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## **COMMON FIXED POINTS OF TWO MAPPINGS**

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# COMMON FIXED POINTS OF TWO MAPPINGS

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Let  $S, T$  be functions on a nonempty complete metric space  $(X, d)$ . The main result of this paper is the following.  $S$  or  $T$  has a fixed point if there exist decreasing functions  $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$  of  $(0, \infty)$  into  $[0, 1)$  such that (a)  $\sum_{i=1}^5 \alpha_i < 1$ ; (b)  $\alpha_1 = \alpha_2$  or  $\alpha_3 = \alpha_4$ , (c)  $\lim_{t \downarrow 0} (\alpha_1 + \alpha_2) < 1$  and  $\lim_{t \downarrow 0} (\alpha_3 + \alpha_4) < 1$  and (d) for any distinct  $x, y$  in  $X$ ,

$$d(S(x), T(y)) \leq \alpha_1 d(x, S(x)) + \alpha_2 d(y, T(y)) + \alpha_3 d(x, T(y)) \\ + \alpha_4 d(y, S(x)) + \alpha_5 d(x, y),$$

where  $\alpha_i = \alpha_i(d(x, y))$ . A number of related results are obtained.

1. Introduction. Let  $(X, d)$  be a nonempty complete metric space and let  $S, T$  be mappings of  $X$  into itself which are not necessarily continuous nor commuting. Suppose that there are nonnegative real numbers  $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$  such that

$$(a) \quad \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1,$$

$$(b) \quad \alpha_1 = \alpha_2 \quad \text{or} \quad \alpha_3 = \alpha_4,$$

and for any  $x, y$  in  $X$ ,

$$(c) \quad d(S(x), T(y)) \leq \alpha_1 d(x, S(x)) + \alpha_2 d(y, T(y)) + \alpha_3 d(x, T(y)) \\ + \alpha_4 d(y, S(x)) + \alpha_5 d(x, y).$$

It is proved in this paper that each of  $S, T$  has a unique fixed point and these two fixed points coincide. Among others, a generalization is obtained by replacing  $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$  with nonnegative real-valued functions on  $(0, \infty)$ . This result generalizes the Banach contraction mapping theorem and some results of G. Hardy and T. Rogers [5], R. Kannan [7], E. Rakotch [8], S. Reich [9], P. Srivastava, and V. K. Gupta [10]. It also gives a different proof for these special cases. Note that even if  $X = [0, 1]$  and if  $T_1, T_2$  are commuting continuous functions of  $X$  into itself,  $T_1, T_2$  need not have a common fixed point [1], [2], and [6].

## 2. Basic results.

**THEOREM 1.** *Let  $S, T$  be mappings of a complete metric space  $(X, d)$  into itself. Suppose that there exist nonnegative real numbers  $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$  which satisfy (a), (b), and (c). Then each of  $S, T$*

has a unique fixed point and these two fixed points coincide.

*Proof.* Let  $x_0 \in X$ . Define

$$x_{2n+1} = S(x_{2n}), x_{2n+2} = T(x_{2n+1}), \quad n = 0, 1, 2, \dots.$$

From (c),

$$\begin{aligned} d(x_1, x_2) &= d(S(x_0), T(x_1)) \\ &\leq (a_1 + a_5)d(x_0, x_1) + a_2d(x_1, x_2) + a_3d(x_0, x_2) \\ &\leq (a_1 + a_5)d(x_0, x_1) + a_2d(x_1, x_2) + a_3(d(x_0, x_1) + d(x_1, x_2)). \end{aligned}$$

So

$$(1) \quad d(x_1, x_2) \leq \frac{a_1 + a_3 + a_5}{1 - a_2 - a_3} d(x_0, x_1).$$

Similarly,

$$(2) \quad d(x_2, x_3) \leq \frac{a_2 + a_4 + a_5}{1 - a_1 - a_4} d(x_1, x_2).$$

Let

$$r = \frac{a_1 + a_3 + a_5}{1 - a_2 - a_3}, \quad s = \frac{a_2 + a_4 + a_5}{1 - a_1 - a_4}.$$

