ERGODICITY IN VON NEUMANN ALGEBRAS

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We investigate the ergodicity of elements of a von Neumann algebra \( \mathcal{A} \) under the action of an arbitrary cyclic group of inner *-automorphisms of \( \mathcal{A} \). A simple corollary of our results is the following characterization: A von Neumann algebra \( \mathcal{A} \) is finite if and only if for each \( A \in \mathcal{A} \) and inner *-automorphism \( \alpha \) of \( \mathcal{A} \), there exists \( \tilde{A} \in \mathcal{A} \) such that

\[
\frac{1}{N} \sum_{n=0}^{N-1} \alpha^n(A) \rightarrow \tilde{A}
\]

in the weak operator topology.

1. Introduction. Our purpose is to explore in a new direction the ergodic theory of von Neumann algebras presented by Kovács and Szűcs [2]. In [2] the essential contribution was the introduction of a certain restriction (called G-finiteness) on a group of *-automorphisms of a von Neumann algebra, fashioned so that all elements of the algebra behave ergodically with respect to the group. Instead we consider the action of a natural class of (cyclic) groups of *-automorphisms, namely the inner ones, and investigate which elements of the algebra behave ergodically with respect to all such groups.

2. Behavior of infinite projections. From the ergodic theory developed in [2], we note the following simple consequence.

**Theorem 0.** (Kovács and Szűcs). Let \( \mathcal{A} \) be a finite von Neumann algebra. For each \( A \in \mathcal{A} \) and each inner *-automorphism \( \alpha \) of \( \mathcal{A} \), there exists \( \tilde{A} \in \mathcal{A} \) such that

\[
\frac{1}{N} \sum_{n=0}^{N-1} \alpha^n(A) \rightarrow \tilde{A}
\]

in the strong operator topology.

Our first result is a complement to this and provides a new characterization of finiteness for von Neumann algebras.

**Theorem 1.** Let \( \mathcal{A} \) be a von Neumann algebra. For each nonzero infinite projection \( P \in \mathcal{A} \) there exists an infinite projection \( \theta \in \mathcal{A} \), \( \theta \leq P \), and a unitary \( U \in \mathcal{A} \), such that

\[
\frac{1}{N} \sum_{n=0}^{N-1} U^n\theta U^{-n}
\]

does not converge in the weak operator topology.

First we need the following lemma.

**Lemma.** There exists a nonzero properly infinite projection \( P' \leq P \).

**Proof.** Let \( S \) be the set of all central projections \( E \) of \( \mathcal{A} \) such
that $EP$ is finite. 0 $\in S$ so $S$ is not empty. Let $\{E_a\}$ be an orthogonal family of elements of $S$. If $\sum_a E_a P \sim Q \leq \sum_a E_a P$ (where $\sim$ is the usual equivalence relation for projections in $\mathfrak{A}$), then $E_a P \sim E_a Q \leq E_a P$ so that $E_a Q = E_a P$ and therefore $Q \leq \sum_a E_a Q = \sum_a E_a P$. Therefore, $Q = \sum_a E_a P$ and so $\sum_a E_a P$ is finite. It follows easily that there exists a (unique) maximal element $F$ in $S$. From [1, III.2.3.5] it follows that $(I-F)P$ is nonzero and infinite. Assume it is not properly infinite. Then from [1, III.2.5.9] there exists a central projection $G$ such that $0 \neq G(I-F)P$ is finite. But then from [1, III.2.3.5] $F < F + G(I-F)P \in S$, which contradiction proves our lemma with $P' = (I-F)P$.

**Proof of Theorem 1.** From [1, III.8.6.2] there exists a set $\{P_n \mid n \in \mathbb{Z}\}$ of nonzero projections $P_n \in \mathfrak{A}$ such that $P_n P_m = \delta_{n,m} P_n$ and $P_n \sim P_m$ for all $m, n \in \mathbb{Z}$, and such that $\sum_{|n| \leq m} P_n \longrightarrow P'$ in the strong operator topology. Therefore, there exist $V_n \in \mathfrak{A}$ such that $V_n^* V_n = P_n$ and $V_n V_n^* = P_{n+1}$ for all $n \in \mathbb{Z}$, so that $P_{n+1} V_n = V_n P_n$ and $P_n V_n^* = V_{n+1}^* P_{n+1}$ for all $n \in \mathbb{Z}$. Define for each $f \in \mathcal{H}$ (the Hilbert space of definition of $\mathfrak{A}$),

$$Uf = (\text{norm lim}_{m \rightarrow \infty} \sum_{|n| \leq m} V_n P_n f) + (I - P') f,$$

where the limit exists since $\|V_n P_n f\| = \|P_n f\|$ and $V_n P_n f = P_{n+1} V_n f$ so that $\{V_n P_n f \mid n \in \mathbb{Z}\}$ are pairwise orthogonal and

$$\sum_{|n| \leq m} \|V_n P_n f\|^2 = \sum_{|n| \leq m} \|P_n f\|^2 \leq \|P' f\|^2.$$

