

Pacific Journal of Mathematics

ON A GENERALIZATION OF MARTINGALES DUE TO BLAKE

RAMACHANDRAN SUBRAMANIAN

ON A GENERALIZATION OF MARTINGALES DUE TO BLAKE

R. SUBRAMANIAN

**It is shown that any uniformly integrable fairer with time
 game (stochastic process) converges in L_1 .**

1. Introduction. Let (Ω, \mathcal{U}, P) be a probability space and $\{\mathcal{U}_n\}_{n \geq 1}$ an increasing family of sub σ -algebras of \mathcal{U} . Let $\{X_n\}_{n \geq 1}$ be a stochastic process adapted to $\{\mathcal{U}_n\}_{n \geq 1}$ (see, [2, p. 65]). Following Blake [1] we refer to $\{X_n\}_{n \geq 1}$ as a game and define

DEFINITION. The game $\{X_n\}_{n \geq 1}$ will be said to become *fairer with time* if for every $\varepsilon > 0$

$$P[|E(X_n|\mathcal{U}_m) - X_m| > \varepsilon] \rightarrow 0$$

as $n, m \rightarrow \infty$ with $n \geq m$. Any martingale is, trivially, a fairer with time game and thus this concept generalizes that of martingales. Blake, in [1], gave a set of sufficient conditions under which any uniformly integrable fairer with time game $\{X_n\}_{n \geq 1}$ is convergent in L_1 . We show that these sufficient conditions are not needed; in fact, we show that any uniformly integrable, fairer with time game converges in L_1 .

2. THEOREM 2.1. *Any uniformly integrable fairer with time game $\{X_n\}_{n \geq 1}$ converges in L_1 .*

Proof. To facilitate understanding, we break up the proof into a few important steps numbered (S1) through (S5). For every m and $n \geq m$ define $Y_{m,n} = E(X_n|\mathcal{U}_m)$. Let Γ stand for the family $\{Y_{m,n}$, for all m and $n \geq m\}$.

(S1) Γ is uniformly integrable.

Since $\{X_n\}_{n \geq 1}$ is uniformly integrable there exists a function f defined on the nonnegative real axis which is positive, increasing and convex, such that

$$\lim_{t \rightarrow \infty} \frac{f(t)}{t} = +\infty$$

and $\sup_n E[f \circ |X_n|] < \infty$. (See [2, II T 22].) Now,

$$\begin{aligned} E[f \circ |Y_{m,n}|] &= E[f \circ |E(X_n|\mathcal{U}_m)|] \\ &\leq E[f \circ E(|X_n|/\mathcal{U}_m)] \text{ (since } f \text{ is nondecreasing)} \\ &\leq E[E(f \circ |X_n|/\mathcal{U}_m)] \\ &= E[f \circ |X_n|]. \end{aligned}$$

Therefore,

$$\sup_{Y_{m,n} \in \Gamma} E[f \circ | Y_{m,n} |] \leq \sup_n E[f \circ | X_n |] < \infty .$$

Another application of II T 22 of [2] ensures that Γ is uniformly integrable. Hence (S1).

(S2) Given $\varepsilon > 0$, there exists M such that for all $m \geq M$, one has

$$E(| X_m - Y_{m,n} |) \leq 2\varepsilon \quad \text{for all } n \geq m .$$

Since Γ is uniformly integrable given $\varepsilon > 0$ there exists $\delta > 0$ such that $P(A) < \delta$ implies $\int_A | Y_{m,n} | dP \leq \varepsilon/2$, for all $Y_{m,n} \in \Gamma$. Choose M so large that $m \geq M$ and $n \geq m$ implies $P[| X_m - E(X_m/U_m) | > \varepsilon] < \delta$. Then, it is not difficult to see that

$$E[| X_m - Y_{m,n} |] \leq 2\varepsilon \quad \text{for all } m \geq M \quad \text{and } n \geq m .$$

(S3) For every fixed m , the sequence $\{Y_{m,n}\}$ converges in L_1 to an \mathcal{U}_m measurable random variable Z_m .

Let $m \leq n < n'$.

$$\begin{aligned} E[| Y_{m,n} - Y_{m,n'} |] &= E[| E(X_n/\mathcal{U}_m) - E(X_{n'}/\mathcal{U}_m) |] \\ &= E[| E(X_n - X_{n'}/\mathcal{U}_m) |] \\ &= E[| E(\{E(X_n - X_{n'}/\mathcal{U}_n)\}/\mathcal{U}_m) |] \\ &\leq E[E(\{E(X_n - X_{n'}/\mathcal{U}_n)\}/\mathcal{U}_m)] \\ &= E[| E(X_n - X_{n'}/\mathcal{U}_n) |] \\ &= E[| X_n - Y_{n,n'} |] . \end{aligned}$$

