ON QUASI-COMPLEMENTS

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Results of H. P. Rosenthal and the author on $w^*$-basic sequences are combined with known techniques and applied to quasi-complementation problems in Banach spaces.

1. Introduction. Recall that (closed, linear) subspaces $Y, Z$ of the Banach space $X$ are quasi-complements (respectively complements) provided $Y \cap Z = \{0\}$ and $Y + Z$ is dense in $X$ (respectively, $Y + Z = X$).

Suppose that $Y, Z$ are quasi-complements, but not complements, for the separable space $X$. We show that there exist closed subspaces $Y_1$ and $Y_2$ of $X$ with $Y_1 \subset Y \subset Y_2$, $\dim Y/Y_1 = \infty = \dim Y_2/Y$, such that $Y_1, Z$ are quasi-complements and $Y_2, Z$ are quasi-complements. This generalizes a theorem of James [5], who proved the existence of $Y_1$ for the case of general separable $X$ and the existence of $Y_2$ for separable, reflexive $X$. Our proof uses James' method (and $w^*$-basic sequences), but seems simpler than James' construction. Also, our argument provides information for some nonseparable spaces.

We show also the following.

**Theorem 2.** Suppose $Y$ is a subspace of $X$ and $Y^*$ is weak*-separable. If $X/Y$ has a separable, infinite dimensional quotient space, then $Y$ is quasi-complemented in $X$.

Theorem 2 was discovered by J. Lindenstrauss and H. P. Rosenthal [unpublished], both of whom apparently use an idea from [3]. Our argument uses $w^*$-basic sequences and Rosenthal's proof of Theorem 2 in the case where $X/Y$ has a reflexive, infinite dimensional quotient (cf. [12]).

The final result of the paper is that every subspace of a separable conjugate space admits a weak*-closed quasi-complement which is spanned by a boundedly complete $w^*$-basic sequence.

The notation and terminology agree with [6]. In particular, subspaces and quotients are assumed to be infinite dimensional and complete. For $A \subset X, A^\perp$ is the annihilator of $A$ in $X^*$, while for $B \subset X^*, B^*$ is the annihilator of $B$ in $X$ and $\bar{B}$ is the weak*-closure of $B$ in $X^*$.

II. THE THEOREMS. We recall the definition of $w^*$-basic sequence
A sequence \((y_n) \subseteq X^*\) is called \(w^*\)-basic provided that there exists \((x_n) \subseteq X\) biorthogonal to \((y_n)\) and, for each \(y\) in the weak*-closure \([\overline{y_n}]\) of the closed linear span \([y_n]\) of \((y_n)\), \(y = w^*\)-lim \(\sum_{i=1}^{n} y(x_i)y_i\).

In [6] it was proved that, when \(X\) is separable, if \((y_n) \subseteq X^*, y_n \xrightarrow{w^*} 0\), but \(\liminf ||y_n|| > 0\), then \((y_n)\) contains a \(w^*\)-basic subsequence. Let us note that the same result is true when \(X\) admits a weakly compact fundamental set. Indeed, in this case there exists by [1] a norm one projection \(P\) on \(X\) with \(PX\) separable and \((y_n) \subseteq P^*X^*\). \(P^*X^*\) is isometric to \((PX)^*\) and the relative weak* topology on \(P^*X^*\) from \(X^*\) agrees with the weak* topology on \(P^*X^*\) considered as the conjugate of \(PX\). Therefore, the above mentioned result from [6] applies to show that \((y_n)\) has a \(w^*\)-basic subsequence.

First we prove the extension of James' theorem:

**Theorem 1.** Suppose that \(Y, Z\) are quasi-complements, but not complements, for \(X\).

(a) If \(Y\) has a weakly compact fundamental subset, then there exists a subspace \(Y_1\) of \(Y\) with \(\dim Y/Y_1 = \infty\) and \(Y, Z\) are quasi-complements.

(b) If \(X/Y\) has a weakly compact fundamental subset (in particular, if \(X\) does), then there exists a subspace \(Y_2\) of \(X\) with \(\dim Y_2/Y = \infty\), and \(Y, Z\) are quasi-complements.

**Proof.** Pick positive numbers \((a_n)\) less than 1 so that \(a_1 + a_2a_3 + \cdots < \infty\). Let \(p\) be a bijection of \(N \times N\) onto \(N\) (\(N\) is the set of natural numbers) so that for each \(n\) and \(j\), \(p(n, j) \geq j\).

