LIMITS FOR MARTINGALE-LIKE SEQUENCES

ANTHONY G. MUCCI
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The concept of a martingale is generalized in two ways. The first generalization is shown to be equivalent to convergence in probability under certain uniform integrability restrictions. The second generalization yields a martingale convergence theorem.

1. Introduction. In what follows \( \{X_n, \mathcal{B}_n\} \) is a sequence of integrable random variables and sub-sigma fields on the probability space \((\Omega, \mathcal{B}, P)\) such that

\[
X_n \text{ is } \mathcal{B}_n\text{-measurable}
\]

\[
\mathcal{B}_n \subseteq \mathcal{B}_{n+1}
\]

\[
\mathcal{B} = \sigma\left(\bigcup_1 \mathcal{B}_n\right).
\]

We call the sequence \( \{X_n, \mathcal{B}_n\} \) an adapted sequence. In [2] Blake defines \( \{X_n, \mathcal{B}_n\} \) as a game which becomes fairer with time provided

\[
E(X_n | \mathcal{B}_m) - X_m \xrightarrow{P} 0 \quad \text{as } n \geq m \to \infty,
\]

i.e., provided, for all \( \varepsilon > 0 \):

\[
\lim_{m \to \infty} P(\{E(X_n | \mathcal{B}_m) - X_m\} > \varepsilon) = 0
\]

It is proven in [1] that if \( \{X_n, \mathcal{B}_n\} \) becomes fairer with time, and if there exists \( Z \in L_1 \) with \( |X_n| \leq Z \) for all \( n \), then \( X_n \xrightarrow{\mathcal{C}} X \), some \( X \in \mathcal{L}_1 \).

In the present paper we will show that \( X_n \xrightarrow{\mathcal{C}} X \) under the less restrictive assumption that \( \{X_n\} \) is uniformly integrable. We will further show that in the presence of uniform integrability \( \{X_n, \mathcal{B}_n\} \) becomes fairer with time if and only if \( \{X_n\} \) converges in probability, i.e.,

\[
E(X_n | \mathcal{B}_m) - X_m \xrightarrow{P} 0 \iff X_n - X_m \xrightarrow{P} 0.
\]

Finally, by using the more restrictive concept that \( \{X_n, \mathcal{B}_n\} \) is a martingale in the limit, namely,

\[
\lim_{m \to \infty} (E(X_n | \mathcal{B}_m) - X_m) = 0 \quad \text{a.e.,}
\]

we will prove (Theorem (2)) a generalization of a standard martingale convergence theorem.
2. PROPOSITION 1. Let the sequence \( \{X_n\} \) be uniformly integrable and assume
\[
\lim_{n \to \infty} \int_A X_n \text{ exists, all } A \in \bigcup_i ^\infty \mathcal{B}_n .
\]
Then there exists \( X \in L_1 \) such that
\[
\lim_{n \to \infty} \int_A X_n = \int_A X , \text{ all } A \in \mathfrak{B} .
\]

Proof. Let \( A \in \mathfrak{B}, \delta > 0 \). There exists \( A_\delta \in \bigcup_i ^\infty \mathcal{B}_n \) with \( P(A \Delta A_\delta) \leq \delta \). This, together with the argument in Neveu [3] (page 117) proves the desired result.

REMARKS. Let \( \Omega = [0,1) \) with Lebesgue measure. Let \( \mathcal{B}_n \) be the \( \sigma \)-field generated by the subintervals \( A_{k,n} = [k/2^n, (k+1)/2^n), k = 0,1, \ldots, 2^n - 1 \). Set \( X_n = \sum_{k=0}^{2^n-1} (-1)^k I_{A_{k,n}} \), where \( I_A \) is the indicator function of \( A \). Then for any \( A \in \bigcup \mathcal{B}_n \) we have \( \lim_{n \to \infty} \int_A X_n = 0 \). Further, \( \{X_n\} \) is uniformly integrable. However, \( \{X_n\} \) does not converge in the \( L^\infty \)-norm.

