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The existence of bounded solutions of nonlinear elliptic variational inequalities is shown. The nonlinear second order elliptic operator involved has at most an exponential growth in u and a polynomial growth in Du. The regularity of the solutions is studied.

Let K be a closed convex subset of a reflexive Banach space V and let A be a pseudo-monotone coercive operator from V into V^*. Then for any f in V^* it is known that there exists u in K such that:

$$(Au - f, v - u) \geqslant 0 \qquad \text{for all } v \text{ in } K.$$

The existence of solutions of nonlinear elliptic variational inequalities has been shown by Brezis [1], Browder [4], Lions-Stampacchia [10] and others. The regularity of the solutions when A is a linear second order elliptic operator written in divergence form has been studied by Brezis [2], Lewy-Stampacchia [8], Lions [9] and an abstract regularity result has been obtained by Brezis-Stampacchia [3] when A is nonlinear.

It is the purpose of this paper to show the existence of *bounded* solutions of nonlinear variational inequalities when A is a pseudo-monotone coercive operator defined on $V \cap L^\infty(G)$ and mapping $V \cap L^\infty(G)$ into V^*. V is a given closed subspace of $W^{1,p}(G)$ with $W_0^{1,p}(G) \subset V \subset W^{1,p}(G)$. The nonlinear elliptic operator A may have an *exponential growth in* u and a polynomial growth in Du. Abstract existence theorems are proved in §2. The applications and the regularity of the solutions are studied in §3. The notations and the basic assumptions are given in §1.

1. Let G be a bounded open subset of R^n with a smooth boundary ∂G. Set: $D_j = \partial/\partial x_j, j = 1, \cdots, n$ and

$$W^{1,p}(G) = \{u: u \text{ in } L^p(G), D_j u \text{ in } L^p(G), j = 1, \cdots, n\}.$$

$W^{1,p}(G)$ is a real reflexive separable Banach space with the norm:

$$\|u\|_{1,p} = \left\{\|u\|_{L^p(G)}^p + \sum_{j=1}^n \|D_j u\|_{L^p(G)}^p\right\}^{1/p}; \ 2 \leqslant p < \infty.$$

$W_0^{1,p}(G)$ is the completion of $C_0^\infty(G)$, the family of all infinitely differentiable functions with compact support in G, in the $\|\cdot\|_{1,p}$ norm.

Let V be a closed linear subspace of $W^{1,p}(G)$ with $W_0^{1,p}(G) \subset V \subset W^{1,p}$ and let $H = L^2(G)$. The pairing between V and its dual V^* is denoted by $(\cdot, \cdot)$.

Let W be a closed linear subspace of $W^{1,p+n}(G)$ with $W_0^{1,p+n}(G) \subset W \subset W^{1,p+n}(G)$. The Sobolev imbedding theorem gives $W \subset C(\mathrm{cl}\, G)$. The pairing between W and W^* is denoted by $((\cdot, \cdot))$. Throughout the paper we shall assume:

(a) $W \subset V$.

(b)   If u is in W then $|u|^{s-2}u$ is also in W for all s with $2 \leqslant s < \infty$. The assumption is verified if $V = W_0^{1,p}(G)$, $W = W_0^{1,p+n}(G)$ and if $V = W^{1,p}(G)$ with $W = W^{1,p+n}(G)$.

$C^{0,\alpha}(G)$, $0 < \alpha < 1$, is the family of all functions u which are Hölder-continuous with exponent α on any compact subset of G. Set:

$$F = V \cap L^\infty(G) \,.$$

In this paper nonlinear operators $A(u, v)$ mapping $F \times V$ into V^* satisfying the following assumption are considered.

Assumption (I):   (i) $A(u, v)$ *maps bounded sets of* $F \times V$ *into bounded sets of* V^*.

(ii) *If* $u_n \to u$ *weakly in* V, $u_n \to u$ *in the weak*-topology of* $L^\infty(G)$ *and* $\limsup (A(u_n, u_n), u_n - u) \leqslant 0$ *then:*

$$\liminf (A(u_n, u_n), u_n - v) \geqslant (A(u, u), u - v)$$

$$\textit{for all } v \textit{ in } V \,.$$

The pseudo-monotone operators from V into V^* satisfy all the conditions of Assumption (I).

