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**A TOPOLOGICAL CHARACTERIZATION OF COMPLETE,  
DISCRETELY VALUED FIELDS**

SETH WARNER

# A TOPOLOGICAL CHARACTERIZATION OF COMPLETE, DISCRETELY VALUED FIELDS

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**It is shown that the topology of a topological field  $F$  is given by a complete, discrete valuation if and only if  $F$  is locally strictly linearly compact. More generally, the topology of a topological division ring  $K$  is given by a complete, discrete valuation and  $K$  is finite dimensional over its center if and only if  $K$  is locally centrally linearly compact, that is, if and only if  $K$  contains an open subring  $B$ , the open left ideals of which form a fundamental system of neighborhoods of zero, such that  $B$ , regarded as a module over its center, is strictly linearly compact.**

In [5], Jacobson showed that the topology of an indiscrete, totally disconnected, locally compact division ring is given by a discrete valuation (that is, a valuation whose value group is isomorphic to the cyclic group of integers). Consequently, an indiscrete topological division ring  $K$  is locally compact and totally disconnected if and only if its topology is given by a complete, discrete valuation whose residue field is finite [4, Prop. 2, p. 118, Prop. 1, p. 156]. From this, it follows rather readily that the center  $C$  of  $K$  is indiscrete, that  $K$  is finite dimensional over  $C$ , and that  $C$  is either a finite extension of the  $p$ -adic number field for some prime  $p$  or the field of formal power series over a finite field [4, Theorem 1, p. 158].

Our purpose here is to generalize Jacobson's theorem by characterizing those topological fields whose topology is given by a complete, discrete valuation, and more generally, those topological division rings  $K$  such that  $K$  is finite dimensional over its center and the topology of  $K$  is given by a complete, discrete valuation.

For this purpose, we assume some familiarity with basic properties of linearly compact and strictly linearly compact modules and rings, as developed in [10] or [3, Exercises 14-22, pp. 108-112]. We recall that a (left) topological  $A$ -module  $E$  (it is not assumed that  $E$  is unitary) is *linearly topologized* if the open submodules of  $E$  form a fundamental system of neighborhoods of zero;  $E$  is *linearly compact* if  $E$  is Hausdorff, linearly topologized, and every filter base of cosets of submodules has an adherent point;  $E$  is *strictly linearly compact* if  $E$  is linearly compact and every continuous epimorphism from  $E$  onto a Hausdorff, linearly topologized  $A$ -module is open (equivalently, if  $E/U$  is an artinian  $A$ -module for every open submodule  $U$  of  $E$ ). A topological ring  $A$  is respectively linearly topologized, linearly compact,

or strictly linearly compact if the associated left  $A$ -module  $A$  is.

DEFINITION. A topological ring  $A$  is *locally strictly linearly compact* if  $A$  has an open subring  $B$  that is strictly linearly compact for its induced topology.

To handle the noncommutative case, we need the following definition:

DEFINITION. A topological ring  $B$  is *centrally linearly compact* if the open left ideals of  $B$  form a fundamental system of neighborhoods of zero and if  $B$ , regarded as a module over its center  $C_B$ , is a strictly linearly compact  $C_B$ -module. A topological ring  $A$  is *locally centrally linearly compact* if  $A$  contains an open subring that is centrally linearly compact for its induced topology.

Thus a commutative topological ring is (locally) centrally linearly compact if and only if it is (locally) strictly linearly compact. Note that if  $B$  is a centrally linearly compact ring, then for any subring  $B_0$  of  $B$  that contains the center  $C_B$ ,  $B$  is a strictly linearly compact  $B_0$ -module (in particular,  $B$  is a strictly linearly compact ring); indeed, since the open left ideals of  $B$  form a fundamental system of neighborhoods of zero,  $B$  is a linearly topologized  $B_0$ -module, and since a  $B_0$ -submodule is also a  $C_B$ -submodule, every filter base of cosets of  $B_0$ -submodules necessarily has an adherent point.

By a topological division ring (field)  $K$  we mean a topological ring that is algebraically a division ring (field); that is, we do not assume that  $x \mapsto x^{-1}$  is continuous on the set  $K^*$  of nonzero elements.