PROPOSITION 2. Let \( \{X_n\} \) be uniformly integrable and assume \( \{X_n\} \) becomes fairer with time:
\[
(*) \quad \lim_{n \to \infty} P(\{|E(X_n | \mathcal{B}_m) - X_m| > \varepsilon\}) = 0 .
\]
Then there exists \( X \in L_1 \) such that \( X_n \overset{d}{\to} X \).

Proof. Let \( A \in \mathcal{B}_m, p \geq q \geq m \). Then
\[
\left| \int_A X_p - \int_A X_q \right| = \left| \int_A E(X_p | \mathcal{B}_q) - X_q \right|
\leq \int_A \int_{A(E(X_p | \mathcal{B}_q) - X_q > \varepsilon)} |E(X_p | \mathcal{B}_q) - X_q| + \varepsilon
\leq 2 \sup_k \int_A \int_{A(E(X_p | \mathcal{B}_q) - X_q > \varepsilon)} |X_k| + \varepsilon .
\]
By uniform integrability and the assumption (*) we see that
\[
\lim_{n \to \infty} \int_A X_n \text{ converges, all } A \in \bigcup_i ^\infty \mathcal{B}_n .
\]
By Proposition 1, there exists \( X \in L_1 \) with
\[
\lim_{n \to \infty} \int_A X_n = \int_A X , \text{ all } A \in \mathfrak{B} .
\]
Note that \( \{E(X|\mathcal{B}_n), \mathcal{B}_n\} \) is a martingale and \( E(X|\mathcal{B}_n) \to X \) both in the \( L_1 \) and the almost sure sense (Levy’s Theorem). Since
\[
\int |X_n - X| \leq \int |X_n - E(X|\mathcal{B}_n)| + \int |E(X|\mathcal{B}_n) - X|,
\]

it will be enough to show \( \int |X_n - E(X|\mathcal{B}_n)| \to 0 \). Now
\[
\int |X_n - E(X|\mathcal{B}_n)| = \int_{\{X_n \geq E(X|\mathcal{B}_n)\}} (X_n - E(X|\mathcal{B}_n)) + \int_{\{X_n < E(X|\mathcal{B}_n)\}} (E(X|\mathcal{B}_n) - X_n).
\]

Letting \( n' \geq n \) and setting \( A = \{ |E(X_{n'}|\mathcal{B}_n) - X_n| > \varepsilon \} \), we have
\[
\int_{\{X_n \geq E(X|\mathcal{B}_n)\}} (X_n - E(X|\mathcal{B}_n)) \leq \int_A |X_n| + \int_A |X_n'|
+ \int_{\{X_n \geq E(X|\mathcal{B}_n)\}} (X_n' - X) + \varepsilon
\leq 2 \sup_k \int_A |X_k|
+ \int_{\{X_n \geq E(X|\mathcal{B}_n)\}} (X_n' - X) + \varepsilon.
\]

By uniform integrability and condition (*) the first integral is small. Letting \( n' \to \infty \), the difference in the remaining integral tends to zero. An identical analysis shows
\[
\int_{\{X_n < E(X|\mathcal{B}_n)\}} (E(X|\mathcal{B}_n) - X_n) \to 0.
\]

**REMARKS.** Suppose \( X_n \overset{d}{\to} X \). Then since
\[
\int_A |X_n| \leq \int |X_n - X| + \int_A |X|,
\]

we see that \( \{X_n\} \) is uniformly integrable. Further
\[
P( |E(X_n|\mathcal{B}_m) - X_m| > \varepsilon) \leq \frac{1}{\varepsilon} \int |E(X_n|\mathcal{B}_m) - X_m|
\leq \frac{1}{\varepsilon} \int |X_n - X_m|,
\]

so \( \{X_n, \mathcal{B}_n\} \) becomes fairer with time. It is shown (Neveu [3], page 52):
\( \{X_n\} \) is Cauchy in the \( L_1 \) norm \( \iff \{X_n\} \) is uniformly integrable and \( \{X_n\} \) is Cauchy in probability.