Let $A(u, v)$ be the operator defined by:

$$(A(u, v), w) = \sum_{j=1}^{n} \int_G (1 + c(x) \exp u)\,|D_j v|^{p-2} D_j v D_j w\, dx$$

$$+ \int_G b(x) u \exp u \cdot w\, dx$$

u is in F, v, w in V, $c(x)$ and $b(x)$ are two nonnegative bounded functions on G. $A(u, v)$ is the prototype of operators considered in this paper. It is not difficult to check that $A(u, v)$ satisfies all the conditions of Assumption (I).

In the proof of the existence theorems we need the following auxiliary nonlinear monotone operator:

$$A_2 v = - \sum_{j=1}^{n} D_j(|D_j v|^{p+n-2} D_j v) + v \,.$$

2. In this section the existence of bounded solutions of nonlinear elliptic variational inequalities is shown.

The following theorem gives a generalization of a result of Browder [4], Hartman-Stampacchia [6] and others when V is a closed linear subspace of $W^{1,p}(G)$.

THEOREM 1. *Let $A(u, v)$ be a nonlinear operator mapping $F \times V$ into V^* and satisfying Assumption* (I). *Let K be a closed convex subset of V with $0 \in K$ and let β be the penalty operator associated with K. Suppose that:*

(i) $(A(u, u), u) \geqslant c \|u\|_V^p$ *for all u in F.*

(ii) $(A(u, u), |u|^{s-2}u) \geqslant c \|u\|_{L^s(G)}^s$ *for all u in W and all s with $2 \leqslant s < \infty$. The positive constant c is independent of s.*

(iii) $(\beta(u), |u|^{s-2}u) \geqslant 0$ *for all u in W and all s, $2 \leqslant s < \infty$. Then for any f in $L^\infty(G)$ there exists u in $F \cap K$ such that:*

$$(A(u, u) - f, v - u) \geqslant 0 \qquad \text{for all } v \text{ in } K .$$

THEOREM 2. *Let $A(u, w)$ be as in Theorem 1 and let K be a closed convex subset of both V and W with $0 \in K$. Let β be the penalty operator associated with K considered as a subset of W. Suppose all the hypotheses of Theorem 1 are satisfied with* (iii) *replaced by:*

$$((\beta(u), u))/\|u\|_W \longrightarrow + \infty \quad as \quad \|u\|_W \longrightarrow + \infty .$$

Then for any f in $L^\infty(G)$ there exists u in $F \cap K$ such that:

$$(A(u, u) - f, v - u) \geqslant 0 \qquad \text{for all } v \text{ in } K .$$

For a smaller class of nonlinear operators $A(u, v)$ we have the following theorem which extends a result of Dubinskii [5].

THEOREM 3. *Let $A(u, v)$ be a nonlinear operator mapping bounded sets of $L^\infty(G) \times V$ into bounded sets of V^*. Suppose that:*

(i) $(A(u, v) - A(u, w), v - w) \geqslant 0$ *for all u in $L^\infty(G)$ and all v, w in V.*

(ii) *For fixed u in $L^\infty(G)$, $A(u, \cdot)$ is continuous from the strong topology of V to the weak topology of V^*.*

(iii) *For fixed v in V, if $u_n \to u$ a.e. on G and $u_n \to u$ in the weak*-topology of $L^\infty(G)$ then: $A(u_n, v) \to A(u, v)$ in V^*.*

(iv) $(A(|u|^{s-2}u, u), u) \geqslant c \|u\|_V^p$ *for all u in W and all s with $1 < s < \infty$.*

(v) $(A(|u|^{s-2}u, u), |u|^{r-2}u) \geqslant c \|u\|_{L^{r+s-2}(G)}^{r+s-2}$ *for all u in W with $1 < s < \infty$ and $2 \leqslant r < \infty$. The constant c is independent of r.*

Then for any f in $L^\infty(G)$ there exists u in $L^\infty(G)$ with $|u|^{\alpha-2}u$ in V such that:

$$A(u, |u|^{\alpha-2}u) = f .$$

α *is any number with* $1 < \alpha < \infty$.

