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**ON A GENERALIZATION OF MARTINGALES DUE TO BLAKE**

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# ON A GENERALIZATION OF MARTINGALES DUE TO BLAKE

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**It is shown that any uniformly integrable fairer with time game (stochastic process) converges in  $L_1$ .**

1. Introduction. Let  $(\Omega, \mathcal{U}, P)$  be a probability space and  $\{\mathcal{U}_n\}_{n \geq 1}$  an increasing family of sub  $\sigma$ -algebras of  $\mathcal{U}$ . Let  $\{X_n\}_{n \geq 1}$  be a stochastic process adapted to  $\{\mathcal{U}_n\}_{n \geq 1}$  (see, [2, p. 65]). Following Blake [1] we refer to  $\{X_n\}_{n \geq 1}$  as a game and define

DEFINITION. The game  $\{X_n\}_{n \geq 1}$  will be said to become *fairer with time* if for every  $\varepsilon > 0$

$$P[|E(X_n/\mathcal{U}_m) - X_m| > \varepsilon] \rightarrow 0$$

as  $n, m \rightarrow \infty$  with  $n \geq m$ . Any martingale is, trivially, a fairer with time game and thus this concept generalizes that of martingales. Blake, in [1], gave a set of sufficient conditions under which any uniformly integrable fairer with time game  $\{X_n\}_{n \geq 1}$  is convergent in  $L_1$ . We show that these sufficient conditions are not needed; in fact, we show that any uniformly integrable, fairer with time game converges in  $L_1$ .

2. THEOREM 2.1. *Any uniformly integrable fairer with time game  $\{X_n\}_{n \geq 1}$  converges in  $L_1$ .*

*Proof.* To facilitate understanding, we break up the proof into a few important steps numbered (S1) through (S5). For every  $m$  and  $n \geq m$  define  $Y_{m,n} = E(X_n/\mathcal{U}_m)$ . Let  $\Gamma$  stand for the family  $\{Y_{m,n}$ , for all  $m$  and  $n \geq m\}$ .

(S1)  $\Gamma$  is uniformly integrable.

Since  $\{X_n\}_{n \geq 1}$  is uniformly integrable there exists a function  $f$  defined on the nonnegative real axis which is positive, increasing and convex, such that

$$\lim_{t \rightarrow \infty} \frac{f(t)}{t} = +\infty$$

and  $\sup_n E[f \circ |X_n|] < \infty$ . (See [2, II T 22].) Now,

$$\begin{aligned} E[f \circ |Y_{m,n}|] &= E[f \circ |E(X_n/\mathcal{U}_m)|] \\ &\leq E[f \circ E(|X_n|/\mathcal{U}_m)] \quad (\text{since } f \text{ is nondecreasing}) \\ &\leq E[E(f \circ |X_n|/\mathcal{U}_m)] \\ &= E[f \circ |X_n|]. \end{aligned}$$

Therefore,

$$\sup_{Y_{m,n} \in \Gamma} E[f \circ | Y_{m,n} |] \leq \sup_n E[f \circ | X_n |] < \infty .$$

Another application of II T 22 of [2] ensures that  $\Gamma$  is uniformly integrable. Hence (S1).

(S2) Given  $\varepsilon > 0$ , there exists  $M$  such that for all  $m \geq M$ , one has

$$E(| X_m - Y_{m,n} |) \leq 2\varepsilon \text{ for all } n \geq m .$$

Since  $\Gamma$  is uniformly integrable given  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $P(A) < \delta$  implies  $\int_A | Y_{m,n} | dP \leq \varepsilon/2$ , for all  $Y_{m,n} \in \Gamma$ . Choose  $M$  so large that  $m \geq M$  and  $n \geq m$  implies  $P[| X_m - E(X_n/U_m) | > \varepsilon] < \delta$ . Then, it is not difficult to see that

$$E[| X_m - Y_{m,n} |] \leq 2\varepsilon \text{ for all } m \geq M \text{ and } n \geq m .$$

(S3) For every fixed  $m$ , the sequence  $\{Y_{m,n}\}$  converges in  $L_1$  to an  $\mathcal{U}_m$  measurable random variable  $Z_m$ .

Let  $m \leq n < n'$ .

$$\begin{aligned} E[| Y_{m,n} - Y_{m,n'} |] &= E[| E(X_n/\mathcal{U}_m) - E(X_{n'}/\mathcal{U}_m) |] \\ &= E[| E(X_n - X_{n'}/\mathcal{U}_m) |] \\ &= E[| E(\{E(X_n - X_{n'}/\mathcal{U}_n)\}/\mathcal{U}_m) |] \\ &\leq E[E(\{ | E(X_n - X_{n'}/\mathcal{U}_n) | \}/\mathcal{U}_m)] \\ &= E[| E(X_n - X_{n'}/\mathcal{U}_n) |] \\ &= E[| X_n - Y_{n,n'} |] . \end{aligned}$$

Now from (S2) it follows that given  $\varepsilon > 0$  for all sufficiently large  $n$  and  $n'$

$$E[| Y_{m,n} - Y_{m,n'} |] \leq E[| (X_n - Y_{n,n'}) |] \leq 2\varepsilon .$$

Hence, for  $m$  fixed, the sequence  $\{Y_{m,n}\}$  is Cauchy in the  $L_1$ -norm. So, there exists, an integrable random variable  $Z_m$ , such that,  $Y_{m,n} \xrightarrow[n \rightarrow \infty]{L_1} Z_m$ . Without loss of generality we can take  $Z_m$  to be  $\mathcal{U}_m$  measurable. (Note that each  $Y_{m,n}$  is  $\mathcal{U}_m$  measurable and there is a subsequence  $\{Y_{m,n'}\}$  converging almost surely to  $Z_m$ .)