LEMMA 1. *If  $B$  is an open, centrally linearly compact subring of an indiscrete topological division ring  $K$ , then there is an open, centrally linearly compact subring  $B_1$  of  $K$  that contains 1.*

*Proof.* Let  $B_1$  be the closure of the subring generated by  $B$  and 1. The open left ideals of  $B$  then form a fundamental system of neighborhoods of zero in  $B_1$ ; each open left ideal  $\alpha$  of  $B$  is a left ideal of  $B_1$ , for as  $\alpha$  is closed,  $\{x \in B_1: x\alpha \subseteq \alpha\}$  is a closed subring of  $B_1$  containing  $B$  and 1 and hence is all of  $B_1$ .

Since  $B$  is open,  $B \neq (0)$ ; let  $b$  be some nonzero element of  $B$ , and let  $c$  be its inverse in  $K$ . Then,  $B_1 = B_1bc \subseteq B_1Bc$ , so  $B_1 \subseteq Bc$  since, as we saw above,  $B$  is a left ideal of  $B_1$ . Thus  $Bc \supseteq B_1 \supseteq B$ , so  $Bc$  is a linearly topologized  $C_B$ -module, where  $C_B$  is the center of  $B$ . Hence  $Bc$  is a strictly linearly compact  $C_B$ -module, as it is the image of the strictly linearly compact  $C_B$ -module  $B$  under the continuous homomorphism  $x \mapsto xc$ . Consequently, the closed  $C_B$ -submodule

$B_1$  of  $B_c$  is strictly compact; as  $C_b$  is contained in the center of  $B_1$ ,  $B_1$  is *a fortiori* strictly linearly compact over its center.

We recall that an element  $a$  of a topological ring is *topologically nilpotent* if  $\lim a^n = 0$ .

**LEMMA 2.** *Let  $K$  be a Hausdorff topological division ring, let  $B$  be an open subring of  $K$  that contains 1, and let  $\mathfrak{r}$  be the radical of  $B$ . If  $B$  is strictly linearly compact, then  $B$  is a (left) noetherian ring,  $B/\mathfrak{r}$  is a division ring, the topology of  $B$  is the  $\mathfrak{r}$ -adic topology, and  $\mathfrak{r}$  is the set of all topological nilpotents of  $B$ .*

*Proof.* As  $B$  is open and as  $y \mapsto yx$  is a homeomorphism for each  $x \in K^*$ ,  $Bx$  is open for every  $x \in K^*$ , and hence every nonzero left ideal of  $B$  is open. Let  $\mathfrak{s} = \bigcap_{n=1}^{\infty} \mathfrak{r}^n$ . Assume that  $\mathfrak{s} \neq (0)$ . Then  $\mathfrak{s}$  is open, so  $B/\mathfrak{s}$  is an artinian  $B$ -module and hence an artinian ring. Consequently, its radical  $\mathfrak{r}/\mathfrak{s}$  is nilpotent, so there exists  $n$  such that  $\mathfrak{r}^n = \mathfrak{s}$ . Hence  $(0) \neq \mathfrak{r}^n = \mathfrak{r}^{n+1} = \dots$ , in contradiction to [10, Theorem 9]. Therefore,  $\bigcap_{n=1}^{\infty} \mathfrak{r}^n = (0)$ .

Since every nonzero left ideal of  $B$  is open and hence closed,  $B$  is a (left) noetherian ring,  $B/\mathfrak{r}$  is an artinian ring, and the topology of  $B$  is its  $\mathfrak{r}$ -adic topology by [13, Theorem 16]. Consequently, every element of  $\mathfrak{r}$  is a topological nilpotent. Therefore, as  $B$  is complete,  $B$  is suitable for building idempotents [11, Lemma 4; 6, Definition 1, p. 53]. Thus every idempotent of  $B/\mathfrak{r}$  is the coset of  $\mathfrak{r}$  determined by an idempotent of  $B$  [6, Proposition 4, p. 54]. But as  $K$  is a division ring,  $B$  has no idempotents other than 0 and 1. Thus  $B/\mathfrak{r}$  is an artinian, semisimple ring whose only idempotents are 0 and 1. By the Wedderburn-Artin theorem, therefore,  $B/\mathfrak{r}$  is a division ring. In particular, if  $x \notin \mathfrak{r}$ , then  $x + \mathfrak{r}$  is not a nilpotent of  $B/\mathfrak{r}$ , so  $x$  is not a topological nilpotent.