Proof of Theorem 1.   (1)   Let $0 < \eta < \varepsilon < 1$ and let $\mathscr{A}(u, v)$ be the nonlinear operator:

$$\mathscr{A}(u, v) = \eta A_2 v + A(u, u) + \varepsilon^{-1}\beta(u) .$$

$\mathscr{A}(u, v)$ maps bounded sets of $W \times W$ into bounded sets of W^* and is coercive.   Being the sum of a monotone operator and a pseudo-monotone operator, $\mathscr{A}$ is a pseudo-monotone operator from W into W^*.   It follows from the theory of coercive pseudo-monotone operators that there exists $u_{\varepsilon\eta}$ in W, solution of the equation:

$$\eta A_2 u_{\varepsilon\eta} + A(u_{\varepsilon\eta}, u_{\varepsilon\eta}) + \varepsilon^{-1}\beta(u_{\varepsilon\eta}) = f .$$

It is clear that:

$$\eta||u_{\varepsilon\eta}||_W^{p+n} + ||u_{\varepsilon\eta}||_V \leqslant C .$$

C is a constant independent of both ε and η.
Since $u_{\varepsilon\eta}$ is in W, $|u_{\varepsilon\eta}|^{s-2}u_{\varepsilon\eta}$ lies also in W for all s with $2 \leqslant s < \infty$.
Thus:

$$\eta((A_2 u_{\varepsilon\eta}, |u_{\varepsilon\eta}|^{s-2}u_{\varepsilon\eta})) + (A(u_{\varepsilon\eta}, u_{\varepsilon\eta}), |u_{\varepsilon\eta}|^{s-2}u_{\varepsilon\eta})$$
$$+ \varepsilon^{-1}(\beta(u_{\varepsilon\eta}), |u_{\varepsilon\eta}|^{s-2}u_{\varepsilon\eta}) = (f, |u_{\varepsilon\eta}|^{s-2}u_{\varepsilon\eta}) .$$

An elementary computation gives:

$$((A_2 u_{\varepsilon\eta}, |u_{\varepsilon\eta}|^{s-2}u_{\varepsilon\eta})) \geqslant 0 .$$

From the hypotheses of the theorem, we get:

$$||u_{\varepsilon\eta}||_{L^s(G)} \leqslant C ||f||_{L^\infty(G)} .$$

Since $u_{\varepsilon\eta}$ is in W hence in $L^\infty(G)$ we may let $s \to +\infty$.
So:

$$||u_{\varepsilon\eta}||_{L^\infty(G)} \leqslant C ||f||_{L^\infty(G)} .$$

C is a constant independent of both ε and η.
   (2)   From the weak compactness of the unit ball in a reflexive Banach space we obtain by taking subsequences if necessary: $u_{\varepsilon\eta} \to u_\varepsilon$ weakly in V, $u_{\varepsilon\eta} \to u_\varepsilon$ in the weak*-topology of $L^\infty(G)$, $\eta^{1/p+n}u_{\varepsilon\eta} \to 0$ weakly in W, $A(u_{\varepsilon\eta}, u_{\varepsilon\eta}) \to g_\varepsilon$ weakly in V^* and $\beta(u_{\varepsilon\eta}) \to h_\varepsilon$ weakly in V^* as $\eta \to 0$.
   So:

$$||u_\varepsilon||_V + ||u_\varepsilon||_{L^\infty(G)} \leqslant C .$$

Moreover:

$$g_\varepsilon + \varepsilon^{-1}h_\varepsilon = f \ .$$

We show that $g_\varepsilon = A(u_\varepsilon, u_\varepsilon)$ and $h_\varepsilon = \beta(u_\varepsilon)$.
We have:

$$\limsup_{\eta \to 0} \ (A(u_{\varepsilon\eta}, u_{\varepsilon\eta}) + \varepsilon^{-1}\beta(u_{\varepsilon\eta}), u_{\varepsilon\eta}) \leqslant (g_\varepsilon + \varepsilon^{-1}h_\varepsilon, u_\varepsilon) \ .$$

On the other hand:

$$(A(u_{\varepsilon\eta}, u_{\varepsilon\eta}), u_{\varepsilon\eta} - u_\varepsilon) = (A(u_{\varepsilon\eta}, u_{\varepsilon\eta}) + \varepsilon^{-1}\beta(u_{\varepsilon\eta}), u_{\varepsilon\eta} - u_\varepsilon)$$
$$- \varepsilon^{-1}(\beta(u_{\varepsilon\eta}) - \beta(u_\varepsilon), u_{\varepsilon\eta} - u_\varepsilon) - \varepsilon^{-1}(\beta(u_\varepsilon), u_{\varepsilon\eta} - u_\varepsilon) \ .$$