Repeating the above argument, we obtain, for each  $n = 0, 1, 2, \dots$ ,

$$(3) \quad d(x_{2n+1}, x_{2n+2}) \leq rd(x_{2n+1}, x_{2n}),$$

$$(4) \quad d(x_{2n+3}, x_{2n+2}) \leq sd(x_{2n+2}, x_{2n+1}).$$

By (3), (4), and induction, we have, for each  $n = 0, 1, 2, \dots$ ,

$$(5) \quad d(x_{2n+1}, x_{2n+2}) \leq r(rs)^nd(x_0, x_1),$$

$$(6) \quad d(x_{2n+2}, x_{2n+3}) \leq (rs)^{n+1}d(x_0, x_1).$$

Since  $rs < 1$  and

$$\sum_{n=0}^{\infty} d(x_n, x_{n+1}) \leq (1 + r) \sum_{n=0}^{\infty} (rs)^nd(x_0, x_1),$$

$\{x_n\}$  is Cauchy. By completeness of  $(X, d)$ ,  $\{x_n\}$  converges to some point  $x$  in  $X$ . We shall now prove that  $x$  is a fixed point of  $S$  and  $T$ . Let  $n$  be given. Then

$$\begin{aligned} (7) \quad d(x, S(x)) &\leq d(x, x_{2n+2}) + d(S(x), x_{2n+2}) \\ &= d(x, x_{2n+2}) + d(S(x), T(x_{2n+1})). \end{aligned}$$

By (c),

$$(8) \quad \begin{aligned} d(S(x), T(x_{2n+1})) &\leq a_1 d(x, S(x)) + a_2 d(x_{2n+1}, x_{2n+2}) + a_3 d(x, x_{2n+2}) \\ &\quad + a_4 d(x_{2n+1}, S(x)) + a_5 d(x, x_{2n+1}) . \end{aligned}$$

Combining (7) and (8) and letting  $n$  tend to infinity, we obtain

$$d(x, S(x)) \leq (a_1 + a_4) d(x, S(x)) .$$

Since  $a_1 + a_4 < 1$ ,  $S(x) = x$ . Similarly  $T(x) = x$ . Let  $y$  be a fixed point of  $T$ . Then from  $d(x, y) = d(S(x), T(y))$  and (c), we obtain

$$d(x, y) \leq (a_3 + a_4 + a_5) d(x, y) .$$

Since  $a_3 + a_4 + a_5 < 1$ ,  $d(x, y) = 0$ . So  $T$  has a unique fixed point. Similarly,  $S$  has a unique fixed point.

When  $a_3 = a_4 = a_5 = 0$ ,  $S = T$  and  $T$  is continuous (or even  $x \rightarrow d(x, T(x))$  is lower semicontinuous) on  $X$ , Theorem 1 can be obtained by an earlier result of the author [11, Theorem 1].

From the proof of Theorem 1, we know that  $S, T$  still have a common fixed point if conditions (a), (b) are replaced by the following conditions:

$$(9) \quad (a_1 + a_3 + a_5)(a_2 + a_4 + a_6) < (1 - a_2 - a_3)(1 - a_1 - a_4) ,$$

$$(10) \quad a_1 + a_4 < 1 .$$

If in addition,

$$(11) \quad a_3 + a_4 + a_5 < 1 ,$$

then the common fixed point of  $S, T$  is the unique fixed point of  $S$  (and  $T$ ). Note that conditions (a) and (b) imply (9), but (a) alone does not. Indeed, for any  $a_1, a_2, a_5$  in  $[0, \infty)$  with  $a_1 \neq a_2$  and  $a_1 + a_2 + a_5 < 1$ , we can always find  $a_3, a_4$  in  $[0, \infty)$  such that (a) holds but (9) does not. This can be seen by considering the affine function  $f$ :

$$f(x, y) = (1 - a_2 - x)(1 - a_1 - y) - (a_1 + x + a_5)(a_2 + y + a_5)$$

defined on the compact convex set

$$K = \{(x, y) \in [0, 1] \times [0, 1]: a_1 + a_2 + x + y + a_5 \leq 1\} .$$

$f$  takes its minimum value at one of the extreme points of  $K$ . With some computation, we conclude that

$$\min f(K) = -|a_1 - a_2|(1 - a_1 - a_2 - a_5) .$$

Since  $a_1 + a_2 + a_5 > 1$ ,  $\min f(K) < 0$  if and only if  $a_1 \neq a_2$ . Thus if  $a_1 \neq a_2$ , then by continuity of  $f$ , there exists a point  $(a_3, a_4)$  in

$$K \setminus \{(x, y) \in K: a_1 + a_2 + x + y + a_3 = 1\}$$

such that  $f(a_3, a_4) < 0$ .