To prove (a), we use the fact that \(Y + Z\) is not closed to select unit vectors \((y_n)\) in \(Y\) with \(d(y_n, Z) = \inf \{ ||y_n + z|| : z \in Z \} \rightarrow 0\). Since \(Y \cap Z = \{0\}\), 0 is the only possible weak cluster point of \((y_n)\), and hence either \(y_n \xrightarrow{w^*} 0\) or the weak closure of \((y_n)\) is not weakly compact. Thus, by either [2] or [11], \((y_n)\) has a basic subsequence, which we also denote by \((y_n)\).

Let \((y_n^*)\) be a bounded sequence of functionals in \(Y^*\) biorthogonal to \((y_n)\). Since \(Y\) admits a weakly compact fundamental set, the unit ball of \(Y^*\) is weak* sequentially compact (cf. [1]), so we may assume, by passing to a subsequence, that \(y_n^* \xrightarrow{w^*} y^*\). \((y_n^* - y^*)\) converges \(w^*\) to 0 and is bounded away from zero, so it has a \(w^*\)-basic subsequence. Thus by passing to a subsequence of \((y_n, y_n^* - y^*)\), we have that there exists a biorthogonal sequence \((x_n, x_n^*)\) in \(Y\) with \(||x_n|| = 1\), \((||x_n^*||)\) bounded, \(d(x_n, Z) \leq \sqrt{n^{-1}a_1a_2a_3 \cdots a_n}\), \((x_n)\) is basic, and \((x_n^*)\) is \(w^*\)-basic.
Let \( Y_i = \{(x_1^i)\} \cup (a_i x_{p(n, i)} - x_{p(n, i+1)})\}_{n=1}^\infty. \) (The annihilator of \( (x_1^i) \) is of course taken in \( Y_i \).) We claim that \( Y_i \cap [x_{p(n, i)}] = \{0\} \). To see this, first note that \( w_*^n = x_{p(n, i)}^* + a_i x_{p(n, 2)}^* + a_i a_2 x_{p(n, 3)}^* + \cdots \) is absolutely convergent, \( w_*^n(x_{p(n, i)}) = 1 \), while \( w_*^n(x_{p(n, i+1)}) = 0 \) when \( n \neq m \). By construction, \( Y_i \subset (w_*^n)^\perp \), and \( (w_*^n) \cap [(x_{p(n, i)}^i)] = \{0\} \) because \( (x_{p(n, i)}^i) \) is basic under some ordering and \( (x_{p(n, i)}, w_*^n) \) is biorthogonal. Hence, \( Y_i \cap [x_{p(n, i)}] = \{0\} \), whence \( \dim Y_i/Y_i = \infty \).

We complete the proof by showing that \( Y_1 + Z \) is dense in \( X \). Now \( (x_1^i)^* + [x_n] \) is dense in \( Y_i \) because \( (x_1^i)^* \) is \( w^* \)-basic, so we need show only that \( (x_{p(n, i)}^i) \subset Y_1 + Z \). But
\[
[(a_1 x_{p(n, i)} - x_{p(n, 2)}) - (a_1 a_2)(a_2 x_{p(n, 3)} - x_{p(n, 4)}) - \cdots - (a_1 a_2 \cdots a_j)(a_j x_{p(n, j)} - x_{p(n, j+1)})]
\]
\[
= (a_1 a_2 \cdots a_j)^{-1} x_{p(n,j+1)}.
\]
Since \( d(x_{p(n,j+1)}, Z) \leq p(n,j+1)^{-1} a_1 a_2 \cdots a_{p(n,j+1)} \leq (j+1)^{-1} a_1 a_2 \cdots a_j \), it follows that \( d(x_{p(n,j)}, Y_1 + Z) \leq (j+1)^{-1} \). Since \( j \) is arbitrary, this completes the proof of (a).

The proof of (b) is very similar to the above: Since \( Y, Z \) are not complements, \( Y^\perp + Z^\perp \) is not closed in \( X^* \). Thus there exists a sequence \( (y_1^*) \) of unit vectors in \( Y^\perp \) with \( d(y_1^*, Z^\perp) \to 0 \). Of necessity, \( y_1^* \xrightarrow{w^*} 0 \). Now \( Y^\perp = (X/Y)^* \) in the canonical way, so \( (y_1^*) \) has a \( w^* \)-basic subsequence. Hence for an appropriate subsequence \( (x_1^*) \) of \( (y_1^*) \), we have that there exists a biorthogonal sequence \( (x_n, x_1^*) \) in \( X \) with \( ||x_1^*|| \) bounded, \( ||x_n^*|| = 1 \), \( (x_n^*) \subset Y^\perp \), \( (x_n^*) \) \( w^* \)-basic, and \( d(x_n^*, Z^\perp) \leq n^{-1} a_1 a_2 \cdots a_n \).