Now from (S2) it follows that given $\varepsilon > 0$ for all sufficiently large n and n'

$$E[| Y_{m,n} - Y_{m,n'} |] \leq E[| (X_n - Y_{n,n'}) |] \leq 2\varepsilon .$$

Hence, for m fixed, the sequence $\{Y_{m,n}\}$ is Cauchy in the L_1 -norm. So, there exists, an integrable random variable Z_m , such that, $Y_{m,n} \xrightarrow[n \rightarrow \infty]{L_1} Z_m$. Without loss of generality we can take Z_m to be \mathcal{U}_m measurable. (Note that each $Y_{m,n}$ is \mathcal{U}_m measurable and there is a subsequence $\{Y_{m,n'}\}$ converging almost surely to Z_m .)

(S4) $\{Z_m, \mathcal{U}_m\}_{m \geq 1}$ is a uniformly integrable martingale.

The fact that $\{Z_m\}_{m \geq 1}$ is uniformly integrable follows trivially because the closure in L_1 of a uniformly integrable collection is uniformly integrable. (See, [2, II T20].) To show $\{Z_m, \mathcal{U}_m\}$ is a martingale it is enough to show that for every m , $E(Z_{m+1}/\mathcal{U}_m) = Z_m$ a.s. Since

$$\begin{aligned}
 E[| E(Y_{m+1,n}/\mathcal{U}_m) - E(Z_{m+1}/\mathcal{U}_m) |] & \\
 &= E[| E\{(Y_{m+1,n} - Z_{m+1})/\mathcal{U}_m\} |] \\
 &\leq E[E\{|(Y_{m+1,n} - Z_{m+1})|/\mathcal{U}_m\}] \\
 &= E[| Y_{m+1,n} - Z_{m+1} |] \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,
 \end{aligned}$$

there exists a subsequence n' of $\{n: n \geq m\}$ such that

$$E(Y_{m+1,n'}/\mathcal{U}_m) \xrightarrow{\text{a.s.}} E(Z_{m+1}/\mathcal{U}_m).$$

We can assume (– if necessary, by choosing a further subsequence, –) that $Y_{m,n'} \xrightarrow{\text{a.s.}} Z_m$. Now,

$$\begin{aligned}
 E(Z_{m+1}/\mathcal{U}_m) &= \lim_{n' \rightarrow \infty} E(Y_{m+1,n'}/\mathcal{U}_m) \quad \text{a.s.} \\
 &= \lim_{n' \rightarrow \infty} E(\{E(X_{n'}/\mathcal{U}_{m+1})\}/\mathcal{U}_m) \quad \text{a.s.} \\
 &= \lim_{n' \rightarrow \infty} E(X_{n'}/\mathcal{U}_m) \quad \text{a.s.} \\
 &= \lim_{n' \rightarrow \infty} Y_{m,n'} \quad \text{a.s.} \\
 &= Z_m \quad \text{a.s.}
 \end{aligned}$$

Hence (S4). (S5) $\{X_n\}_{n \geq 1}$ converges in L_1 .

Since $\{Z_n, \mathcal{U}_n\}_{n \geq 1}$ is an uniformly integrable martingale, there exists an integrable random variable Z_∞ such that $Z_n \xrightarrow[n \rightarrow \infty]{L_1} Z_\infty$. We shall show that $X_n \xrightarrow[n \rightarrow \infty]{L_1} Z_\infty$. From (S3) and (S2) it is easy to check that given $\varepsilon > 0$ there exists M such that for all $m \geq M$

$$\int |X_m - Z_m| dP \leq 2\varepsilon.$$

Therefore, for sufficiently large m ,

$$\int |X_m - Z_\infty| dP \leq \int |X_m - Z_m| dP + \int |Z_m - Z_\infty| dP \leq 3\varepsilon,$$

say. Hence (S5) and the theorem.

Since any game (stochastic process) $\{X_n\}_{n \geq 1}$ converging in L_1 can be taken to be a game fairer with time, by setting $\mathcal{U}_n \equiv \mathcal{U}$ in n , we get the following corollary.

COROLLARY 2.1. *Let $\{X_n\}_{n \geq 1}$ be a game. It converges in L_1 if and only if it is uniformly integrable and fairer with time with respect to some increasing family of sub σ -algebras $\{\mathcal{U}_n\}_{n \geq 1}$ to which it is adapted.*

Let $p > 1$.