Taking into account the monotonicity of the penalty operator, we get:

$$(A(u_{\varepsilon\eta}, u_{\varepsilon\eta}), u_{\varepsilon\eta} - u_\varepsilon) \leqslant (A(u_{\varepsilon\eta}, u_{\varepsilon\eta}) + \varepsilon^{-1}\beta(u_{\varepsilon\eta}), u_{\varepsilon\eta} - u_\varepsilon)$$
$$- \varepsilon^{-1}(\beta(u_\varepsilon), u_{\varepsilon\eta} - u_\varepsilon) \ .$$

Thus:

$$\limsup_{\eta \to 0} \ (A(u_{\varepsilon\eta}, u_{\varepsilon\eta}), u_{\varepsilon\eta} - u_\varepsilon) \leqq 0 \ .$$

Assumption (I) gives:

$$A(u_\varepsilon, u_\varepsilon) = g_\varepsilon \ .$$

It is now easy to show that $h_\varepsilon = \beta(u_\varepsilon)$.

(3) Again from the weak compactness of the unit ball in a reflexive Banach space we get by taking subsequences if necessary: $u_\varepsilon \to u$ weakly in V, $u_\varepsilon \to u$ in the weak*-topology of $L^\infty(G)$ and $A(u_\varepsilon, u_\varepsilon) \to g$ weakly in V^* as $\varepsilon \to 0$.

Since:

$$A(u_\varepsilon, u_\varepsilon) + \varepsilon^{-1}\beta(u_\varepsilon) = f$$

it is clear that

$$\beta(u_\varepsilon) \longrightarrow 0 \quad \text{in} \quad V^* \quad \text{as} \quad \varepsilon \longrightarrow 0 \ .$$

The penalty operator β is a monotone hemi-continuous operator from V into V^*. A standard argument gives:

$$\beta(u) = 0 \text{ i.e., } u \in K \ .$$

We have:

$$(A(u_\varepsilon, u_\varepsilon), u_\varepsilon - u) = (f - \varepsilon^{-1}\beta(u_\varepsilon), u_\varepsilon - u)$$
$$= (f, u_\varepsilon - u) - \varepsilon^{-1}(\beta(u_\varepsilon) - \beta(u), u_\varepsilon - u) \ .$$

Thus:

$$(A(u_\varepsilon, u_\varepsilon), u_\varepsilon - u) \leqq (f, u_\varepsilon - u) \,.$$

Hence:

$$\limsup_{\varepsilon \to 0} (A(u_\varepsilon, u_\varepsilon), u_\varepsilon - u) \leqq 0 \,.$$

Assumption (I) gives:

$$A(u, u) = g$$

and

$$\liminf_{\varepsilon \to 0} (A(u_\varepsilon, u_\varepsilon), u_\varepsilon - v) \geqslant (A(u, u), u - v) \,.$$

Let $v \in K$ and we have:

$$(A(u_\varepsilon, u_\varepsilon) - f, v - u_\varepsilon) = -\varepsilon^{-1}(\beta(u_\varepsilon), v - u_\varepsilon)$$
$$= \varepsilon^{-1}(\beta(v) - \beta(u_\varepsilon), v - u_\varepsilon) \,.$$

So:

$$(A(u_\varepsilon, u_\varepsilon) - f, v - u_\varepsilon) \geqslant 0 \qquad \text{for all } v \text{ in } K \,.$$

Let $\varepsilon \to 0$ and we obtain:

$$(A(u, u) - f, v - u) \geqq 0 \qquad \text{for all } v \text{ in } K \,.$$

The theorem is proved.

Proof of Theorem 2. The proof is similar to that of Theorem 1, we shall not reproduce it.

Proof of Theorem 3. (1) Let $0 < \varepsilon < 1$ and consider the nonlinear operator $\mathscr{A}(u, v)$ defined by:

$$\mathscr{A}(u, v) = \varepsilon A_2 v + A(|u|^{\alpha'-2}u, u)$$

u and v are in W and $1/\alpha + 1/\alpha' = 1$.

$\mathscr{A}$ is coercive and maps bounded sets of $W \times W$ into bounded sets of W^*. Being the sum of a monotone operator and a pseudo-monotone operator, $\mathscr{A}$ is pseudo-monotone. Therefore, there exists v_ε in W, solution of the equation:

$$\varepsilon A_2 v_\varepsilon + A(|v_\varepsilon|^{\alpha'-2}v_\varepsilon, v_\varepsilon) = f \,.$$

It is easy to show that:

$$\varepsilon \|v_\varepsilon\|_W^{p+n} + \|v_\varepsilon\|_V \leqslant C \,.$$

C is a constant independent of ε.