We define \( Y_2^\perp \) to be the weak*-closure of \( [Y^\perp \cap (x_n^*) \cup (a_i x_{p(m, i)} - x_{p(m, i+1)})]_{i=1}^\infty \). Since \( Y_2^\perp \subset Y^\perp \), we have \( Y_2 \supset Y \). To show that \( \dim Y_2/Y = \infty \), it clearly suffices to prove that \( Y_2^\perp \cap [x_{p(n, i)}^i] = \{0\} \). But note that \( y_n \equiv x_{p(n, i)} + a_i a_2 x_{p(n, 2)} + a_i a_2 a_3 x_{p(n, 3)} + \cdots \) is absolutely convergent, \( x_{p(n, i)}^i(y_n) = 1 \), while \( x_{p(n, i)}^i(y_n) = 0 \) when \( m \neq n \). By construction, \( (x_n^i) \supset (a_i x_{p(m, i)} - x_{p(m, i+1)})_{i=1}^\infty \) and \( (y_n) \supset (x_n^i) \), hence \( (y_n) \supset Y_2^\perp \). But \( (y_n) \cap [x_{p(n, i)}^i] = \{0\} \) because \( (x_{p(n, i)}^i) \) is \( w^* \)-basic in some ordering and \( (y_n, x_{p(n, i)}^i) \) is biorthogonal.

Since \( Y_2^\perp \cap Z^\perp \subset Y^\perp \cap Z^\perp = \{0\} \), we have that \( Y_2 + Z \) is dense in \( X \). To show that \( Y_2 \cap Z = \{0\} \), we prove the equivalent fact that \( Y_2^\perp + Z^\perp \) is \( w^* \) dense in \( X^* \). But \( Y_2^\perp \cap (x_n^i) + [x_n^i] \) is \( w^* \) dense in \( Y^\perp \) because \( (x_n^i) \) is \( w^* \)-basic, so we need only show that each \( x_{p(n, i)}^i \) is in the closure of \( Y_2^\perp + Z \). To see that this last statement is true, write
\[ x_{p(n,1)}^* - a_1^{-1}[a_2x_{p(n,1)}^* - x_{p(n,2)}^*] - (a, a_2)^{-1}[a_3x_{p(n,2)}^* - x_{p(n,3)}^*] - \cdots \]
\[ = (a, a_2 \cdots a_j)^{-1}x_{p(n,j+1)}^*. \]
Since \( d(x_{p(n,j+1)}^*, Z) \leq p(n, j + 1)^{-1}a_1 \cdots a_{p(n,j+1)} \leq (j + 1)^{-1}a_1 \cdots a_j \), we have \( d(x_{p(n,1)}^*, Y_{a1} + Z) \leq (j + 1)^{-1} \) for arbitrary \( j \).

Next we prove the result of Lindenstrauss and Rosenthal.

**Proof of Theorem 2.** Since \( X/Y \) has a separable quotient, there exists a biorthogonal sequence \((x_n, x_n^*)\) in \( X \) with \((x_n^*) \subset Y^\perp\), \((x_n^*)\) \( w^*\)-basic, and normalized so that \( \|x_n\| = 1 \). Since \( Y^* \) is \( w^*\)-separable, a biorthogonalization argument (cf., e.g., [8] or [7]) shows that there exists a biorthogonal sequence \((y_n, y_n^*)\) for \( Y \) with \((y_n^*) \subset X^*\), \( Y \cap (y_n^*)^r = \{0\} \), and normalized so that \( \|y_n\| = 1 \).

Define \( T: X \rightarrow X \) by \( Tx = \sum_{n=1}^\infty 2^{-n-1}x_n^*(x)x_n \). Then \( \|T\| \leq 1/2 \), so \( I + T \) is an isomorphism. Hence \((I + T)^*\) is a \( w^*\)-isomorphism on \( X^* \), whence \((x_n^* + T^*x_n^*)\) is a \( w^*\)-basic sequence \( w^*\)-equivalent to \((x_n^*)\).

Computing \( T^*x_n^* \), we have \( T^*x_n^*(x) = x_n^*Tx = x_n^* \sum_{m=1}^\infty 2^{-m-1}y_m^*(x)x_m = 2^{-n-1}y_n^*(x) \); i.e., \( T^*x_n^* = 2^{-n-1}y_n^* \).

