential to get that  $Z$  is weak\*-closed. Indeed, regard  $m = l_1^*$ . Rosenthal [12] showed that  $c_0$  is quasi-complemented in  $m$ . However, if  $Z$  is a quasi-complement for  $c_0$  in  $m$ , then  $Z$  cannot be weak\*-closed. For if  $Z$  were  $w^*$ -closed, then  $m/Z$  would be isomorphic to  $(Z^\tau)^*$ . But  $m/Z$  is separable, hence reflexive (cf. [4]). Thus  $Z^\tau$  would be a reflexive subspace of  $l_1$ , a contradiction (cf., e.g., [10]).

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