We tie these results together with Proposition 2 to get
**THEOREM 1.** Let \( \{X_n, \mathcal{F}_n\} \) be an adapted sequence. Then the following three statements are equivalent:

(a) There exists \( X \in \mathcal{L}_1 \) and \( X_n \xrightarrow{L_1} X \).

(b) \( \{X_n\} \) is uniformly integrable and \( E(X_n|\mathcal{F}_m) - X_m \xrightarrow{P} 0 \).

(c) \( \{X_n\} \) is uniformly integrable and \( X_n - X_m \xrightarrow{P} 0 \).

**COROLLARY 1.** Let the adapted sequence \( \{X_n, \mathcal{F}_n\} \) be uniformly integrable. Then

\[
E(X_n|\mathcal{F}_m) - X_m \xrightarrow{P} 0 \iff X_n - X_m \xrightarrow{P} 0 .
\]

**REMARKS.** In the absence of uniform integrability we have neither implication. Consider the following two examples:

1. Set \( X_n = \sum_{k=1}^{n} y_k \) where \( \{y_k\} \) is a sequence of independent identically distributed random variables with zero means. Set \( \mathcal{F}_n = \sigma(y_1, y_2, \cdots, y_n) \). Clearly \( \{X_n, \mathcal{F}_n\} \) is a martingale, so \( E(X_n|\mathcal{F}_m) - X_m \xrightarrow{P} 0 \). But, if, for instance

\[
y_k = \begin{cases} 
1 & \text{with probability } \frac{1}{2} \\
-1 & \text{with probability } \frac{1}{2} \end{cases}
\]

then

\[
P(|X_n - X_m| \geq 1) = P \left( \left| \sum_{k=1}^{n-m} y_k \right| \geq 1 \right)
\]

\[
= 1 - P \left( \sum_{k=1}^{n-m} y_k = 0 \right) \sim 1 - \frac{c}{\sqrt{n-m}} \xrightarrow{} 0 ,
\]

so \( X_n - X_m \xrightarrow{P} 0 \).

2. Let \( \{y_k\} \) independent where \( P(y_k = k^2) = 1/k^2 \) and \( P(y_k = 0) = 1 - 1/k^2 \). Then, setting \( X_n = \sum_{k=1}^{n} y_k \) we have

\[
|E(X_n|\mathcal{F}_m) - X_m| = E \sum_{n-m}^{n} y_k \geq 1
\]

while

\[
P(|X_n - X_m| \geq \varepsilon) = P \left( \sum_{n-m}^{n} y_k \geq \varepsilon \right) = P \left( \bigcup_{n-m}^{n} (y_k \geq \varepsilon) \right)
\]

\[
\leq \sum_{n-m}^{n} P(y_k \geq \varepsilon) = \sum_{n-m}^{n} \frac{1}{k^2} \xrightarrow{} 0 ,
\]

so in this case \( X_n - X_m \xrightarrow{P} 0 \) while \( E(X_n|\mathcal{F}_m) - X_m \xrightarrow{P} 0 \).
Recall now the definition that \( \{X_n, \mathcal{B}_n\} \) be a martingale in the limit, namely:

\[
(\star) \quad E(X_n | \mathcal{B}_n) - X \to 0 \text{ almost everywhere.}
\]

**Theorem 2.** Let the adapted sequence \( \{X_n, \mathcal{B}_n\} \) be uniformly integrable and a martingale in the limit. Then there exists \( X \in \mathcal{L}_1 \) such that

\[
X_n \to X \text{ almost everywhere and in the } \mathcal{L}_1\text{-norm.}
\]

**Proof.** Clearly, \( \{X_n, \mathcal{B}_n\} \) becomes fairer with time, so from Theorem 1 there exists \( X \in \mathcal{L}_1 \) with \( X_n \xrightarrow{\mathcal{L}_1} X \). Now, for an arbitrary subsequence \( \{n'\} \),

\[
|X_m - X| \leq |X_m - E(X_{n'} | \mathcal{B}_m)| + |E(X_{n'} - X | \mathcal{B}_m)| + |E(X | \mathcal{B}_m) - X|.
\]

