

Pacific Journal of Mathematics

RELATIONALLY INDUCED SEMIGROUPS

EUGENE MICHAEL NORRIS

RELATIONALLY INDUCED SEMIGROUPS

EUGENE M. NORRIS

This paper gives sufficient conditions, of a relation-theoretic nature, in order that a quotient of the state space of a recursion (or topological machine) be a topological semigroup isomorphic to the endomorphism semigroup of the recursion, generalizing recent function-theoretic results.

Relations. By a *relation* R from a set A to a set B , we mean that R is a subset of $A \times B$. If A and B are topological spaces, we say that R is *closed* to mean that it is a closed subset of the product space. If R is a relation from A to B and S is a relation from B to C , their *composition* is the relation $S \circ R$ from A to C defined by $(a, c) \in S \circ R$ if and only if there is some $b \in B$ with $(a, b) \in R$ and $(b, c) \in S$. This is contrary to the notation in [1], but agrees with the usual (non-algebraist's) notation for the composition of functions. The *inverse* of a relation R is the relation R^{-1} defined by $(b, a) \in R^{-1}$ if and only if $(a, b) \in R$. A relation from A to A is reflexive if R contains $\Delta_A = \{(a, a) : a \in A\}$, symmetric if $R^{-1} \subseteq R$ (whence follows $R^{-1} = R$), and *transitive* if $R \circ R \subseteq R$. R is an *equivalence relation* if it is reflexive, symmetric, and transitive. For any relation R from A to B and any subsets $A' \subseteq A, B' \subseteq B, A'R$ denotes the set $\{b \in B : (a, b) \in R \text{ for some } a \in A'\}$; RB' is then defined to be the set $B'R^{-1}$. We write aR rather than $\{a\}R$ and Rb for $R\{b\}$, for simplicity's sake. It is known that if A' is compact and R is closed then $A'R$ is closed; if A, B , and C are all compact Hausdorff spaces and R and S are closed relations from A to B and B to C respectively then $S \circ R$ is also closed. It is also known that if A is compact and R is a closed equivalence on A then the quotient space $A/R = \{aR : a \in A\}$ is compact Hausdorff. See Kelley [3] for topological details.

After Riguet [5, 6], a relation R from A to B is called *difunctional* if $R \circ R^{-1} \circ R \subseteq R$; we observe that any function is difunctional and any symmetric, transitive relation is difunctional; in elementary geometry, the relation of orthogonality is difunctional, as Riguet noted. We use Riguet's 1950 results freely [6] and note in particular that if R is a difunctional relation from A to B satisfying $A = RB$ and $B = AR$, then $R^{-1} \circ R$ and $R \circ R^{-1}$ are equivalence relations on A and B , respectively, closed if R is closed and A and B are compact Hausdorff. Furthermore, $A/(R^{-1} \circ R) = \{Rb : b \in B\}$ and $B/(R \circ R^{-1}) = \{aR : a \in A\}$. For any difunctional relation R , the *slices* aR and $a'R$ either coincide or are disjoint, a property well-known for equivalence relations; the same property holds for slices Rb, Rb' , since R^{-1} is

difunctional if and only if R is difunctional. In fact, this property of slices characterizes difunctional relations. Unfortunately, the composition of difunctional relations need not be difunctional.

Recursions. A *recursion* is a triple (T, X, \cdot) , where T and X are spaces and $T \times X \xrightarrow{\cdot} X$ is a continuous binary operation, the value $t \cdot x$ of which at the point (t, x) is usually denoted by juxtaposition, unless emphasis seems wise. For $T' \subseteq T$ and $X' \subseteq X$, we write $T'X'$ (or occasionally $T' \cdot X'$) to denote the set $\{tx: t \in T' \text{ and } x \in X'\}$. We frequently avoid the use of curly brackets, writing Tx for $T\{x\}$ and so forth. In particular, if $R \subseteq T \times X$ and t, z are elements of T , then $t(zR) = \{t\} \cdot (zR)$, the translate of the slice zR . A recursion is *c.o.d.* if both spaces are compact Hausdorff or both are discrete.

For the sake of completeness we state below an easily established folkloric lemma that A. D. Wallace attribute to G. E. Schweigert [7], and a generalization, the Induced Function Theorem (IFT for short), proved in [1]. The lemma is frequently used in what follows.