THEOREM 2.2. *Let $\{X_n\}_{n \geq 1}$ be a fairer with time game with $\{|X_n|^p\}_{n \geq 1}$ uniformly integrable. Then $\{X_n\}_{n \geq 1}$ converges in L_p .*

Proof. Noting that the function f defined on the nonnegative real axis by $f(t) = t^p$ is positive, increasing and convex and $\lim_{t \rightarrow \infty} (f(t)/t) = +\infty$, in view of II T 22 of [2], it is clear that $\{X_n\}_{n \geq 1}$ is uniformly integrable. Hence by Theorem 2.1 it converges in L_1 ; in particular, $\{X_n\}_{n \geq 1}$ converges in probability. Therefore, $\{X_n\}_{n \geq 1}$ converges in L_p . (See Proposition II 6.1 of [3].)

COROLLARY 2.2. *The game $\{X_n\}_{n \geq 1}$ converges in L_p if and only if $\{|X_n|^p\}_{n \geq 1}$ is uniformly integrable and $\{X_n\}_{n \geq 1}$ is fairer with time with respect to some increasing family of sub σ -algebras $\{\mathcal{Z}_n\}_{n \geq 1}$ to which it is adapted.*

REMARK. In view of our Theorem 2.1, the second convergence theorem of Blake in [1] becomes redundant.

REFERENCES

1. L. H. Blake, *A generalization of martingales and two consequent convergence theorems*, Pacific J. Math., **35** (1970), 279-283.
2. P. A. Meyer, *Probability and Potentials*, Blaisdell Publishing Company, Waltham, Massachusetts, 1966.
3. J. Neveu, *Mathematical Foundations of the Calculus of Probability*, Holden-Day, Inc., 1965.

Received March 10, 1972.

INDIAN STATISTICAL INSTITUTE
CALCUTTA-35, INDIA

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)
University of California
Los Angeles, California 90024

J. DUGUNDJI*
Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT
University of Washington
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM
Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

* C. R. DePrima California Institute of Technology, Pasadena, CA 91109, will replace J. Dugundji until August 1974.

Pacific Journal of Mathematics

Vol. 48, No. 1

March, 1973

Jan Aarts and David John Lutzer, <i>Pseudo-completeness and the product of Baire spaces</i>	1
Gordon Owen Berg, <i>Metric characterizations of Euclidean spaces</i>	11
Ajit Kaur Chilana, <i>The space of bounded sequences with the mixed topology</i>	29
Philip Throop Church and James Timourian, <i>Differentiable open maps of $(p + 1)$-manifold to p-manifold</i>	35
P. D. T. A. Elliott, <i>On additive functions whose limiting distributions possess a finite mean and variance</i>	47
M. Solveig Espelie, <i>Multiplicative and extreme positive operators</i>	57
Jacques A. Ferland, <i>Domains of negativity and application to generalized convexity on a real topological vector space</i>	67
Michael Benton Freeman and Reese Harvey, <i>A compact set that is locally holomorphically convex but not holomorphically convex</i>	77
Roe William Goodman, <i>Positive-definite distributions and intertwining operators</i>	83
Elliot Charles Gootman, <i>The type of some C^* and W^*-algebras associated with transformation groups</i>	93
David Charles Haddad, <i>Angular limits of locally finitely valent holomorphic functions</i>	107
William Buhmann Johnson, <i>On quasi-complements</i>	113
William M. Kantor, <i>On 2-transitive collineation groups of finite projective spaces</i>	119
Joachim Lambek and Gerhard O. Michler, <i>Completions and classical localizations of right Noetherian rings</i>	133
Kenneth Lamar Lange, <i>Borel sets of probability measures</i>	141
David Lowell Lovelady, <i>Product integrals for an ordinary differential equation in a Banach space</i>	163
Jorge Martinez, <i>A hom-functor for lattice-ordered groups</i>	169
W. K. Mason, <i>Weakly almost periodic homeomorphisms of the two sphere</i>	185
Anthony G. Mucci, <i>Limits for martingale-like sequences</i>	197
Eugene Michael Norris, <i>Relationally induced semigroups</i>	203
Arthur E. Olson, <i>A comparison of c-density and k-density</i>	209
Donald Steven Passman, <i>On the semisimplicity of group rings of linear groups. II</i>	215
Charles Radin, <i>Ergodicity in von Neumann algebras</i>	235
P. Rosenthal, <i>On the singularities of the function generated by the Bergman operator of the second kind</i>	241
Arthur Argyle Sagle and J. R. Schumi, <i>Multiplications on homogeneous spaces, nonassociative algebras and connections</i>	247
Leo Sario and Cecilia Wang, <i>Existence of Dirichlet finite biharmonic functions on the Poincaré 3-ball</i>	267
Ramachandran Subramanian, <i>On a generalization of martingales due to Blake</i>	275
Bui An Ton, <i>On strongly nonlinear elliptic variational inequalities</i>	279
Seth Warner, <i>A topological characterization of complete, discretely valued fields</i>	293
Chi Song Wong, <i>Common fixed points of two mappings</i>	299