**THEOREM 1.** *If  $K$  is an indiscrete, Hausdorff topological field, then the topology of  $K$  is given by a complete, discrete valuation if and only if  $K$  is locally strictly linearly compact.*

*Proof. Necessity.* It is well known that a complete, semilocal noetherian ring, equipped with its natural  $\mathfrak{r}$ -adic topology, is strictly linearly compact [cf. 13, Corollary of Lemma 2]. In particular, the valuation ring of a complete discrete valuation is strictly linearly compact.

*Sufficiency.* By Lemma 1, there is an open, strictly linearly compact subring  $B$  of  $K$  that contains 1. By Lemma 2,  $B$  is a complete, local noetherian domain, and its topology is its natural  $\mathfrak{m}$ -adic topology, where  $\mathfrak{m}$  is the maximal ideal of  $B$ . In particular,  $B$  is not

a field since  $B$  is not discrete. Therefore, as  $B$  is open in the topological field  $K$ , the topology of  $K$  is defined by a complete, discrete valuation [12, Theorem 6].

**THEOREM 2.** *If  $K$  is an indiscrete, Hausdorff topological division ring, then the topology of  $K$  is given by a complete, discrete valuation and  $K$  is finite-dimensional over its center  $C$  if and only if  $K$  is locally centrally linearly compact; in this case,  $C$  is indiscrete, and hence its topology is given by a complete, discrete valuation.*

*Proof. Necessity.* As  $K$  is finite-dimensional over  $C$ , the valuation induced on  $C$  by that of  $K$  is not the improper valuation; hence as  $C$  is closed, the topology of  $C$  is given by a complete, discrete valuation  $v$ . Let  $e_1, \dots, e_n$  be a basis of  $K$  over  $C$  such that  $e_1 = 1$ , and let  $e_i e_j = \sum_{k=1}^n \alpha_{ijk} e_k$ . Let  $\lambda \in C$  be such that  $v(\lambda) \geq 0$  and  $v(\lambda) \geq -\min \{v(\alpha_{ijk}) : 1 \leq i, j, k \leq n\}$ . Let  $f_1 = 1$  and  $f_k = \lambda e_k$  for  $2 \leq k \leq n$ . Let  $V$  be the valuation ring of  $C$ , and for each  $m \geq 0$  let  $V_m = \{x \in V : v(x) \geq m\}$ . Let  $B = Vf_1 + \dots + Vf_n$ , and for each  $m \geq 0$  let  $b_m = V_m f_1 + \dots + V_m f_n$ . Easy calculations establish that  $B$  is a ring and that  $b_m$  is an ideal of  $B$  for each  $m \geq 0$ . By [2, Theorem 2, p. 18],  $F: (\lambda_1, \dots, \lambda_n) \mapsto \sum_{i=1}^n \lambda_i f_i$  is a topological isomorphism from the  $C$ -vector space  $C^n$  onto the  $C$ -vector space  $K$ . Hence  $B$  is an open subring of  $K$ , and  $(b_m)_{m \geq 0}$  is a fundamental system of neighborhoods of zero in  $B$ , each an ideal of  $B$ . We saw earlier that  $V$  is strictly linearly compact; hence as  $B = F(V^n)$ ,  $B$  is a strictly linearly compact  $V$ -module and, *a fortiori*, is a centrally linearly compact ring.

