PRODUCT INTEGRALS FOR AN ORDINARY DIFFERENTIAL EQUATION IN A BANACH SPACE

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Let $Y$ be a Banach space with norm $|||$, and let $R^+$ be the interval $[0, \infty)$. Let $A$ be a function on $R^+$ having the properties that if $t$ is in $R^+$ then $A(t)$ is a function from $Y$ to $Y$ and that the function from $R^+ \times Y$ to $Y$ described by $(t, x) \mapsto A(t)[x]$ is continuous. Suppose there is a continuous real-valued function $\alpha$ on $R^+$ such that if $t$ is in $R^+$ then $A(t) - \alpha(t)I$ is dissipative. Now it is known that if $z$ is in $Y$, the differential equation $u'(t) = A(t)[u(t)]; \ u(0) = z$ has exactly one solution on $R^+$. It is shown in this paper that if $t$ is in $R^+$ then $u(t) = \exp \left[ \int_0^t A(s) \right][z] = \exp \left[ \int_0^t (ds) A(s) \right][z]$, where the exponentials are defined by the solutions of the associated family of autonomous equations.

The dissipativity condition on $A$ is simply that if $(t, x, y)$ is in $R^+ \times Y \times Y$ and $c$ is a positive number then

\begin{equation}
[I - cA(t)][x] - [I - cA(t)][y] \geq [1 - c\alpha(t)]|x - y| .
\end{equation}

The author and R. H. Martin, Jr. [5] have shown that if (1) holds, and $z$ is in $Y$, then there is exactly one continuously differentiable function $u$ from $R^+$ to $Y$ such that

\begin{equation}
u(0) = z
\end{equation}

and

\begin{equation}
u'(t) = A(t)[u(t)]
\end{equation}

whenever $t$ is in $(0, \infty)$. In the present article we shall show that $u$ can be expressed as a product integral in each of two forms:

\begin{equation}
u(t) = \prod_0^t \exp \left[ (ds) A(s) \right][z]
\end{equation}

and

\begin{equation}
u(t) = \prod_0^t \left[ I - (ds) A(s) \right]^{-1}[z] .
\end{equation}

Our work is related to results of J. V. Herod [2, §6] and G. F. Webb [7], [8]. Herod showed that representation (5) is valid if the mapping $(t, x) \mapsto A(t)[x]$ is bounded on bounded subsets of $R^+ \times Y$. Webb obtained in [7] a representation similar to (4) under a set of hypotheses different from, and independent of, those used here. In
[8], Webb showed that (5) is valid if \( A \) is independent of \( t \). (Actually
Webb in [8] restricted his attention to the case \( \alpha = 0 \), but his proofs
adapt easily to the general time-independent case.)

II. Product integrals. We shall assume throughout that \( A \) and \( \alpha \) are as in our introduction, and that (1) is true whenever \((t, x, y)\)
is in \( \mathbb{R}^+ \times Y \times Y \) and \( c \) is a positive number. Now it follows from
either of [5] and [6] that if \((t, x)\) is in \( \mathbb{R}^+ \times Y \) then there is exactly
one solution \( v \) of the problem

\[
(6) \quad v'(s) = A(t)[v(s)]; \quad v(0) = x .
\]

Furthermore, this problem generates an operator semigroup, which
we shall denote \( \{\exp[sA(t)]; \, s \text{ is in } \mathbb{R}^+\} \), i.e., if \( s \) is in \( \mathbb{R}^+ \) then
\( \exp[sA(t)] \) is a function from \( Y \) to \( Y \) such that if \( x \) is in \( Y \) then
\( \exp[sA(t)][x] = v(s) \), where \( v \) solves (6).

It is clear from (1) that there is no loss in assuming \( \alpha \) to be
\( \mathbb{R}^+ \)-valued, and we shall. It follows from [6] that if \((c, t)\) is in
\( \mathbb{R}^+ \times \mathbb{R}^+ \) and \( c\alpha(t) < 1 \) then \( I - cA(t) \) is a bijection on \( Y \), and

\[
|\{I - cA(t)\}^{-1}[x] - \{I - cA(t)\}^{-1}[y]| \leq \{1 - c\alpha(t)\}^{-1}|x - y|
\]

whenever \((x, y)\) is in \( Y \times Y \). If \( \{B_1, \cdots, B_s\} \) is a set of functions
from \( Y \) to \( Y \), and \( x \) is in \( Y \), then \( \prod_{j=1}^{k} B_j[x] = x \) and \( \prod_{j=1}^{k} B_j[x] = B_k[\prod_{j=1}^{k} B_j[x]] \)
even when \( k \) is an integer in \([1, n]\). If \((t, x, y)\) is in
\( \mathbb{R}^+ \times Y \times Y \) then the statement

\[
y = \prod_{0}^{t} \{I - (ds)A(s)\}^{-1}[x]\]

means that if \( \varepsilon \) is a positive number then there is a chain \( \{r_j\}_{j=0}^{n} \) from \( 0 \)
to \( t \) such that if \( \{s_{j}\}_{j=0}^{n} \) is a refinement of \( \{r_{j}\}_{j=0}^{n} \), and \( \{\tilde{s}_{k}\}_{k=1}^{n} \) is a
\([0, t]\)-valued sequence such that if \( k \) is an integer in \([1, n]\) then \( \tilde{s}_{k} \)
is in \([s_{k-1}, s_{k}]\), then

\[
|y - \prod_{k=1}^{n} \{I - (s_{k} - s_{k-1})A(\tilde{s}_{k})\}^{-1}[x]| < \varepsilon .
\]