LEMMA 0. *If A, B , and C are all compact or all discrete spaces, if $f: A \rightarrow B$ and $g: A \rightarrow C$ are continuous functions with f surjective and if the condition $f(a) = f(a')$ implies $g(a) = g(a')$ for all a, a' holds then there is a unique continuous function $h: B \rightarrow C$, satisfying $h(f(a)) = g(a)$ for all a in A .*

Induced Function Theorem. *Let A and B be both compact Hausdorff or both discrete spaces, $R \subseteq A \times B$ a closed relation from A to B , and E and F closed equivalence relations on A and B , respectively. If $A = RB$ and $R \circ E \circ R^{-1} \subseteq F$ then there is a unique continuous function h making the following diagram of projection and quotient functions analytic:*

$$\begin{array}{ccc} A & \xleftarrow{R} & B \\ \downarrow & & \downarrow \\ A/E & \xrightarrow{h} & B/F \end{array}$$

Furthermore, if in addition to the previous hypothesis $B = AR$ and $R^{-1} \circ F \circ R \subseteq E$, then h is a homeomorphism.

Results.

THEOREM 1. *Suppose (T, X, \cdot) is a c.o.d. recursion and $R \subseteq T \times X$ is a closed difunctional relation satisfying, for all t', t'', t and $s \in T$,*

- (1) $t'R = t''R \Rightarrow t'(tR) = t''(tR)$
- (2) $tR = t'(t'R) \Rightarrow t(sR) = t'(t''(sR))$

(3) $T = RX$ and $X = TR$

(4) for each t, t' in T there is some t'' in T with $t(t'R) = t''R$. Then $X/(R \circ R^{-1})$ is a topological semigroup with multiplication $*$ satisfying $tR * t'R = t(t'R)$ identically.

Proof. From difunctionality and hypothesis (3), $R^{-1} \circ R$ and $R \circ R^{-1}$ are equivalence relations on T and on X , respectively, and are closed if T and X are compact. The Induced Function Theorem implies that there is a unique homeomorphism h making the following diagram of projection and quotient maps analytic.

$$\begin{array}{ccc} T & \longleftarrow R & \longrightarrow X \\ \downarrow & & \downarrow \\ T/(R^{-1} \circ R) & \longrightarrow & X/(R \circ R^{-1}) \end{array}$$

In the following diagram,

$$\begin{array}{ccc} T \times X & \xrightarrow{\cdot} & X \\ h p \times q \downarrow & & \downarrow q \\ X/(R \circ R^{-1}) \times X/(R \circ R^{-1}) & \longrightarrow & X/(R \circ R^{-1}) \end{array}$$

we note that $(t, x) \in R$ iff $tR = q(x)$ and $Rx = p(t)$ iff $h(Rx) = tR$. If (t, x) and (t', x') satisfy $[hp \times q](t, x) = [hp \times q](t', x')$ then $h(p(t)) = h(p(t'))$ and $q(x) = q(x')$, so that $tR = t'R$. If $t'' \in Rx$ then $x \in t''R$, hence $tx \in t(t''R) = t'(t''R)$ by hypothesis (1). We also have $t'x' \in t'(t''R)$ since $Rx = Rx'$. Hypothesis (4) allows us to conclude that $(tx, t'x') \in R \circ R^{-1}$, i.e., $q(tx) = q(t'x')$. Hence Lemma 0 applies to give a unique continuous function $*$ making the diagram analytic. We observe that $tR * q(x) = tq(x)$ for all $t \in T$ and all $x \in X$. Now $*$ is associative, for if $t, t'' \in T$, then there is some $s \in T$ such that $t(t'R) = sR$, and hence $(tR * t'R) * t''R = t(t'R) * t''R = sR * t''R = s(t''R) = t(t'(t''R)) = tR * t'(t''R) = tR * (t'R * t''R)$, using hypothesis (2).

THEOREM 2. Suppose (T, X, \cdot) is a c.o.d. recursion and $R \subseteq T \times X$ is a closed difunctional relation satisfying

- (1) $T = RX$ and $X = TR$
- (2) the set $Z = \{z \in T: tR = t'R \Rightarrow t(zR) = t'(zR)\}$ is not empty
- (3) for each $t, t' \in T$ there is some $t'' \in T$ with $t(t'R) = t''R$
- (4) if $tR = t'(t''R)$ then for any $z \in Z$, $t(zR) = t'(t''(zR))$.

Then $\{zR: z \in Z\}$ is a topological semigroup in the quotient topology with multiplication $*$ satisfying $zR * z'R = z(z'R)$ for all $z, z' \in Z$.

Proof. For simplicity, let $\bar{Z} = \{zR: z \in Z\}$ be the subspace of the quotient space $A/(R \circ R^{-1})$. We dispose topological considerations first.

One verifies easily that if T and X are compact, then Z is closed, and it follows by standard results that ZR and finally \bar{Z} are compact. Of course, if T and X are discrete, so is \bar{Z} .