*Sufficiency.* By Lemma 1, there is an open, centrally linearly compact subring  $B$  that contains 1. Let  $\mathfrak{r}$  be the radical of  $B$ . As the  $\mathfrak{r}$ -adic topology is the given indiscrete topology of  $B$  by Lemma 2, there exists a nonzero  $a \in B$  such that  $\lim a^n = 0$ . Let  $K_0$  be the closed subfield generated by  $C$  and  $a$ , let  $B_0 = K_0 \cap B$ , and let  $\mathfrak{r}_0$  be the radical of  $B_0$ . Since the open left ideals of  $B$  form a fundamental system of neighborhoods of zero for  $B$ , the open ideals of  $B_0$  form a fundamental system of neighborhoods of zero for  $B_0$ . Moreover, the center  $C_B$  of  $B$  is simply  $C \cap B$ ; indeed, if  $c \in C_B$  and if  $x \in K$ , then  $a^n x \in B$  for some  $n$  as  $\lim a^n x = 0$ , whence  $(a^n x)c = c(a^n x) = (ca^n)x = (a^n c)x$ , so  $xc = cx$ . Thus  $C_B = C \cap B \subseteq K_0 \cap B = B_0$ , so  $B_0$  is a closed  $C_B$ -submodule of  $B$  and hence is a strictly linearly compact  $C_B$ -module. Consequently,  $B_0$  is a strictly linearly compact ring, so by Lemma 2, the topology of  $B_0$  is the  $\mathfrak{r}_0$ -adic topology, and  $\mathfrak{r}$  and  $\mathfrak{r}_0$  are respectively the set of topological nilpotents in  $B$  and  $B_0$ , whence  $\mathfrak{r}_0 = \mathfrak{r} \cap B_0$ . Hence  $\bigcap_{n=1}^{\infty} (\mathfrak{r}_0^n B) \subseteq \bigcap_{n=1}^{\infty} \mathfrak{r}^n = (0)$ . As the topology of  $B_0$  is indiscrete,  $\mathfrak{r}_0^n \neq (0)$ , so  $\mathfrak{r}_0^n B$  is open as it contains a nonzero left ideal of  $B$ . By

[13, Theorem 10],  $r_0B$  is a finitely generated  $B_0$ -module; let  $r_0B = B_0x_1 + \cdots + B_0x_m$ . Also as  $B$  is a strictly linearly compact  $C_B$ -module and as  $r_0B$  is open,  $B/r_0B$  is an artinian  $C_B$ -module, hence an artinian  $B_0$ -module; now  $B/r_0B$  admits the structure of  $B_0/r_0B$ -module, and  $B_0/r_0B$  is a field by Lemma 2; consequently  $B/r_0B$  is an artinian, therefore, finite-dimensional, and hence noetherian  $B_0/r_0B$ -vector space; thus  $B/r_0B$  is a noetherian  $B_0$ -module. Let  $x_{m+1}, \dots, x_n \in B$  be such that  $B = B_0x_{m+1} + \cdots + B_0x_n + r_0B$ . Then  $B = B_0x_1 + \cdots + B_0x_n$ . Consequently,  $x_1, \dots, x_n$  is a set of generators of the  $K_0$ -vector space  $K$ , for if  $z \in K$ , there exists  $t$  such that  $a^t z \in B$ , whence  $a^t z = b_1x_1 + \cdots + b_nx_n$  where  $b_i \in B_0$ , and thus  $z = (a^{-t}b_1)x_1 + \cdots + (a^{-t}b_n)x_n \in K_0x_1 + \cdots + K_0x_n$ . By [1, Theorem 16], the centralizer  $K'_0$  of  $K_0$  has degree  $\leq n$  over  $C$ . But  $K'_0 \cong K_0$  as  $K_0$  is commutative. Moreover, the topology of  $K_0$  is given by a discrete valuation by Theorem 1, as  $B_0$  is an open, strictly linearly compact subring. Therefore, as  $[K_0: C] \leq n$ , the valuation induced on  $C$  is not the improper valuation; hence the topology of  $C$  is given by a complete, discrete valuation. As

$$[K: C] = [K: K_0][K_0: C] \leq n^2,$$

the given topology of  $K$  is the only topology for which  $K$  is a Hausdorff topological vector space over  $C$  [2, Theorem 2, p. 18]; by valuation theory, that topology is given by a complete, discrete valuation.

The idea of using [1, Theorem 16] is suggested by Kaplansky's treatment of locally compact division rings in [8].

Jacobson's theorem concerning totally disconnected locally compact division rings follows at once from Theorem 2. Indeed, if  $K$  is an indiscrete, totally disconnected, locally compact division ring, then  $K$  contains a compact open subring  $B$  [9, Lemma 4]; the open ideals of  $B$  form a fundamental system of neighborhoods of zero [7, Lemmas 9 and 10], and therefore the compact ring  $B$  is clearly centrally linearly compact; by Theorem 2,  $K$  is finite-dimensional over its center, which is indiscrete, and the topology of  $K$  is given by a complete, discrete valuation.

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