**COROLLARY 1.** R. Kannan [7, Theorem 1]. *Let  $S$  be a mapping of a complete metric space  $(X, d)$  into itself. Suppose that there exists a number  $r$  in  $[0, 1/2)$  such that*

$$d(S(x), S(y)) \leq r(d(x, S(x)) + d(y, S(y)))$$

*for all  $x, y$  in  $X$ . Then  $S$  has a unique fixed point.*

**COROLLARY 2.** P. Srivastava and V. K. Gupta [10, Theorem 1]. *Let  $S, T$  be mappings of a complete metric space  $(X, d)$  into itself. Suppose that there exists nonnegative real numbers  $a_1, a_2$  such that*

$$(a) \quad a_1 + a_2 < 1$$

and

$$(b) \quad d(S(x), T(y)) \leq a_1 d(x, S(x)) + a_2 d(y, T(y))$$

*for all  $x, y$  in  $X$ .*

*Then  $S, T$  have a unique common fixed point.*

Srivastava and Gupta stated the above result in a more general form with  $S, T$  replaced by  $S^p, T^q$  for some positive integers  $p, q$ . Since the unique fixed point of  $S^p$  (similarly  $T^q$ ) is the unique fixed point of  $S$ , this result is equivalent to Corollary 2.

For Corollaries 1 and 2, we have the following related result.

**PROPOSITION.** *Let  $S, T$  be self-maps of a nonempty complete metric space  $(X, d)$ . Suppose that there exist nonnegative real numbers  $a_1, a_2$  such that  $a_1 + a_2 < 1$  and*

$$(*) \quad d(S(x), T(y)) \leq a_1 d(x, S(x)) + a_2 d(y, T(y)), \quad x, y \in X.$$

*Then either  $(*)$  is true when all of its  $S$  are replaced by  $T$  or  $(*)$  is true when all of its  $T$  are replaced by  $S$ .*

The following example proves that our result is actually more general than that of Srivastava and Gupta.

**EXAMPLE.** Let  $X = \{1, 2, 3\}$ . Let  $d$  be the metric for  $X$  determined by

$$d(1, 2) = 1, \quad d(2, 3) = \frac{4}{7}, \quad d(1, 3) = \frac{5}{7}.$$

Let  $S, T$  be the function on  $X$  such that

$$S(1) = S(2) = S(3) = 1;$$

$$T(1) = T(3) = 1, \quad T(2) = 3.$$

Let  $a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 5/7, a_5 = 0$ . Then the conditions of Theorem 1 are satisfied. However, no nonnegative real numbers  $a_1, a_2, a_3, a_5$  can be chosen such that  $a_1 + a_2 + a_3 + a_5 < 1$  and for  $x, y \in X$ ,

$$d(S(x), T(y)) \leq a_1 d(x, S(x)) + a_2 d(y, T(y)) + a_3 d(x, T(y)) + a_5 d(x, y).$$

For if there exist such  $a_1, a_2, a_3, a_5$ , then

$$d(S(3), T(2)) \leq a_1 d(3, S(3)) + a_2 d(2, T(2)) + a_3 d(3, T(2)) + a_5 d(3, 2).$$

So

$$\frac{5}{7} \leq \frac{5a_1}{7} + \frac{4a_2}{7} + \frac{4a_5}{7} \leq \frac{5}{7} (a_1 + a_2 + a_5) < \frac{5}{7},$$

a contradiction.