Since v_ε is in W, $|v_\varepsilon|^{s-2}v_\varepsilon$ lies in W for all s with $2 \leqslant s < \infty$. Thus:

$$\varepsilon((A_2 v_\varepsilon, |v_\varepsilon|_\varepsilon^{s-2}v_\varepsilon)) + (A(|v_\varepsilon|^{\alpha'-2}v_\varepsilon, v_\varepsilon), |v_\varepsilon|^{s-2}v_\varepsilon) = (f, |v_\varepsilon|^{s-2}v_\varepsilon).$$

An elementary computation shows that the first term of the left hand side of the above equation is positive. It follows from the hypotheses of the theorem that:

$$\|v_\varepsilon\|_{L^{s+\alpha'-2}(G)}^{\alpha'-1} \leqslant C \|f\|_{L^\infty(G)}.$$

Since v_ε is in W we may let $s \to +\infty$. Hence:

$$\|v_\varepsilon\|_{L^\infty(G)} \leqslant M.$$

M is independent of ε.

(2) The weak compactness of the unit ball in a reflexive Banach space gives by taking subsequences if necessary: $v_\varepsilon \to v$ weakly in V, $v_\varepsilon \to v$ in the weak*-topology of $L^\infty(G)$, $\varepsilon^{1/p+n}v_\varepsilon \to 0$ weakly in W, $|v_\varepsilon|^{\alpha'-2}v_\varepsilon \to h$ weakly in $L^\alpha(G)$ and

$$A(|v_\varepsilon|^{\alpha'-2}v_\varepsilon, v_\varepsilon) \longrightarrow g \quad \text{weakly in} \quad V^* \quad \text{as} \quad \varepsilon \longrightarrow 0.$$

Since the injection mapping of V into H is compact, the Lebesgue convergence theorem yields:

$$v_\varepsilon \longrightarrow v \quad \text{in} \quad L^s(G) \quad \text{for any } s \text{ with } 1 < s < \infty.$$

Thus:

$$|v_\varepsilon|^{\alpha'-2}v_\varepsilon \longrightarrow |v|^{\alpha'-2}v \text{ a.e. on } G \quad \text{and} \quad h = |v|^{\alpha'-2}v.$$

We have:

$$\limsup (A(|v_\varepsilon|^{\alpha'-2}v_\varepsilon, v_\varepsilon), v_\varepsilon) \leqslant (g, v).$$

So:

$$\limsup (A(|v_\varepsilon|^{\alpha'-2}v_\varepsilon, \varphi), v_\varepsilon - \varphi) \leqslant (g, v - \varphi)$$
$$\text{for all } \varphi \text{ in } V.$$

It follows from the hypotheses of the theorem that:

$$(g - A(|v|^{\alpha'-2}v, \varphi), v - \varphi) \geqslant 0 \qquad \text{for all } \varphi \text{ in } V.$$

Hence:

$$A(|v|^{\alpha'-2}v, v) = g = f.$$

Set:

$$u = |v|^{\alpha'-2}v.$$

Then:

$$u \in L^{\infty}(G) \quad \text{and} \quad v = |u|^{\alpha-2}u \in V \, .$$

Moreover:

$$A(u, |u|^{\alpha-2}u) = f \, .$$

The theorem is proved.

3.   We shall now give some applications of the theorems proved in the previous section and study the regularity of the solutions. First consider the case when the constraint is in G.

THEOREM 4.   *Let* $V = W_0^{1,p}(G)$, $K = \{u: u \text{ in } L^p(G), u \geqslant 0 \text{ a.e. on } G\}$ *and let* $A(u, v) = -\sum_{j=1}^{n} D_j A_j(x, u, Dv)$.
Suppose that:
   (i)   *For each* j, $A_j(x, u, Dv)$ *is continuous with respect to* x, u, Dv.
   (ii) $|A_j(x, u, Dv)| \leqslant C\{1 + \exp(|u|) + |Dv|^{p-1} \exp(|u|)\}$.
   (iii) $\sum_{j=1}^{n} (A_j(x, u, Dv) - A_j(x, u, Dw))(D_j v - D_j w) \geqslant 0$.
   (iv) $\sum_{j=1}^{n} A_j(x, u, Du)D_j u \geqslant c \sum_{j=1}^{n} |D_j u|^p$.
Then for any f *in* $L^{\infty}(G)$ *there exists* u *in* $K \cap V \cap L^{\infty}(G)$ *such that:*

$$(A(u, u) - f, v - u) \geqslant 0 \quad \text{for all } v \text{ in } V \cap K \, .$$