On the algebraic side, we observe that $Z \cdot ZR \subseteq ZR$, for if $z, z' \in Z$ and $tR = z(z'R)$, then it will be seen that $t \in Z$ (such t exists by hypothesis (3)). To this end, suppose that $t'R = t''R$, and let $t'(zR) = sR$ to infer that $t'(tR) = t'(z(z'R)) = s(z'R)$, by hypothesis (4). Since $z \in Z$, then $t'(zR) = t''(zR)$ and hence $sR = t''(zR)$; it then follows from hypothesis (4) that $s(z'R) = t''(z(z'R))$, so that $t'(tR) = t''(tR)$, implying that $t \in Z$. Hence $Z \cdot ZR \subseteq ZR$.

If x and x' are points in ZR satisfying $(x, x') \in R \circ R^{-1}$, then $(zx, zx') \in R \circ R^{-1}$ also, and hence we may infer from Lemma 0 that the function $Z \times \bar{Z} \xrightarrow{*} \bar{Z}$ given by $z^*z'R = z \cdot (z'R)$ is continuous.

Finally, if R' is the relation from Z to \bar{Z} defined by $(z, z'R) \in R'$ if $\{z\} \times z'R \subseteq R$, then we can easily see that R' is closed and difunctional, so that R' and the compact or discrete recursion $(Z, \bar{Z}, *)$ satisfy the hypothesis of Theorem 1. Theorem 2 now follows.

Representation. Assuming the hypothesis of Theorem 2, let S be the semigroup (with compact open topology) of all continuous functions from the quotient space $A/(R \circ R^{-1})$ into itself, and let end denote the subsemigroup of S defined by $f \in end$ if $t \cdot f(\bar{x}) = f(t \cdot \bar{x})$ for all $t \in T$ and all \bar{x} in $X/(R \circ R^{-1})$. The function $F: T \rightarrow S$, given by $F_t(t'R) = t' \cdot (tR)$, is easily seen to be continuous and maps Z into end ; let F' denote the restriction of F to Z . In a similar way, the map $G: Z \rightarrow ZR/(R \circ R^{-1})$ given by $G(z) = zR$ is a continuous surjection. Lemma 0 is seen easily to apply, giving a continuous function $H: ZR/(R \circ R^{-1}) \rightarrow end$ satisfying $H \circ G = F'$, from which we see that for any $z \in Z$ and any $t \in T$, $[H(zR)](tR) = t(zR)$. Routine computation, using hypothesis (3) and (4), shows that $H(zR^*z'R) = H(z'R) \circ H(zR)$, so that H is an anti-homorphism.

THEOREM 3. *If, in addition to the hypothesis of Theorem 2, for some $z_0 \in Z$ and all $t \in T$ it is the case that $t(z_0R) = tR = z_0(tR)$, then H is an anti-isomorphism and $ZR/(R \circ R^{-1})$ is a monoid with z_0R its identity; furthermore, the set z_0R is a set of generators for X , i.e., $T(z_0R) = X$.*

Proof. That z_0R generates X is clear from the equations $t(z_0R) = tR$ and $TR = X$. That z_0R is the identity follows from the fact that for any $z \in Z$, $zR^*z_0R = z(z_0R) = zR = z_0(zR) = z_0R^*zR$. If $H(zR) = H(z'R)$ then $zR = z_0(zR) = z_0(z'R) = z'R$, so that H is injective. To see that H is also surjective, let $f \in end$, and suppose $f(z_0R) = t_0R$;

we will see that $t_0 \in Z$. To see this suppose $t'R = t''R$ and compute: $t'(t_0R) = t'f(z_0R) = f(t'(z_0R)) = f(t'R)$; similarly, $t''(t_0R) = f(t''R)$; it follows that $t_0 \in Z$. Now for any $t \in T$, we see that $f(tR) = f(t(z_0R)) = tf(z_0R) = t(t_0R) = [H(t_0R)](tR)$, implying that H is surjective.

REMARKS. Theorem 2 obviously generalizes Theorem 1 and also contains a previous result of the author [4]. When R is a continuous function from T onto X it is a closed, difunctional relation and $R \circ R^{-1} = \Delta_X$, so that $X/(R \circ R^{-1})$ is homeomorphic to X , and the set ZR is just the image of Z ; R is surjective just in case $X = TR$. Hence Theorem 1 generalizes the theorem of [7] and Theorem 2, the theorem of [8], which in turn elegantly generalize theorems of Aczel-Wallace, Hosszu, Barnes, Fleck, Weeg, Oehmke *et. al.* (see [8] for references). Other applications will be announced elsewhere. In view of recent results of Fay [2], the present work allows one to induce semigroups "in" the objects of many categories. The details of this extension will be left for another time.