In fact $U$ is clearly a linear and norm preserving surjection, and therefore unitary. Now since

$$\left(\sum_{|n| \leq l} V_n P_n\right) \text{ norm lim } \sum_{m \rightarrow \infty} P_m f = \sum_{|n| \leq l} V_n P_n f$$

it follows that $U_l = I - P' + \sum_{|n| \leq l} V_n P_n$ has $U$ as a strong operator limit as $l \rightarrow \infty$. Therefore, $U \in \mathfrak{A}$. It also follows that $U P_n U^{-1} = P_{n+1}$ for all $n \in \mathbb{Z}$, and so by induction $U^m P_n U^{-m} = P_{n+m}$ for all $m, n \in \mathbb{Z}$.

Now define $g: \mathbb{N} \rightarrow [0, 1]$ by

$$g(n) = \begin{cases} 1 & \text{if } 3^m \leq n < 3^{m+1} \text{ for some } m \in \mathbb{N} \\ 0 & \text{if } 3^{m+1} \leq n < 3^{m+2} \text{ for some } m \in \mathbb{N}. \end{cases}$$

Then define $\theta$ as the strong operator limit as

$$K \rightarrow - \infty \text{ of } \sum_{m=0}^0 g(-m) P_m,$$

and let $\psi$ be a unit vector in $P_0 \mathcal{H}$. Now consider
\[ \langle \psi, \frac{1}{N} \sum_{n=0}^{N-1} U^n \theta U^{-n} \psi \rangle = \frac{1}{N} \sum_{n=0}^{N-1} \langle \psi, U^n \theta U^{-n} P_0 \psi \rangle \]
\[ = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=-\infty}^{0} g(-m) \langle \psi, P_{n+m} P_0 \psi \rangle \]
\[ = \frac{1}{N} \sum_{n=0}^{N-1} g(n) . \]

It is easy to see that for all \( M \in N, \frac{1}{3^{2M+1}} \sum_{n=0}^{2^{M+1}-1} g(n) \geq \frac{2}{3} \) yet \( \frac{1}{3^{2M+2}} \sum_{n=0}^{2^{M+1}-1} g(n) \leq \frac{1}{3} \), and the theorem is proven.

Using Theorem 0, we have immediately,

**COROLLARY 1** (resp.2). A von Neumann algebra \( \mathfrak{A} \) is finite if and only if for each \( A \in \mathfrak{A} \) and inner *-automorphism \( \alpha \) of \( \mathfrak{A} \), there exists \( \bar{A} \in \mathfrak{A} \) such that \( \frac{1}{N} \sum_{n=0}^{N-1} \alpha^n(A) \xrightarrow{N \to \infty} \bar{A} \) in the weak (resp. strong) operator topology.

3. Finite elements. Theorem 1 raises the question of the ergodic behavior, under arbitrary inner *-automorphisms, of "finite elements" of infinite von Neumann algebras. The following theorem gives some information in this direction.

**THEOREM 2.** Let \( \mathfrak{A} \) be a von Neumann algebra and \( \tau \) a faithful normal semi-finite trace on \( \mathfrak{A}^+ \) invariant under the *-automorphism \( \alpha \) of \( \mathfrak{A} \). Then for each \( A \in \mathfrak{A} \) such that \( \tau(A^*A) < \infty \), there exists \( \bar{A} \in \mathfrak{A} \) such that \( \frac{1}{N} \sum_{n=0}^{N-1} \alpha^n(A) \xrightarrow{N \to \infty} \bar{A} \) in the strong operator topology.

**Proof.** First we define the following (standard) objects: see e.g. [1, 1.6.2.2]

\[ ||| \|_2 : A \in \mathfrak{A} \longrightarrow [\tau(A^*A)]^{1/2} \]
\[ \mathcal{N}^- = \{ A \in \mathfrak{A} \mid \| A \|_2 < \infty \} . \]

Let \( L_2 \) be the abstract completion of \( \mathcal{N}^- \) in the norm \( || \|_2 \), and extend \( || \|_2 \) to \( L_2 \) in the usual way. Let \( i \) be the isometric embedding of \( \mathcal{N}^- \) into \( L_2 \). \( L_2 \) is a Hilbert space with the obvious addition and scalar multiplication, and inner product \( \langle, \rangle \) defined as the extension to \( L_2 \times L_2 \) of

\[ \tau : A \times B \in \mathcal{N}^- \times \mathcal{N} \longrightarrow \tau(A^*B) . \]