**COROLLARY 3.** G. Hardy and T. Rogers [5, Theorem 1]. *Let  $S$  be a mapping of a nonempty complete metric space  $(X, d)$  into itself. Suppose that there exist nonnegative real numbers  $a_1, a_2, a_3, a_4, a_5$  such that*

$$(a) \quad a_1 + a_2 + a_3 + a_4 + a_5 < 1$$

and

$$(b) \quad d(S(x), S(y)) \leq a_1 d(x, S(x)) + a_2 d(y, S(y)) + a_3 d(x, S(y)) \\ + a_4 d(y, S(x)) + a_5 d(x, y) \\ \text{for all } x, y \text{ in } X.$$

Then  $S$  has a unique fixed point.

Note that in the above case, we may without loss of generality assume that  $a_1 = a_2, a_3 = a_4$  (replace  $a_1, a_2, a_3, a_4, a_5$  respectively by

$$\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, \frac{a_3 + a_4}{2}, \frac{a_3 + a_4}{2}, a_5$$

if necessary). So the above result follows from Theorem 1. The above example shows that there is no such symmetry ( $a_1 = a_2, a_3 = a_4$ ) for the general case. Indeed, we cannot even assume  $a_3 = a_4$ . For if  $a_3 = a_4$ , then for the above example, we have

$$\begin{aligned}
\frac{5}{7} = d(S(3), T(3)) &\leq \frac{5}{7} a_1 + \frac{4}{7} a_2 + a_4 + \frac{4}{7} a_5 . \\
&= \frac{5}{7} a_1 + \frac{4}{7} a_2 + \frac{1}{2} a_3 + \frac{1}{2} a_4 + \frac{4}{7} a_5 \\
&< \frac{5}{7} (a_1 + a_2 + a_3 + a_4 + a_5) < \frac{5}{7} ,
\end{aligned}$$

a contradiction.

2. Extensions and some related results. The following result generalizes Theorem 1. Its proof is different from the one we gave for Theorem 1.

**THEOREM 2.** *Let  $S, T$  be functions on a nonempty complete metric space  $(X, d)$ . Suppose that there exist decreasing functions  $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$  of  $(0, \infty)$  into  $[0, 1)$  such that*

- (a)  $\sum_{i=1}^5 \alpha_i < 1$ ;
- (b)  $\alpha_1 = \alpha_2$  or  $\alpha_3 = \alpha_4$ ;
- (c)  $\lim_{t \downarrow 0} (\alpha_2 + \alpha_3) < 1$  and  $\lim_{t \downarrow 0} (\alpha_1 + \alpha_4) < 1$ ;
- (d) for any distinct  $x, y$  in  $X$ ,

$$\begin{aligned}
d(S(x), T(y)) &\leq a_1 d(x, S(x)) + a_2 d(y, T(y)) + a_3 d(x, T(y)) \\
&\quad + a_4 d(y, S(x)) + a_5 d(x, y) ,
\end{aligned}$$

where  $a_i = \alpha_i(d(x, y))$ .

Then at least one of  $S, T$  has a fixed point. If both  $S$  and  $T$  have fixed points, then each of  $S, T$  has a unique fixed point and these two fixed points coincide.

*Proof.* Let  $x_0 \in X$ . Define for each  $n = 0, 1, 2, \dots$ ,

$$x_{2n+1} = S(x_{2n}) , \quad x_{2n+2} = T(x_{2n+1}) , \quad b_n = d(x_n, x_{n+1}) .$$

We may assume that  $b_n > 0$  for each  $n$ , for otherwise some  $x_n$  is a fixed point of  $S$  or  $T$ . Let

$$r(t) = \frac{\alpha_1(t) + \alpha_3(t) + \alpha_5(t)}{1 - \alpha_2(t) - \alpha_3(t)} , \quad t > 0 ,$$

$$s(t) = \frac{\alpha_2(t) + \alpha_4(t) + \alpha_5(t)}{1 - \alpha_1(t) - \alpha_4(t)} , \quad t > 0 .$$

