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**ERGODICITY IN VON NEUMANN ALGEBRAS**

CHARLES RADIN

# ERGODICITY IN VON NEUMANN ALGEBRAS

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**We investigate the ergodicity of elements of a von Neumann algebra  $\mathfrak{A}$  under the action of an arbitrary cyclic group of inner \*-automorphisms of  $\mathfrak{A}$ . A simple corollary of our results is the following characterization: A von Neumann algebra  $\mathfrak{A}$  is finite if and only if for each  $A \in \mathfrak{A}$  and inner \*-automorphism  $\alpha$  of  $\mathfrak{A}$ , there exists  $\bar{A} \in \mathfrak{A}$  such that  $1/N \sum_{n=0}^{N-1} \alpha^n(A) \xrightarrow{N \rightarrow \infty} \bar{A}$  in the weak operator topology.**

1. Introduction. Our purpose is to explore in a new direction the ergodic theory of von Neumann algebras presented by Kovács and Szücs [2]. In [2] the essential contribution was the introduction of a certain restriction (called  $G$ -finiteness) on a group of \*-automorphisms of a von Neumann algebra, fashioned so that all elements of the algebra behave ergodically with respect to the group. Instead we consider the action of a natural class of (cyclic) groups of \*-automorphisms, namely the inner ones, and investigate which elements of the algebra behave ergodically with respect to all such groups.

2. Behavior of infinite projections. From the ergodic theory developed in [2], we note the following simple consequence.

**THEOREM 0.** (Kovács and Szücs). *Let  $\mathfrak{A}$  be a finite von Neumann algebra. For each  $A \in \mathfrak{A}$  and each inner \*-automorphism  $\alpha$  of  $\mathfrak{A}$ , there exists  $\bar{A} \in \mathfrak{A}$  such that  $1/N \sum_{n=0}^{N-1} \alpha^n(A) \xrightarrow{N \rightarrow \infty} \bar{A}$  in the strong operator topology.*

Our first result is a complement to this and provides a new characterization of finiteness for von Neumann algebras.

**THEOREM 1.** *Let  $\mathfrak{A}$  be a von Neumann algebra. For each nonzero infinite projection  $P \in \mathfrak{A}$  there exists an infinite projection  $\theta \in \mathfrak{A}$ ,  $\theta \leq P$ , and a unitary  $U \in \mathfrak{A}$ , such that  $1/N \sum_{n=0}^{N-1} U^n \theta U^{-n}$  does not converge in the weak operator topology.*

First we need the following lemma.

**LEMMA.** *There exists a nonzero properly infinite projection  $P' \leq P$ .*

*Proof.* Let  $S$  be the set of all central projections  $E$  of  $\mathfrak{A}$  such

that  $EP$  is finite.  $0 \in S$  so  $S$  is not empty. Let  $\{E_\alpha\}$  be an orthogonal family of elements of  $S$ . If  $\sum_\alpha E_\alpha P \sim Q \leq \sum_\alpha E_\alpha P$  (where  $\sim$  is the usual equivalence relation for projections in  $\mathfrak{A}$ ), then  $E_\alpha P \sim E_\alpha Q \leq E_\alpha P$  so that  $E_\alpha Q = E_\alpha P$  and therefore  $Q \geq \sum_\alpha E_\alpha Q = \sum_\alpha E_\alpha P$ . Therefore,  $Q = \sum_\alpha E_\alpha P$  and  $\sum_\alpha E_\alpha P$  is finite. It follows easily that there exists a (unique) maximal element  $F$  in  $S$ . From [1, III.2.3.5] it follows that  $(I-F)P$  is nonzero and infinite. Assume it is not properly infinite. Then from [1, III.2.5.9] there exists a central projection  $G$  such that  $0 \neq G(I-F)P$  is finite. But then from [1, III.2.3.5]  $F < F + G(I-F) \in S$ , which contradiction proves our lemma with  $P' \equiv (I-F)P$ .

*Proof of Theorem 1.* From [1, III.8.6.2] there exists a set  $\{P_n \mid n \in \mathbb{Z}\}$  of nonzero projections  $P_n \in \mathfrak{A}$  such that  $P_n P_m = \delta_{n,m} P_n$  and  $P_n \sim P_m$  for all  $m, n \in \mathbb{Z}$ , and such that  $\sum_{|n| \leq m} P_n \xrightarrow{m \rightarrow \infty} P'$  in the strong operator topology. Therefore, there exist  $V_n \in \mathfrak{A}$  such that  $V_n^* V_n = P_n$  and  $V_n V_n^* = P_{n+1}$  for all  $n \in \mathbb{Z}$ , so that  $P_{n+1} V_n = V_n P_n$  and  $P_n V_n^* = V_n^* P_{n+1}$  for all  $n \in \mathbb{Z}$ . Define for each  $f \in \mathcal{H}$  (the Hilbert space of definition of  $\mathfrak{A}$ ),

$$Uf = (\text{norm lim}_{m \rightarrow \infty} \sum_{|n| \leq m} V_n P_n f) + (I - P')f,$$

where the limit exists since  $\|V_n P_n f\| = \|P_n f\|$  and  $V_n P_n f = P_{n+1} V_n f$  so that  $\{V_n P_n f \mid n \in \mathbb{Z}\}$  are pairwise orthogonal and

$$\sum_{|n| \leq m} \|V_n P_n f\|^2 = \sum_{|n| \leq m} \|P_n f\|^2 \leq \|P' f\|^2.$$