Moreover:

$$A(u, u) \in L^{\infty}(G) \, .$$

Proof.   (1) K is a closed convex subset of V and $0 \in K$. Let $\beta(v) = -|v^-|^{p-2}v^-$ where $v^- = 0$ if $v(x) \geqslant 0$ and $v^- = -v(x)$ otherwise. It is easy to check that β verifies the hypothesis of Theorem 1. Let $0 < \eta < 1$ and consider the operator:

$$A_1(u, v) = A(u, v) + \eta u \, .$$

$A_1(u, v)$ and $\eta A_2 v + A_1(u, v)$ satisfy all the hypotheses of Theorem 1. Therefore, from Theorem 1, we get:

$$\eta A_2 u_{\varepsilon\eta} + A_1(u_{\varepsilon\eta}, u_{\varepsilon\eta}) + \varepsilon^{-1}\beta(u_{\varepsilon\eta}) = f \, . \quad 0 < \eta < \varepsilon < 1 \, .$$

Moreover, from the proof of Theorem 1 we have:

$$\eta \|u_{\varepsilon\eta}\|_W^{p+n} + \|u_{\varepsilon\eta}\|_V + \eta \|u_{\varepsilon\eta}\|_{L^{\infty}(G)} \leqslant C \, .$$

C is a constant independent of both ε and η.
   Since $u_{\varepsilon\eta}$ is in W and hence in $C(\text{cl } G)$ we get:

$$\operatorname*{ess\,sup}_{\partial G} |u_{\varepsilon\eta}| = 0 \, .$$

It follows from Theorem 7.1 of [7], p. 287 that $\|u_{\varepsilon\eta}\|_{L^\infty(G)}$ is majorized by $\|u_{\varepsilon\eta}\|_V$, c, the $L^\infty(G)$-norm of f.
Hence:

$$\|u_{\varepsilon\eta}\|_{L^\infty(G)} \leqslant C .$$

C is again a constant independent of both ε and η.

(2) We have:

$$\eta((A_2 u_{\varepsilon\eta}, - |u_{\varepsilon\eta}^-|^{s-2} u_{\varepsilon\eta}^-)) + (A_1(u_{\varepsilon\eta}, u_{\varepsilon\eta}), - |u_{\varepsilon\eta}^-|^{s-2} u_{\varepsilon\eta}^-)$$
$$+ \varepsilon^{-1} \|u_{\varepsilon\eta}^-\|_{L^{s+p-2}(G)}^{s+p-2} \leqslant \|f\|_{L^\infty(G)} \|u_{\varepsilon\eta}^-\|_{L^{s+p-2}(G)}^{s-1} .$$

s is any number with $2 \leqslant s < \infty$.

It is not difficult to see that the first two expressions of the left hand side of the above inequality are nonnegative.
Hence:

$$\varepsilon^{-1} \|u_{\varepsilon\eta}^-\|_{L^{s+p-2}(G)}^{p-1} \leqslant \|f\|_{L^\infty(G)} .$$

Since $u_{\varepsilon\eta}^- \in L^\infty(G)$, we may let $s \to +\infty$ and get:

$$\varepsilon^{-1} \|u_{\varepsilon\eta}^-\|_{L^\infty(G)}^{p-1} \leqslant C .$$

C is independent of both ε and η.

(3) Let $\eta \to 0$ and the proof of Theorem 1 gives:

$$A(u_\varepsilon, u_\varepsilon) + \varepsilon^{-1} \beta(u_\varepsilon) = f .$$

Moreover:

$$\|u_\varepsilon\|_{L^\infty(G)} + \|u_\varepsilon\|_V + \varepsilon^{-1} \|u_\varepsilon^-\|_{L^\infty(G)}^{p-1} \leqslant C .$$

u_ε is the weak limit in V of $u_{\varepsilon\eta}$ as $\eta \to 0$.
Let $\varepsilon \to 0$ then again the proof of Theorem 1 gives:

$$(A(u, u) - f, v - u) \geqslant 0 \qquad \text{for all } v \text{ in } K\, V .$$

Since $\varepsilon^{-1} \beta(u_\varepsilon) \to g$ in the weak*-topology of $L^\infty(G)$, we have:

$$A(u, u) = f - g .$$

The theorem is proved.

With some further hypotheses on A_j, we have a regularity result.