(S4)  $\{Z_m, \mathcal{U}_m\}_{m \geq 1}$  is a uniformly integrable martingale.

The fact that  $\{Z_m\}_{m \geq 1}$  is uniformly integrable follows trivially because the closure in  $L_1$  of a uniformly integrable collection is uniformly integrable. (See, [2, II T20].) To show  $\{Z_m, \mathcal{U}_m\}$  is a martingale it is enough to show that for every  $m$ ,  $E(Z_{m+1}/\mathcal{U}_m) = Z_m$  a.s. Since

$$\begin{aligned}
E[|E(Y_{m+1,n}/\mathcal{U}_m) - E(Z_{m+1}/\mathcal{U}_m)|] \\
&= E[|E\{(Y_{m+1,n} - Z_{m+1})/\mathcal{U}_m\}|] \\
&\leq E[E\{|(Y_{m+1,n} - Z_{m+1})|/\mathcal{U}_m\}] \\
&= E[|Y_{m+1,n} - Z_{m+1}|] \longrightarrow 0 \text{ as } n \longrightarrow \infty,
\end{aligned}$$

there exists a subsequence  $n'$  of  $\{n: n \geq m\}$  such that

$$E(Y_{m+1,n'}/\mathcal{U}_m) \xrightarrow{\text{a.s.}} E(Z_{m+1}/\mathcal{U}_m).$$

We can assume (— if necessary, by choosing a further subsequence, —) that  $Y_{m,n'} \xrightarrow{\text{a.s.}} Z_m$ . Now,

$$\begin{aligned}
E(Z_{m+1}/\mathcal{U}_m) &= \lim_{n' \rightarrow \infty} E(Y_{m+1,n'}/\mathcal{U}_m) \text{ a.s.} \\
&= \lim_{n' \rightarrow \infty} E(\{E(X_{n'}/\mathcal{U}_{m+1})\}/\mathcal{U}_m) \text{ a.s.} \\
&= \lim_{n' \rightarrow \infty} E(X_{n'}/\mathcal{U}_m) \text{ a.s.} \\
&= \lim_{n' \rightarrow \infty} Y_{m,n'} \text{ a.s.} \\
&= Z_m \text{ a.s.}
\end{aligned}$$

Hence (S4). (S5)  $\{X_n\}_{n \geq 1}$  converges in  $L_1$ .

Since  $\{Z_n, \mathcal{U}_n\}_{n \geq 1}$  is an uniformly integrable martingale, there exists an integrable random variable  $Z_\infty$  such that  $Z_n \xrightarrow[n \rightarrow \infty]{L_1} Z_\infty$ . We shall show that  $X_n \xrightarrow[n \rightarrow \infty]{L_1} Z_\infty$ . From (S3) and (S2) it is easy to check that given  $\varepsilon > 0$  there exists  $M$  such that for all  $m \geq M$

$$\int |X_m - Z_m| dP \leq 2\varepsilon.$$

Therefore, for sufficiently large  $m$ ,

$$\int |X_m - Z_\infty| dP \leq \int |X_m - Z_m| dP + \int |Z_m - Z_\infty| dP \leq 3\varepsilon,$$

say. Hence (S5) and the theorem.

Since any game (stochastic process)  $\{X_n\}_{n \geq 1}$  converging in  $L_1$  can be taken to be a game fairer with time, by setting  $\mathcal{U}_n \equiv \mathcal{U}$  in  $n$ , we get the following corollary.

**COROLLARY 2.1.** *Let  $\{X_n\}_{n \geq 1}$  be a game. It converges in  $L_1$  if and only if it is uniformly integrable and fairer with time with respect to some increasing family of sub  $\sigma$ -algebras  $\{\mathcal{U}_n\}_{n \geq 1}$  to which it is adapted.*

Let  $p > 1$ .

**THEOREM 2.2.** *Let  $\{X_n\}_{n \geq 1}$  be a fairer with time game with  $\{\|X_n\|^p\}_{n \geq 1}$  uniformly integrable. Then  $\{X_n\}_{n \geq 1}$  converges in  $L_p$ .*

*Proof.* Noting that the function  $f$  defined on the nonnegative real axis by  $f(t) = t^p$  is positive, increasing and convex and  $\lim_{t \rightarrow \infty} (f(t)/t) = +\infty$ , in view of II T 22 of [2], it is clear that  $\{X_n\}_{n \geq 1}$  is uniformly integrable. Hence by Theorem 2.1 it converges in  $L_1$ ; in particular,  $\{X_n\}_{n \geq 1}$  converges in probability. Therefore,  $\{X_n\}_{n \geq 1}$  converges in  $L_p$ . (See Proposition II 6.1 of [3].)

**COROLLARY 2.2.** *The game  $\{X_n\}_{n \geq 1}$  converges in  $L_p$  if and only if  $\{\|X_n\|^p\}_{n \geq 1}$  is uniformly integrable and  $\{X_n\}_{n \geq 1}$  is fairer with time with respect to some increasing family of sub  $\sigma$ -algebras  $\{\mathcal{Z}_n\}_{n \geq 1}$  to which it is adapted.*

**REMARK.** In view of our Theorem 2.1, the second convergence theorem of Blake in [1] becomes redundant.

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