We note the simple inequalities
\[ || AB ||_2 \leq || A || || B ||_2 \quad \text{for all} \ B \in \mathcal{N}, \ A \in \mathfrak{A} \]
\[ || AB ||_2 \leq || A ||_2 || B || \quad \text{for all} \ B \in \mathcal{N}, \ A \in \mathfrak{A} . \]
We then define the $C^*$-representation $\pi$ of $\mathcal{A}$ on $L_2$ by
\[
\pi(A)i(B) = i(AB)
\]
and noting that $\|\pi(A)i(B)\|_2 = \|AB\|_2 \leq \|A\| \|B\|_2$, so that $\pi(A)$ extends uniquely to $L_2$ by continuity. It is easy to see that $\pi$ is faithful and normal and that
\[
U: i(B) \longrightarrow i(\alpha[B]) \quad \text{for } B \in \mathcal{N}
\]
extends to a unitary operator on $L_2$. Defining, for $B \in \mathcal{A}$,
\[
B_N = \frac{1}{N} \sum_{n=0}^{N-1} \alpha^n(B),
\]
we know by von Neumann's mean ergodic theorem that for each $A \in \mathcal{N}$, $i(A_N)$ is $\|\cdot\|_2$-Cauchy. Define for each $B \in \mathcal{N}$,
\[
D_A: i(B) \longrightarrow \text{norm lim}_{N \to \infty} \pi(A_N)i(B)
\]
which limit exists since
\[
\|\pi(A_N - A_M)i(B)\|_2 \leq \|A_N - A_M\|_2 \|B\|.
\]
$D_A$ is obviously linear. Furthermore,
\[
\|D_A i(B)\|_2 = \lim_{N \to \infty} \|\pi(A_N)i(B)\|_2 \leq \|A\| \|B\|_2
\]
so $D_A$ extends uniquely to a bounded operator on $L_2$ by continuity. It is easy to see that $\pi(A_N)$ converges to $D_A$ in the strong operator topology. Since $\pi$ is normal, $\pi(\mathcal{A})$ is strong operator closed [1, I.4.3.2] so there exists $\tilde{A} \in \mathcal{A}$ such that $D_A = \pi(\tilde{A})$. Since $\pi$ is faithful, $A_N \to \tilde{A}$ in the strong operator topology [1, I.4.3.1].

**COROLLARY 1.** Let $\mathcal{A}$ be a countably decomposable von Neumann algebra. For each finite projection $P \in \mathcal{A}$ and inner *-automorphism $\alpha$ of $\mathcal{A}$, there exists $\tilde{P} \in \mathcal{A}$ such that
\[
\frac{1}{N} \sum_{n=0}^{N-1} \alpha^n(P) \overset{N \to \infty}{\longrightarrow} \tilde{P} \quad \text{in the strong operator topology}.
\]

**Proof.** Let
\[
A \in \mathcal{A} \longrightarrow A_1 \oplus A_2 \in \mathcal{A}_1 \oplus \mathcal{A}_2
\]
be the canonical decomposition of $\mathcal{A}$ into its countably decomposable semi-finite and purely infinite components. From [1, I.6.7.9] we know that any finite countably decomposable von Neumann algebra has a faithful, normal, tracial state. Inserting this fact into the proof of
[3, 2.5.3], we see that there exists a countable faithful family \( \{\tau_n \mid n \in \mathbb{N}\} \) of normal semi-finite traces on \( \mathfrak{A}^+ \) with pairwise orthogonal supports such that \( \tau_n(P_i) < \infty \) for all \( n \in \mathbb{N} \). Define

\[
\tau' = \sum_{n=0}^{\infty} \tau_n/[\tau_n(P_i) + 2]^a
\]

on \( \mathfrak{A}^+ \); it is faithful, normal and semi-finite. Since \( \alpha \) is also inner for \( \mathfrak{A} \) and therefore leaves \( \tau' \) invariant, we may apply Theorem 2 to \( \mathfrak{A} \). Since \( P_i = 0 \) from [1, III.2.4.8], we are finished.

In the countably decomposable case, Theorem 2 gives us an essentially different proof of Theorem 0, namely

**Corollary 2.** Let \( \mathfrak{A} \) be a finite countably decomposable von Neumann algebra. For each \( A \in \mathfrak{A} \) and inner \( * \)-automorphism \( \alpha \) of \( \mathfrak{A} \), there exists \( \overline{A} \in \mathfrak{A} \) such that

\[
\frac{1}{N} \sum_{\pi=0}^{N-1} \alpha^n(A) \xrightarrow{N \to \infty} \overline{A} \text{ in the strong operator topology}.
\]

**Proof.** Just combine the existence of a faithful finite normal trace on \( \mathfrak{A}^+ \) [1, I.6.7.9] with Theorem 2.

**References**


Received June 6, 1972. Research supported by AFOSR under Contract F44620-71-C-0108.

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Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan
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