Then  $r, s$  are decreasing. From (a) and (c), the limits

$$r_0 = \lim_{t \downarrow 0} r(t) , \quad s_0 = \lim_{t \downarrow 0} s(t)$$

are nonnegative real numbers. Let

$$f(t) = r(t)s(t) , \quad t > 0 .$$

Then  $f$  is decreasing and  $f(t) < 1$  for each  $t > 0$ . As in the proof of Theorem 1, we have for each  $n = 0, 1, 2, \dots$ ,

$$(12) \quad b_{2n+1} \leq r(b_{2n})b_{2n} ,$$

$$(13) \quad b_{2n+2} \leq s(b_{2n+1})b_{2n+1} .$$

Let  $n$  be given. Then

$$(14) \quad b_{2n+3} \leq r(b_{2n+2})s(b_{2n+1})b_{2n+1} ,$$

$$(15) \quad b_{2n+2} \leq s(b_{2n+1})r(b_{2n})b_{2n} .$$

Since  $r, s$  are decreasing,

$$(16) \quad b_{2n+3} \leq f(\min \{b_{2n+2}, b_{2n+1}\})b_{2n+1} ,$$

$$(17) \quad b_{2n+2} \leq f(\min \{b_{2n+1}, b_{2n}\})b_{2n} .$$

Since  $f(t) < 1$  for each  $t > 0$ ,  $\{b_{2n+1}\}$ ,  $\{b_{2n}\}$  are decreasing sequences. So  $\{b_{2n+1}\}$ ,  $\{b_{2n}\}$  converge respectively to some points  $c_1, c_2$ . We shall prove that  $c_1 = 0, c_2 = 0$ . From (12) and (13),

$$c_1 \leq r_0 c_2 , \quad c_2 \leq s_0 c_1 .$$

So either both  $c_1, c_2$  are zero or both  $c_1, c_2$  are not zero. Suppose to the contrary that  $c_1 \neq 0, c_2 \neq 0$ . Then from (16) and (17),

$$(18) \quad b_{n+2} \leq f(\min \{c_1, c_2\})b_n , \quad n = 0, 1, 2, \dots .$$

By induction,

$$(19) \quad b_{2n} \leq (f(\min \{c_1, c_2\}))^n b_0 \quad n = 0, 1, 2, \dots .$$

So  $c_2 = 0$ , a contradiction. Therefore,  $c_1 = c_2 = 0$ . This proves that  $\{b_n\}$  converges to 0.

Now we shall prove that  $\{x_n\}$  is Cauchy. Suppose not. Then there exist  $\varepsilon \in (0, \infty)$  and sequences  $\{p(n)\}$ ,  $\{q(n)\}$  such that for each  $n \geq 0$ ,

$$(20) \quad p(n) > q(n) > n ,$$

$$(21) \quad d(x_{p(n)}, x_{q(n)}) \geq \varepsilon ,$$

and (by the well-ordering principle),

$$(22) \quad d(x_{p(n)-1}, x_{q(n)}) < \varepsilon .$$

Let  $n \geq 0$  be given,  $c_n = d(x_{p(n)}, x_{q(n)})$ . Then



$$\begin{aligned}
 (23) \quad \varepsilon &\leq c_n \\
 &\leq d(x_{p(n)}, x_{p(n)-1}) + d(x_{p(n)-1}, x_{q(n)}) < b_{p(n)-1} + \varepsilon .
 \end{aligned}$$

From  $c_1 = c_2 = 0$ , we conclude that  $\{c_n\}$  converges to  $\varepsilon$  from the right. Let

$$\begin{aligned}
 I_1 &= \{n: p(n), q(n) \text{ are odd}\} , \\
 I_2 &= \{n: p(n) \text{ is odd, } q(n) \text{ is even}\} . \\
 I_3 &= \{n: p(n) \text{ is even, } q(n) \text{ is odd}\} , \\
 I_4 &= \{n: p(n), q(n) \text{ are even}\} .
 \end{aligned}$$