The statement

\[
y = \prod_{0}^{t} \exp[(ds)A(s)][x]
\]
is defined analogously.

**Theorem.** Let \( z \) be in \( Y \), and let \( u \) solve (2) and (3). Then
each of (4) and (5) is true whenever \( t \) is in \( \mathbb{R}^+ \).
Let \( m_- \) be that function from \( Y \times Y \) to the real numbers given by

\[
m_-[x, y] = \lim_{\delta \to 0^+} \frac{1}{\delta} (|x + \delta y| - |x|).
\]

Now (1) is equivalent to requiring that

\[
m_-[x - y, A(t)[x] - A(t)[y]] \leq \alpha(t) |x - y|
\]

whenever \((t, x, y)\) is in \( R^+ \times Y \times Y \) (compare [1, p. 3]). Also, if \( f \) is a function from a subset of \( R^+ \) to \( Y \), if \( c \) is in the domain of \( f \), if \( f_-'(c) \) (the left derivative of \( f \) at \( c \)) exists, and if \( P \) is given on the domain of \( f \) by \( P(t) = |f(t)| \), then \( P_-'(c) \) exists and \( P_-'(c) = m_-[f(c), f_-'(c)] \) (compare [1, p. 3]). If \((x, y, z)\) is in \( Y \times Y \times Y \) then \( m_-[x, y + z] \leq m_-[x, y] + |z| \) (see [4, Lemma 6]). We are now prepared to prove our theorem.

**Proof of the theorem.** Let \( b \) be a positive number, and let \( \beta \) be a positive upper bound for the set \( \{a(t): t \text{ is in } [0, b]\} \). Let \( \epsilon \) be a positive number, and let \( \delta \) be a positive number such that \((\delta/\beta)(e^{\delta b} - 1) < \epsilon\). Now \( \{u(t): t \text{ is in } [0, b]\} \) is a compact subset of \( Y \), so the function described by \((t, x) \to A(t)[x]\) is uniformly continuous on \([0, b] \times \{u(t): t \text{ is in } [0, b]\}\). In particular, there is a positive number \( \gamma \) such that if \((r, s, t)\) is in \([0, b] \times [0, b] \times [0, b]\) and \( |r - s| < \gamma \) then \( |A(r)[u(t)] - A(s)[u(t)]| < \delta \). Let \( \{t_k\}_{k=0}^n \) be a chain from 0 to \( b \) such that \( t_k - t_{k-1} < \gamma \) whenever \( k \) is an integer in \([1, n]\), and let \( \{\tilde{t}_k\}_{k=1}^n \) be a \([0, b]\)-valued sequence such that if \( k \) is an integer in \([1, n]\) then \( \tilde{t}_k \) is in \([t_{k-1}, t_k]\). Let \( v \) be that function from \([0, b]\) to \( Y \) having the property that if \( k \) is an integer in \([1, n]\) and \( t \) is in \([t_{k-1}, t_k]\) then

\[
v(t) = \exp \left[ (t - t_{k-1})A(\tilde{t}_{k-1}) \right] \prod_{j=1}^{k-1} \exp \left[ (t_j - t_{j-1})A(\tilde{t}_j) \right] [x].
\]

Clearly now \( v \) is continuous. Also, \( v \) is left differentiable on \((0, b]\): if \( k \) is an integer in \([1, n]\) and \( t \) is in \((t_{k-1}, t_k]\) then

\[
v_-'(t) = A(\tilde{t}_{k-1})[v(t)].
\]

Let \( P \) be given on \([0, b]\) by \( P(t) = |v(t) - u(t)| \). Now \( P(0) = 0 \). Suppose that \( t \) is in \((0, b]\) and \( k \) is an integer in \([1, n]\) and \( t \) is in \((t_{k-1}, t_k]\). Now

\[
P_-'(t) = m_-[v(t) - u(t), A(\tilde{t}_{k-1})[v(t)] - A(t)[u(t)]]
\]

\[
= m_-[v(t) - u(t), A(\tilde{t}_{k-1})[v(t)] - A(\tilde{t}_{k-1})[u(t)] + A(\tilde{t}_{k-1})[u(t)] - A(t)[u(t)]]
\]
Hence [3, Theorem 1.4.1, p. 15],

\[ P(t) \leq \int_0^t \delta e^{\beta t-s} ds = (\delta/\beta)(e^{\beta t} - 1) \]

whenever \( t \) is in \( [0, b] \). In particular,

\[
\left| u(b) - \prod_{k=1}^n \exp \left( (t_k - t_{k-1}) A(\bar{t}_k) \right) [z] \right|
\]

\[
= \left| u(b) - v(b) \right|
\]

\[
= P(b)
\]

\[
\leq (\delta/\beta)(e^{\beta b} - 1) < \varepsilon.
\]

Thus we have proved that representation (4) is valid.