By Levy's theorem, the third term is less than \( \varepsilon/3 \) for large enough \( m \). The first term is also bounded by \( \varepsilon/3 \) for large \( m, n' \) since \( \{X_n, \mathcal{B}_n\} \) is a martingale in the limit. We must now show that the second term is small. Note first that for an arbitrary \( \sigma \)-field \( \mathcal{X} \) we have

\[
E(X_n | \mathcal{X}) \xrightarrow{\mathcal{X}} E(X | \mathcal{X}).
\]

Now start with the \( \sigma \)-field \( \mathcal{B}_i \) and note that the convergence \( E(X_n | \mathcal{B}_i) \xrightarrow{\mathcal{B}_i} E(X | \mathcal{B}_i) \) implies the existence of subsequence \( \{n_i\} \subset \{n\} \) with \( E(X_{n_i} | \mathcal{B}_i) \to E(X | \mathcal{B}_i) \) almost everywhere. Continuing, we have \( E(X_{n_i} | \mathcal{B}_i) \xrightarrow{\mathcal{B}_i} E(X | \mathcal{B}_i) \), and we can extract \( \{n_i\} \subset \{n_i\} \) with \( E(X_{n_i} | \mathcal{B}_i) \to E(X | \mathcal{B}_i) \) almost everywhere. Thus, there exists a subsequence \( \{\tilde{n}\} \subset \{n\} \) with \( E(X_{\tilde{n}} | \mathcal{B}_{\tilde{n}}) \to E(X | \mathcal{B}_{\tilde{n}}) \) a.e. for all \( m \), namely the diagonal subsequence. Now choose \( \{n'\} \) as a subsequence of \( \{\tilde{n}\} \), and we can bound the second term above by \( \varepsilon/3 \).

**Applications.** 1. Let \( \{y_k\} \) be a sequence of independent random variables such that

\[
\lim_{m \to \infty} \left| \sum_{n=m}^{\infty} y_k \right| = 0.
\]

Then \( \sum_{n} y_k \) exists a.e. and in the \( \mathcal{L}_1 \)-norm.

**Proof.** Set \( S_n = \sum_{i=0}^{n} y_k \). Then

\[
\int_{\mathcal{A}} |S_n| \leq \int_{\mathcal{A}} |S_m| + \int_{m+1}^{\infty} |y_k|,
\]

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so it is clear that \( \{S_n\} \) is uniformly integrable. Further, setting \( \mathcal{B}_n = \sigma(y_1, y_2, \ldots, y_n) \), we have

\[
|E(S_n | \mathcal{B}_m) - S_m| = \left| \sum_{k=m+1}^{n} y_k \right| \leq \sum_{k=m+1}^{n} y_k,
\]

so \( \{S_n, \mathcal{B}_n\} \) is uniformly integrable martingale in the limit.

2. Let \( \{X_n, \mathcal{B}_n\} \) be an adapted uniformly integrable sequence with \( |E(X_{n+1} | \mathcal{B}_n) - X_n| \leq c_n \) where \( \{c_n\} \) is a sequence of constants with \( \sum_{n}^{\infty} c_n < \infty \). Then there exists \( x \in L_1 \) with \( X_n \to X \) almost everywhere and in the \( L_1 \)-norm.

**Proof.** We have

\[
E(X_n | \mathcal{B}_m) - X_m = \sum_{k=m}^{n-1} E(X_{k+1} - X_k | \mathcal{B}_m)
\]

\[
= \sum_{k=m}^{n-1} E(E_{k+1} - X_k | \mathcal{B}_k | \mathcal{B}_m).
\]

Thus

\[
|E(X_n | \mathcal{B}_m) - X_m| \leq \sum_{m}^{n-1} c_k.
\]


**REFERENCES**

1. L. Blake, A note concerning a class of games which become fairer with time, to appear, Glasgow Math. J.

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