THEOREM 5.   *Let* V, K *be as in Theorem 4 and let*

$$A(u, v) = - \sum_{j=1}^{n} D_j A_j(x, u, Dv) .$$

Suppose that:

(i) $A_j(x, u, Dv)$ *is continuously differentiable with respect to* x, *u and Dv.*

(ii) $(1 + |Du|) (|A_j(x, u, Du)| + |\partial A_j/\partial u| + |A_{jk}(x, u, Du)|) + |\partial A_j/\partial x_k| \leqq C(1 + \exp(|u|) + |Du|^{p-1} \exp(|u|)).$

$$A_{jk}(x, u, Du) = \partial A_j(x, u, Du)/\partial(D_k u) . \quad j, k = 1, \cdots, n .$$

(iii) $\sum_{j,k=1}^{n} A_{jk}(x, u, Dv) D_j v D_k v \geqslant c \sum_{j=1}^{n} (1 + |D_j u|^{p-2}) |D_j v|^2.$

Then for any f in $L^\infty(G)$, *there exists u in* $K \cap V \cap L^\infty(G)$ *such that:*

$$(A(u, u) - f, v - u) \geqslant 0 \qquad \textit{for all v in } K V .$$

Moreover:

$$u \in C^{0,\alpha}(G) \cap C^{1,\lambda}(G') \cap W^{2,p'}(G) .$$

$0 < \alpha, \gamma < 1$ *and* G' *is any subset of* G *with* $\operatorname{cl} G' \subset G$.

Proof. With the above hypotheses on A_j, $A(u, v)$ is a semi-monotone operator and satisfies all the hypotheses of Theorem 4. Therefore, there exists u in $K \cap V \cap L^\infty(G)$ such that:

$$(A(u, u) - f, v - u) \geqslant 0 \qquad \text{for all } v \text{ in } V .$$

Moreover:

$$A(u, u) = f - g \in L^\infty(G) .$$

From the theory of second order elliptic equations, we get:

$$u \in C^{0,\alpha}(G) .$$

E.g. cf. [7] Theorem 1.1, p. 251.

It has been shown by Dubinskii [5] that the solution of $A(u, u) = f - g$ is in $W^{2,p'}(G)$ and moreover:

$$\sum_{j,k=1}^{n} \int_{G'} (1 + |\operatorname{grad} u|)^{p-2} |D_j D_k u|^2 dx < \infty .$$

The above relation together with the Hölder-continuity of u gives:

$$\int_{G'} |\operatorname{grad} u|^{p+2} dx < \infty .$$

G' is any subset of G with $\operatorname{cl} G' \subset G$.

It follows from [7] that $\operatorname{ess\,sup}_{G'} |\operatorname{grad} u| < \infty$ and therefore Theorem 6.2 of [7], p. 282 gives:

$$u \in C^{1,\lambda}(G') \cap W^{2,2}(G') .$$

The theorem is proved.

If the principal part of A is linear, stronger results could be obtained.

PROPOSITION 1.   *Let V and K be as in Theorem 4 with $p = 2$. Let:*

$$A(u, v) = -\sum_{j,k=1}^{n} D_j(a_{jk}(x)D_k v) + u \exp(u).$$

Suppose that:
- (i) $a_{jk} \in C^2 \,(\mathrm{cl}\,G)$.
- (ii) $\sum_{j,k=1}^{n} a_{jk} D_j v D_k v \geqslant c \sum_{j=1}^{n} |D_j v|^2$.

Then for any f in $L^\infty(G)$ there exists u in $K \cap V \cap W^{2,s}(G)$ such that:

$$(A(u, u) - f, v - u) \geqslant 0 \qquad \text{for all } v \text{ in } K\,V.$$

s is any number with $1 < s < \infty$.

Proof.   We may write

$$A(u, v) = \mathscr{A}v + u \exp(u).$$

$A(u, v)$ satisfies all the hypotheses of Theorem 4. Thus there exists u in $L^\infty(G) \cap K \cap V$ such that:

$$(A(u, u) - f, v - u) \geqslant 0 \qquad \text{for all } v \text{ in } K\,V.$$

Moreover:

$$\mathscr{A}u + u \exp(u) = g \in L^\infty(G).$$

So:

$$\mathscr{A}u = g - u \exp(u) \in L^\infty(G).$$

It follows from the theory of linear elliptic operators that $u \in W^{2,s}(G)$ for any s with $1 < s < \infty$.