In fact  $U$  is clearly a linear and norm preserving surjection, and therefore unitary. Now since

$$\left(\sum_{|k| \leq l} V_k P_k\right) \text{norm lim}_{m \rightarrow \infty} \sum_{|n| \leq m} P_n f = \sum_{|n| \leq l} V_n P_n f$$

it follows that  $U_l \equiv I - P' + \sum_{|k| \leq l} V_k P_k$  has  $U$  as a strong operator limit as  $l \rightarrow \infty$ . Therefore,  $U \in \mathfrak{A}$ . It also follows that  $U P_n U^{-1} = P_{n+1}$  for all  $n \in \mathbb{Z}$ , and so by induction  $U^m P_n U^{-m} = P_{n+m}$  for all  $m, n \in \mathbb{Z}$ . Now define  $g: N \rightarrow \{0, 1\}$  by

$$g(n) = \begin{cases} 1 & \text{if } 3^{2m} \leq n < 3^{2m+1} \text{ for some } m \in N \\ 0 & \text{if } 3^{2m+1} \leq n < 3^{2m+2} \text{ for some } m \in N. \end{cases}$$

Then define  $\theta$  as the strong operator limit as

$$K \rightarrow -\infty \quad \text{of} \quad \sum_{m=K}^0 g(-m) P_m,$$

and let  $\psi$  be a unit vector in  $P_0 \mathcal{H}$ . Now consider

$$\begin{aligned} \left\langle \psi_f, \frac{1}{N} \sum_{n=0}^{N-1} U^n \theta U^{-n} \psi_f \right\rangle &= \frac{1}{N} \sum_{n=0}^{N-1} \left\langle \psi_f, U^n \theta U^{-n} P_0 \psi_f \right\rangle \\ &= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=-\infty}^0 g(-m) \left\langle \psi_f, P_{n+m} P_0 \psi_f \right\rangle \\ &= \frac{1}{N} \sum_{n=0}^{N-1} g(n) . \end{aligned}$$

It is easy to see that for all  $M \in \mathbb{N}$ ,  $\frac{1}{3^{2M+1}} \sum_{n=0}^{3^{2M+1}-1} g(n) \geq 2/3$  yet  $\frac{1}{3^{2M+2}} \sum_{n=0}^{3^{2M+2}-1} g(n) \leq 1/3$ , and the theorem is proven.

Using Theorem 0, we have immediately,

**COROLLARY 1 (resp.2).** *A von Neumann algebra  $\mathfrak{A}$  is finite if and only if for each  $A \in \mathfrak{A}$  and inner \*-automorphism  $\alpha$  of  $\mathfrak{A}$ , there exists  $\bar{A} \in \mathfrak{A}$  such that  $\frac{1}{N} \sum_{n=0}^{N-1} \alpha^n(A) \xrightarrow[N \rightarrow \infty]{} \bar{A}$  in the weak (resp. strong) operator topology.*

**3. Finite elements.** Theorem 1 raises the question of the ergodic behavior, under arbitrary inner \*-automorphisms, of “finite elements” of infinite von Neumann algebras. The following theorem gives some information in this direction.

**THEOREM 2.** *Let  $\mathfrak{A}$  be a von Neumann algebra and  $\tau$  a faithful normal semi-finite trace on  $\mathfrak{A}^+$  invariant under the \*-automorphism  $\alpha$  of  $\mathfrak{A}$ . Then for each  $A \in \mathfrak{A}$  such that  $\tau(A^*A) < \infty$ , there exists  $\bar{A} \in \mathfrak{A}$  such that  $\frac{1}{N} \sum_{n=0}^{N-1} \alpha^n(A) \xrightarrow[N \rightarrow \infty]{} \bar{A}$  in the strong operator topology.*

*Proof.* First we define the following (standard) objects: see e.g. [1, I.6.2.2]

$$\begin{aligned} || \cdot ||_2 : A \in \mathfrak{A} &\longrightarrow [\tau(A^*A)]^{1/2} \\ \mathcal{N} &= \{A \in \mathfrak{A} \mid || A ||_2 < \infty\} . \end{aligned}$$

Let  $L_2$  be the abstract completion of  $\mathcal{N}$  in the norm  $|| \cdot ||_2$ , and extend  $|| \cdot ||_2$  to  $L_2$  in the usual way. Let  $i$  be the isometric embedding of  $\mathcal{N}$  into  $L_2$ .  $L_2$  is a Hilbert space with the obvious addition and scalar multiplication, and inner product  $\langle , \rangle$  defined as the extension to  $L_2 \times L_2$  of

$$\tau : A \times B \in \mathcal{N} \times \mathcal{N} \longrightarrow \tau(A^*B) .$$