We claim that \((x_n^* + 2^{-n-1}y_n^*)^r\) is a quasi-complement to \( Y \). First we show that \( Y^\perp \cap [x_n^* + 2^{-n-1}y_n^*] = \{0\} \) (so that \( Y + (x_n^* + 2^{-n-1}y_n^*)^r \) is dense). But if \( x^* \in [x_n^* + 2^{-n-1}y_n^*] \), then, since \((x_n^* + 2^{-n-1}y_n^*)\) is \( w^*\)-equivalent to \((x_n^*)\), we can write \( x^* = w^*\lim_{n \rightarrow \infty} \sum_{i=1}^n \alpha_i x_i^* + \sum_{i=1}^\infty 2^{-i-1}\alpha_i y_i^* \) for some sequence \((\alpha_i)\) of scalars. Thus for each \( n \), \( x^*(y_n) = 2^{-n-1}\alpha_n \), hence, since \( x^* \in Y^\perp \), \( \alpha_n = 0 \).

We complete the proof by showing that \( Y \cap (x_n^* + 2^{-n-1}y_n^*)^r = \{0\} \). For suppose \( y \) is in this intersection. Since \( y \in Y \), \( x_n^*(y) = 0 \) for each \( n \). Hence \( y_n^*(y) = 0 \) for each \( n \), whence \( y \in (y_n^*)^r \cap Y = \{0\} \).

**Theorem 3.** Suppose \( X^* \) is separable and \( Y \) is a subspace of \( X^* \) with \( \dim X^*/Y = \infty \). Then there exists a weak*-closed subspace \( Z \) of \( X^* \) with \( Y, Z \) quasi-complements and \( Z = [z_n] \) for some boundedly complete, \( w^*\)-basic sequence \((z_n)\).

**Proof.** Mackey [8] showed that \( Y \) has a quasi-complement, say, \( W \). Let \((w_n, w_n^*)\) be a biorthogonal sequence in \( W \) with \( \|w_n\| = 1 \) and \([w_n] = W \) (cf. [9]). By Theorem III. 2 of [6], there exists a
biorthogonal sequence \((x_n, x_n^*)\) in \(X\) with \((x_n^*) \subset Y\), \((x_n^*)\) boundedly complete and \(w^*\)-basic, normalized so that \(\|x_n\| = 1\).

Define \(T: X \to X\) by \(Tx = \sum_{n=1}^{\infty} 2^{-n-1} w_n(x)x_n\). Then \(\|T\| \leq 1/2\), so \(I + T\) is an isomorphism and hence \((I + T)^*\) is a weak*-isomorphism. One checks that \(T^*x_n^* = 2^{-n-1}w_n\), so that \((x_n^* + 2^{-n-1}w_n)\) is a \(w^*\)-basic sequence \(w^*\)-equivalent to \((x_n^*)\). Letting \(Z = [x_n^* + 2^{-n-1}w_n]\), we have by Proposition 1 of [6] that \(Z\) is weak*-closed.

Certainly \(Z + Y \supset (w_n)\), so \(Z + Y \supset Y + W\) and thus is dense. Suppose that \(z \in Z \cap Y\). Then \(z = \sum_{n=1}^{\infty} \alpha_n (x_n^* + 2^{-n-1}w_n)\) for some scalars \((\alpha_n)\) because \((x_n^* + 2^{-n-1}w_n)\) is basic. Hence also \(\sum_{n=1}^{\infty} \alpha_n x_n^*\) converges, whence \(z - \sum_{n=1}^{\infty} \alpha_n x_n^* = \sum_{n=1}^{\infty} \alpha_n 2^{-n-1}w_n\) is again in \(Y\). Certainly \(\sum_{n=1}^{\infty} \alpha_n 2^{-n-1}w_n\) is also in \(W\) so that \(\sum_{n=1}^{\infty} \alpha_n 2^{-n-1}w_n = 0\). Thus \(\alpha_n 2^{-n-1} = w_n^* (\sum_{m=1}^{\infty} \alpha_m 2^{-m-1}w_m) = 0\), so that \(z = 0\).

REMARK. Separability of \(X^*\) in Theorem 3 is essential to get that \(Z\) is weak*-closed. Indeed, regard \(m = l_P^*\). Rosenthal [12] showed that \(c_0\) is quasi-complemented in \(m\). However, if \(Z\) is a quasi-complement for \(c_0\) in \(m\), then \(Z\) cannot be weak*-closed. For if \(Z\) were \(w^*\)-closed, then \(m/Z\) would be isomorphic to \((Z')^*\). But \(m/Z\) is separable, hence reflexive (cf. [4]). Thus \(Z^*\) would be a reflexive subspace of \(l^*_p\), a contradiction (cf., e.g., [10]).

REFERENCES

5. R. C. James, Quasi-complements, J. Approximation Theory, 6 (1972), 147-160.
Received May 16, 1972 and in revised form April 10, 1973. The author was supported in part by NSF GP 28719. Most of the research for this paper was done at the University of Houston.

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