Then at least one of  $I_1, I_2, I_3, I_4$  is infinite. Suppose first that  $I_1$  is infinite. Let

$$d_n = d(x_{p(n)-1}, x_{q(n)}) , \quad n = 0, 1, 2, \dots .$$

Since  $\{c_n\}$  converges to  $\varepsilon$  and  $\{b_n\}$  converges to 0, we conclude from (22) that  $\{d_n\}$  converges to  $\varepsilon$  from the left. Thus

$$J_1 = \{n \in I_1: x_{p(n)-1} \neq x_{q(n)}\}$$

is infinite. Let  $n \in J_1$ ,  $u_n = d(x_{p(n)-1}, x_{q(n)+1})$ . Then

$$\begin{aligned}
 (24) \quad c_n &= d(x_{p(n)}, x_{q(n)}) \leq d(x_{p(n)}, x_{q(n)+1}) + d(x_{q(n)+1}, x_{q(n)}) \\
 &\leq d(S(x_{p(n)-1}), T(x_{q(n)})) + b_{q(n)} .
 \end{aligned}$$

From (d),

$$\begin{aligned}
 (25) \quad d(S(x_{p(n)-1}), T(x_{q(n)})) &\leq \alpha_1(d_n)b_{p(n)-1} + \alpha_2(d_n)b_{q(n)} + \alpha_3(d_n)u_n \\
 &\quad + \alpha_4(d_n)c_n + \alpha_5(d_n)d_n .
 \end{aligned}$$

From (24) and (25),

$$\begin{aligned}
 (26) \quad c_n &\leq \alpha_1(d_n)b_{p(n)-1} + \alpha_2(d_n)b_{q(n)} + \alpha_3(d_n)u_n + \alpha_4(d_n)c_n \\
 &\quad + \alpha_5(d_n)d_n + b_{q(n)} .
 \end{aligned}$$

Without loss of generality, we may assume that each  $\alpha_i$  is continuous from the left, for we can replace the  $\alpha_i$ 's by

$$\beta_i(t) = \lim_{s \uparrow t} \alpha_i(s) , \quad t > 0 , \quad i = 1, 2, 3, 4, 5$$

and conditions (a), (b), (c), and (d) still hold. Thus

$$\lim_{n \rightarrow \infty} \alpha_i(d_n) = \alpha_i(\varepsilon) , \quad i = 1, 2, 3, 4, 5 .$$

So from (26),

$$\varepsilon \leq (\alpha_3(\varepsilon) + \alpha_4(\varepsilon) + \alpha_5(\varepsilon))\varepsilon < \varepsilon ,$$

a contradiction. Now suppose that  $I_2$  is infinite. By a similar argument,  $J_2 = \{n \in I_2: x_{p(n)-1} \neq x_{q(n)-1}\}$  is infinite. Let  $n \in J_2$ ,

$$v_n = d(x_{p(n)-1}, x_{q(n)-1}), \quad w_n = d(x_{p(n)}, x_{q(n)-1}).$$

Then

$$(27) \quad \begin{aligned} c_n &= d(S(x_{p(n)-1}), T(x_{q(n)-1})) \\ &\leq \alpha_1(v_n)b_{p(n)-1} + \alpha_2(v_n)b_{q(n)-1} + \alpha_3(v_n)d_n + \alpha_4(v_n)w_n + \alpha_5(v_n)v_n. \end{aligned}$$

Since  $\{v_n\}$  converges to  $\varepsilon$  (not necessarily from the left or right), we obtain the same contradiction from (27). The other two cases are similar to the above two except the roles of  $S$ ,  $T$  interchange. Hence  $\{x_n\}$  is Cauchy. By completeness,  $\{x_n\}$  converges to a point  $x$  in  $X$ . Since  $b_n > 0$  for each  $n$ ,  $J = \{n: x \neq x_{2n+1}\}$  or  $K = \{n: x \neq x_{2n}\}$  is infinite. Suppose that  $K$  is infinite. Let  $n \in K$ ,

$$l_n = d(x, x_{2n}), \quad h_n = d(x, x_{2n+1}).$$

Then

$$\begin{aligned} d(x, T(x)) &\leq d(x, x_{2n+1}) + d(x_{2n+1}, T(x)) \\ &= h_n + d(S(x_{2n}), T(x)) \\ &\leq h_n + \alpha_1(l_n)b_{2n} + \alpha_2(l_n)d(x, T(x)) + \alpha_3(l_n)d(x_{2n}, T(x)) \\ &\quad + \alpha_4(l_n)h_n + \alpha_5(l_n)l_n \\ &\leq h_n + \alpha_1(l_n)b_{2n} + \alpha_2(l_n)d(x, T(x)) + \alpha_3(l_n)[l_n + d(x, T(x))] \\ &\quad + \alpha_4(l_n)h_n + \alpha_5(l_n)l_n. \end{aligned}$$