Now let \( b \) and \( \beta \) be as before. Let \( c \) be a positive number such that \( c\beta < 1/2 \). Now if \( t \) is in \( [0, b] \) and \( r \) is in \( [0, c] \) then

\[
\left| [I - rA(t)]^{-1} [x] - [I - rA(t)]^{-1} [y] \right|
\]

\[
\leq (1 - r\beta)^{-1} |x - y|
\]

\[
\leq (1 + 2r\beta) |x - y|
\]

\[
\leq e^{r\beta} |x - y|
\]

whenever \((x, y)\) is in \( Y \times Y \).

Now let \( K = \{ u(t) : t \) is in \([0, b]\) \}, and recall that \( K \) is compact. Let \( \varepsilon \) be a positive number. By the aforementioned uniform continuity, there is a positive number \( \gamma_1 \) such that if \((s, t, x, y)\) is in \([0, b] \times [0, b] \times K \times K \) and \( |s - t| < \gamma_1 \) and \( |x - y| < \gamma_1 \) then \( |A(s)[x] - A(t)[y]| < (\varepsilon/b)e^{-2\beta} \).

Let \( \gamma_2 \) be a positive number such that if \((s, t)\) is in \([0, b] \times [0, b] \) and \( |s - t| < \gamma_2 \) then \( |u(s) - u(t)| < \gamma_2 \). Let \( \delta = \min \{ \gamma_1, \gamma_2, c \} \). Suppose that \( 0 \leq r \leq s \leq t \leq b \) and \( t - r < \delta \). Let \( \{ \xi_k \}^n_{k=1} \) be a \([r, t]\)-valued sequence such that if \( k \) is an integer in \([1, n]\) then \( \xi_k \) is in \([\xi_{k-1}, \xi_k]\). Now

\[
\sum_{k=1}^n (\xi_k - \xi_{k-1}) A(\bar{\xi}_k)[u(\bar{\xi}_k)] - (t - r)A(s)[u(t)]
\]

\[
\leq \sum_{k=1}^n (\xi_k - \xi_{k-1}) |A(\bar{\xi}_k)[u(\bar{\xi}_k)] - A(s)[u(t)]|
\]

\[
\leq \sum_{k=1}^n (\xi_k - \xi_{k-1})(\varepsilon/b)e^{-2\beta} = (t - r)(\varepsilon/b)e^{-2\beta}.
\]
It is now clear that
\[
\int A(\xi)[u(\xi)]d\xi - (t - r)A(s)[u(t)] \leq (t - r)(\varepsilon/b)e^{-2\varepsilon b}.
\]

Let \(\{t_k\}_k\) be a chain from 0 to \(b\), and suppose that \(t_k - t_{k-1} < \delta\) whenever \(k\) is an integer in \([1, n]\). Let \(\{\tilde{t}_k\}_{k=1}^n\) be a \([0, b]\)-valued sequence such that if \(k\) is an integer in \([1, n]\) then \(\tilde{t}_k\) is in \([t_{k-1}, t_k]\).

Now
\[
\left| \prod_{k=1}^n [I - (t_k - t_{k-1})A(\tilde{t}_k)]^{-1}[\varepsilon] - u(b) \right|
\leq \sum_{k=1}^n \left| \prod_{j=k+1}^n [I - (t_j - t_{j-1})A(\tilde{t}_j)]^{-1}[u(t_k)] 
- \prod_{j=k}^n [I - (t_j - t_{j-1})A(\tilde{t}_j)]^{-1}[u(t_{k-1})] \right|
\leq e^{2\beta(b-t_k)} |u(t_k) - [I - (t_k - t_{k-1})A(\tilde{t}_k)]^{-1}[u(t_{k-1})]|
\leq e^{2\beta b} \sum_{k=1}^n |u(t_k) - u(t_{k-1}) - (t_k - t_{k-1})A(\tilde{t}_k)[u(t_k)]|
= e^{2\beta b} \sum_{k=1}^n \left| \int_{t_{k-1}}^{t_k} u'(\xi)d\xi - (t_k - t_{k-1})A(\tilde{t}_k)[u(t_k)] \right|
\leq e^{2\beta b} \sum_{k=1}^n (t_k - t_{k-1})(\varepsilon/b)e^{-2\varepsilon b} = \varepsilon.
\]

The proof of the theorem is complete.

REFERENCE


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