THEOREM 6.   *Let $V = W_0^{1,p}(G)$, $K = \{u: u \text{ in } V, |\operatorname{grad} u| \leqslant 1 \text{ a.e. on } G\}$ and let $A(u, v)$ be as in Theorem 4.   Then for any f in $L^\infty(G)$ and for any $\lambda > 0$, there exists u in $K \cap L^\infty(G)$ such that:*

$$(A(u, u) + \lambda u - f, v - u) \geqslant 0 \qquad \text{for all } v \text{ in } K.$$

Proof. K is a closed convex subset of both V and W with $0 \in K$. Let

$$\beta(v) = -\sum_{j=1}^{n} D_j\{(1 - |\operatorname{grad} v|^{p+n-2})^- D_j v\}.$$

β maps bounded sets of $W = W_0^{1,p+n}(G)$ into bounded sets of W^* and is a monotone hemi-continuous operator. Cf. Lions [9]. Moreover:

$$((\beta(v), v)) \geqslant \|v\|_W^{p+n} - (c_1 + c_2 \|v\|_V^{p+n-2}) .$$

It is not difficult to check that β satisfies all the hypotheses of Theorem 2. Thus the result follows from Theorem 2.

Along the lines of Theorem 2, we may consider the case when

$$K = \{u : u \text{ in } W_0^{1,p}(G), u \geqslant 0 \text{ a.e. on } G, |\operatorname{grad} u| \leqslant 1 \text{ a.e. on } G\} .$$

It suffices to apply Theorem 2 with

$$\beta(v) = - |v^-|^{p-2}v^- - \sum_{j=1}^{n} D_j\{(1 - |\operatorname{grad} v|^{p+n-2})^- D_j v\} .$$

We shall now consider the case when constraint imposed on the solution is at the boundary of G.

THEOREM 7. *Let* $V = W^{1,p}(G)$, $W = W^{1,p+n}(G)$, $K = \{u : u \text{ in } V, u \geqslant 0 \text{ a.e. on } \partial G\}$ *and let* $A(u, v)$ *be as in Theorem 5. Then for any* $\lambda > 0$ *and for any* f *in* $L^\infty(G)$ *there exists* $u \in C^{0,\alpha}(G) \cap C^{1,\mu}(G') \cap K$ *such that:*

$$(A(u, u) - f, v - u) \geqslant 0 \qquad \text{for all } v \text{ in } K .$$

G' *is any subset of* G *with* $\operatorname{cl} G' \subset G$ *and* $0 < \alpha, \mu < 1$.

Proof. (1) Let β be the penalty operator defined by:

$$(\beta(v), \varphi) = - \int_{\partial G} v^- \varphi \, dx , \qquad\qquad \varphi \text{ in } V .$$

β satisfies all the hypotheses of Theorem 1. Cf. Lions [9]. Let $0 < \varepsilon < 1$, then from Theorem 1 we have:

$$A(u_\varepsilon, u_\varepsilon) + \lambda u_\varepsilon = f \text{ on } G, \partial u_\varepsilon/\partial \gamma_A = - \varepsilon^{-1} u_\varepsilon^- \text{ on } \partial G .$$

Moreover, from the proof of Theorem 1 we get:

$$\|u_\varepsilon\|_V + \|u_\varepsilon\|_{L^\infty(G)} \leqslant C .$$

C is a constant independent of ε.

(2) Let $\varepsilon \to 0$ then:

$$(A(u, u) + \lambda u - f, v - u) \geqslant 0 \qquad \text{for all } v \in K .$$

Moreover:

$$A(u, u) + \lambda u = f \text{ on } G .$$

u is the weak limit in V of u_ε as $\varepsilon \rightarrow 0$.

Thus:

$$A(u, u) = f - \lambda u \in L^\infty(G) .$$

Hence:

$$u \in C^{0,\alpha}(G) .$$

A standard argument of the theory of elliptic operators gives:

$$\sum_{j,k=1}^{n} \int_{G'} (1 + |\operatorname{grad} u|^{p-2}) |D_j D_k u|^2 dx < \infty .$$

G' is any subset of G with $\operatorname{cl} G' \subset G$.
The Hölder-continuity of u together with the above relation implies that:

$$\int_{G'} |\operatorname{grad} u|^{p+2} dx < \infty .$$

Thus:

$$u \in C^{1,\mu}(G'') \quad \text{where} \quad \operatorname{cl} G'' \subset G' \subset \operatorname{cl} G' \subset G .$$

The theorem is proved.

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