We note the simple inequalities

$$\begin{aligned} || AB ||_2 &\leq || A || || B ||_2 && \text{for all } B \in \mathcal{N}, A \in \mathfrak{A} \\ || AB ||_2 &\leq || A ||_2 || B || && \text{for all } B \in \mathcal{N}, B \in \mathfrak{A} . \end{aligned}$$

We then define the  $C^*$ -representation  $\pi$  of  $\mathfrak{A}$  on  $L_2$  by

$$\pi(A)i(B) \equiv i(AB)$$

and noting that  $\|\pi(A)i(B)\|_2 = \|AB\|_2 \leq \|A\| \|B\|_2$  so that  $\pi(A)$  extends uniquely to  $L_2$  by continuity. It is easy to see that  $\pi$  is faithful and normal and that

$$U: i(B) \longrightarrow i(\alpha[B]) \quad \text{for } B \in \mathcal{N}$$

extends to a unitary operator on  $L_2$ . Defining, for  $B \in \mathfrak{A}$ ,

$$B_N = \frac{1}{N} \sum_{n=0}^{N-1} \alpha^n(B), \quad \text{we know by von Neumann's}$$

mean ergodic theorem that for each  $A \in \mathcal{N}$ ,  $i(A_N)$  is  $\|\cdot\|_2$ -Cauchy. Define for each  $B \in \mathcal{N}$ ,

$$D_A: i(B) \longrightarrow \text{norm } \lim_{N \rightarrow \infty} \pi(A_N)i(B)$$

which limit exists since

$$\|\pi(A_N - A_M)i(B)\|_2 \leq \|A_N - A_M\|_2 \|B\|_2.$$

$D_A$  is obviously linear. Furthermore,

$$\|D_A i(B)\|_2 = \lim_{N \rightarrow \infty} \|\pi(A_N)i(B)\|_2 \leq \|A\| \|B\|_2$$

so  $D_A$  extends uniquely to a bounded operator on  $L_2$  by continuity. It is easy to see that  $\pi(A_N)$  converges to  $D_A$  in the strong operator topology. Since  $\pi$  is normal,  $\pi(\mathfrak{A})$  is strong operator closed [1, I.4.3.2] so there exists  $\bar{A} \in \mathfrak{A}$  such that  $D_A = \pi(\bar{A})$ . Since  $\pi$  is faithful,  $A_N \xrightarrow[N \rightarrow \infty]{} \bar{A}$  in the strong operator topology [1, I.4.3.1].

**COROLLARY 1.** *Let  $\mathfrak{A}$  be a countably decomposable von Neumann algebra. For each finite projection  $P \in \mathfrak{A}$  and inner  $*$ -automorphism  $\alpha$  of  $\mathfrak{A}$ , there exists  $\bar{P} \in \mathfrak{A}$  such that*

$$\frac{1}{N} \sum_{M=0}^{N-1} \alpha^M(P) \xrightarrow[N \rightarrow \infty]{} \bar{P} \quad \text{in the strong operator topology.}$$

*Proof.* Let

$$A \in \mathfrak{A} \longrightarrow A_1 \oplus A_2 \in \mathfrak{A}_1 \oplus \mathfrak{A}_2$$

be the canonical decomposition of  $\mathfrak{A}$  into its countably decomposable semi-finite and purely infinite components. From [1, I.6.7.9] we know that any finite countably decomposable von Neumann algebra has a faithful, normal, tracial state. Inserting this fact into the proof of

[3, 2.5.3], we see that there exists a countable faithful family  $\{\tau_n \mid n \in \mathbb{N}\}$  of normal semi-finite traces on  $\mathfrak{A}_1^+$  with pairwise orthogonal supports such that  $\tau_n(P_1) < \infty$  for all  $n \in \mathbb{N}$ . Define

$$\tau' = \sum_{n=0}^{\infty} \tau_n / [\tau_n(P_1) + 2]^n$$

on  $\mathfrak{A}_1^+$ ; it is faithful, normal and semi-finite. Since  $\alpha$  is also inner for  $\mathfrak{A}_1$  and therefore leaves  $\tau'$  invariant, we may apply Theorem 2 to  $\mathfrak{A}_1$ . Since  $P_2 = 0$  from [1, III.2.4.8], we are finished.

In the countably decomposable case, Theorem 2 gives us an essentially different proof of Theorem 0, namely

**COROLLARY 2.** *Let  $\mathfrak{A}$  be a finite countably decomposable von Neumann algebra. For each  $A \in \mathfrak{A}$  and inner  $*$ -automorphism  $\alpha$  of  $\mathfrak{A}$ , there exists  $\bar{A} \in \mathfrak{A}$  such that*

$$\frac{1}{N} \sum_{n=0}^{N-1} \alpha^n(A) \xrightarrow{N \rightarrow \infty} \bar{A} \quad \text{in the strong operator topology.}$$

*Proof.* Just combine the existence of a faithful finite normal trace on  $\mathfrak{A}^+$  [1, I.6.7.9] with Theorem 2.

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