So

$$(28) \quad \begin{aligned} d(x, T(x)) &\leq \frac{1 + \alpha_4(l_n)}{1 - \alpha_2(l_n) - \alpha_3(l_n)} h_n + \frac{\alpha_3(l_n) + \alpha_5(l_n)}{1 - \alpha_2(l_n) - \alpha_3(l_n)} l_n \\ &\quad + \frac{\alpha_1(l_n)}{1 - \alpha_2(l_n) - \alpha_3(l_n)} b_{2n}. \end{aligned}$$

From (a) and (c), the sequences

$$\frac{1 + \alpha_4(l_n)}{1 - \alpha_2(l_n) - \alpha_3(l_n)}, \quad \frac{\alpha_3(l_n) + \alpha_5(l_n)}{1 - \alpha_2(l_n) - \alpha_3(l_n)}, \quad \frac{\alpha_1(l_n)}{1 - \alpha_2(l_n) - \alpha_3(l_n)}$$

are bounded. So from (28),  $T(x) = x$ . Similarly,  $S(x) = x$  if  $J$  is infinite. Hence  $S$  or  $T$  has a fixed point.

The following result follows easily from Theorem 2.

**THEOREM 3.** *With the conditions of Theorem 2, if further,*

$$d(S(x), T(x)) \leq \alpha [d(x, S(x)) + d(x, T(x))], \quad x \in X$$

for some  $\alpha \in [0, 1)$ , then each of  $S, T$  has a unique fixed point and these two fixed points coincide.

We remark that the conditions of Theorem 1 imply the conditions of Theorem 3. Also, G. Hardy and T. Rogers [5, Theorem 2] gave a different proof for the case  $S = T$ . Their proof cannot be modified for the general case. To see that the conclusion of Theorem 2 is best possible, we note that if  $X = \{0, 1\}$  with the usual distance and if  $S, T$  are two distinct functions of  $X$  onto  $X$ , then  $S, T$  satisfy the conditions of Theorem 2 (and Theorem 3 with  $\alpha = 1$ ), but one has two fixed points and the other has none.

**THEOREM 4.** *Let  $(X, d)$  be a nonempty compact metric space. Let  $S, T$  be functions of  $X$  into itself. Suppose that  $S$  or  $T$  is continuous. Suppose further that there exist nonnegative real-valued decreasing functions  $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$  on  $(0, \infty)$  such that*

- (a)  $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 \leq 1$ ,
- (b)  $\alpha_1 = \alpha_2$  and  $\alpha_3 = \alpha_4$ ,
- (c) for any distinct  $x, y$  in  $X$ ,

$$d(S(x), T(y)) < a_1 d(x, S(x)) + a_2 d(y, T(y)) + a_3 d(x, T(y)) + a_4 d(y, S(x)) + a_5 d(x, y),$$

where  $a_i = \alpha_i(d(x, y))$ .

Then  $S$  or  $T$  has a fixed point. If both  $S$  and  $T$  have fixed points, then each of  $S$  and  $T$  has a unique fixed point and these two fixed points coincide.

*Proof.* By symmetry, we may assume that  $S$  is continuous. Let  $f$  be the function on  $X$  such that

$$f(x) = d(x, S(x)), \quad x \in X.$$

Then  $f$  is continuous (we merely need the fact that  $f$  is lower semi-continuous) on  $X$ . So  $f$  takes its minimum value at some  $x_0$  in  $X$ . We claim that  $x_0$  is a fixed point of  $S$  or  $S(x_0)$  is a fixed point of  $T$ . Suppose not. Let

$$\begin{aligned} x_1 &= S(x_0), & x_2 &= T(x_1), & x_3 &= S(x_2), \\ b_0 &= d(x_0, x_1), & b_1 &= d(x_2, x_3), & b_2 &= d(x_2, x_3). \end{aligned}$$