REFERENCES

1. A. R. Bednarek and A. D. Wallace, *A relation-theoretic result in topological algebra*, Mathematical Systems Theory, 1 No. 3 (1967), 217-224.
2. T. H. Fay, *The Induced Morphism Theorem: A Result in Categorical Relation Theory*, to appear.
3. J. L. Kelley, *General Topology*, D. Van Nostrand and Company, Princeton, New Jersey, 1955.
4. Eugene M. Norris, *More on externally induced operations*, Notices Amer. Math. Soc., # 689-A15, October, 1971.
5. J. Riguet, *Relations binaires, Fermetures, correspondances de Galois*, Bull. Soc. Math. France, **76** (1948), 114-155.
6. ———, *Quelque propriétés des relations difonctionnelles*, C. R. Acad. Sci. Paris, **230** (1950), 1999-2000.
7. A. D. Wallace, *Recursions with semigroup state spaces*, Revue Roumaine de Mathématiques Pures et Appliquées, Tome XII, No. 9 (1967), 1411-1415.
8. ———, *Externally Induced Operations*, Jahresbericht der Deutschen Mathematiker-Vereinigung, Bd. 73, Heft 1 (1971), 48-52.

Received June 6, 1972. This research was supported initially by the Center for Informatics Research of the University of Florida. The author is grateful to A. R. Bednarek for alerting him to Riguet's work.

UNIVERSITY OF SOUTH CAROLINA

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)
University of California
Los Angeles, California 90024

J. DUGUNDJI*
Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT
University of Washington
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM
Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

* C. R. DePrima California Institute of Technology, Pasadena, CA 91109, will replace J. Dugundji until August 1974.

Pacific Journal of Mathematics

Vol. 48, No. 1

March, 1973

Jan Aarts and David John Lutzer, <i>Pseudo-completeness and the product of Baire spaces</i>	1
Gordon Owen Berg, <i>Metric characterizations of Euclidean spaces</i>	11
Ajit Kaur Chilana, <i>The space of bounded sequences with the mixed topology</i>	29
Philip Throop Church and James Timourian, <i>Differentiable open maps of $(p + 1)$-manifold to p-manifold</i>	35
P. D. T. A. Elliott, <i>On additive functions whose limiting distributions possess a finite mean and variance</i>	47
M. Solveig Espelie, <i>Multiplicative and extreme positive operators</i>	57
Jacques A. Ferland, <i>Domains of negativity and application to generalized convexity on a real topological vector space</i>	67
Michael Benton Freeman and Reese Harvey, <i>A compact set that is locally holomorphically convex but not holomorphically convex</i>	77
Roe William Goodman, <i>Positive-definite distributions and intertwining operators</i>	83
Elliot Charles Gootman, <i>The type of some C^* and W^*-algebras associated with transformation groups</i>	93
David Charles Haddad, <i>Angular limits of locally finitely valent holomorphic functions</i>	107
William Buhmann Johnson, <i>On quasi-complements</i>	113
William M. Kantor, <i>On 2-transitive collineation groups of finite projective spaces</i>	119
Joachim Lambek and Gerhard O. Michler, <i>Completions and classical localizations of right Noetherian rings</i>	133
Kenneth Lamar Lange, <i>Borel sets of probability measures</i>	141
David Lowell Lovelady, <i>Product integrals for an ordinary differential equation in a Banach space</i>	163
Jorge Martinez, <i>A hom-functor for lattice-ordered groups</i>	169
W. K. Mason, <i>Weakly almost periodic homeomorphisms of the two sphere</i>	185
Anthony G. Mucci, <i>Limits for martingale-like sequences</i>	197
Eugene Michael Norris, <i>Relationally induced semigroups</i>	203
Arthur E. Olson, <i>A comparison of c-density and k-density</i>	209
Donald Steven Passman, <i>On the semisimplicity of group rings of linear groups. II</i>	215
Charles Radin, <i>Ergodicity in von Neumann algebras</i>	235
P. Rosenthal, <i>On the singularities of the function generated by the Bergman operator of the second kind</i>	241
Arthur Argyle Sagle and J. R. Schumi, <i>Multiplications on homogeneous spaces, nonassociative algebras and connections</i>	247
Leo Sario and Cecilia Wang, <i>Existence of Dirichlet finite biharmonic functions on the Poincaré 3-ball</i>	267
Ramachandran Subramanian, <i>On a generalization of martingales due to Blake</i>	275
Bui An Ton, <i>On strongly nonlinear elliptic variational inequalities</i>	279
Seth Warner, <i>A topological characterization of complete, discretely valued fields</i>	293
Chi Song Wong, <i>Common fixed points of two mappings</i>	299