Then  $b_0 > 0, b_1 > 0$ . From (c), we can prove that

$$(29) \quad (1 - \alpha_2(b_0) - \alpha_3(b_0))b_1 < (\alpha_1(b_0) + \alpha_3(b_0) + \alpha_5(b_0))b_0.$$

Let

$$p(t) = 1 - \alpha_2(t) - \alpha_3(t), \quad q(t) = \alpha_1(t) + \alpha_3(t) + \alpha_5(t), \quad t > 0.$$

From (a) and (b),  $p(b_0) > 0$ . So

$$(30) \quad b_1 < \frac{q(b_0)}{p(b_0)} b_0.$$

Similarly,

$$(31) \quad b_2 < \frac{v(b_1)}{u(b_1)} b_1,$$

where

$$u(t) = 1 - \alpha_1(t) - \alpha_4(t), \quad v(t) = \alpha_2(t) + \alpha_4(t) + \alpha_5(t), \quad t > 0.$$

From (30) and (31),

$$(32) \quad b_2 < \frac{v(b_1)}{u(b_1)} \frac{q(b_0)}{p(b_0)} b_0.$$

It suffices to prove that  $(v(b_1)q(b_0)/u(b_1)p(b_0)) < 1$ , for then,  $b_2 < b_0$ , a contradiction to the minimality of  $b_0$ . Let  $b = \min\{b_0, b_1\}$ . Then

$$v(b_1)q(b_0) - u(b_1)p(b_0) \leq v(b)q(b) - u(b)p(b) < 0$$

if  $\alpha_1 = \alpha_2$  and  $\alpha_3 = \alpha_4$ . So  $S$  or  $T$  has a fixed point. Now suppose that  $x$  is a fixed point of  $S$  and  $y$  is a fixed point of  $T$ . Then  $x = y$ , otherwise, from (c),

$$d(x, y) = d(S(x), T(y)) < d(x, y),$$

a contradiction.

The following result is stated without proof.

**THEOREM 5.** *Let  $(X, d)$  be complete metric space. Let  $\{S_n\}, \{T_n\}$  be sequence of functions of  $X$  into  $X$  which converge pointwise to  $S, T$  respectively. Suppose that the pairs  $(S_n, T_n)$  satisfy the conditions of Theorem 3 with the same  $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ . Then  $S, T$  have a unique common fixed point  $x$  and  $x$  is the limit of the sequence  $\{x_n\}$  of the fixed points  $x_n$  of  $S_n$ .*

**THEOREM 6.** *Let  $(X, d)$  be a nonempty compact metric space. Let  $\{S_n\}, \{T_n\}$  be sequences of functions of  $X$  into itself which converge pointwise to the functions  $S, T$  on  $X$  respectively. Suppose that for each  $n$ , there exist decreasing functions  $\alpha_1^n, \alpha_2^n, \alpha_3^n, \alpha_4^n, \alpha_5^n$  of  $(0, \infty)$  into  $[0, \infty)$  such that*

- (a)  $\alpha_1^n + \alpha_2^n + \alpha_3^n + \alpha_4^n + \alpha_5^n \leq 1$ ,
- (b)  $\alpha_1^n = \alpha_2^n$  and  $\alpha_3^n = \alpha_4^n$ ,
- (c) for any distinct  $x, y$  in  $X$ ,

$$d(S_n(x), T_n(y)) < \alpha_1^n d(x, S_n(x)) + \alpha_2^n d(y, T_n(y)) + \alpha_3^n d(x, T_n(y)) \\ + \alpha_4^n d(y, S_n(x)) + \alpha_5^n d(x, y) ,$$

where

$$\alpha_i^n = \alpha_i^n(d(x, y)) .$$

Then  $S$  or  $T$  has a fixed point. Indeed, every cluster point of a sequence  $\{x_n\}$  of fixed points  $x_n$  of  $S_n$  or  $T_n$  is a fixed point of  $S$  or  $T$ .

*Proof.* By Theorem 4, for each  $n$ , either  $S_n$  or  $T_n$  has a fixed point. By symmetry, we may assume that  $S_n$  has a fixed point for infinitely many of  $n$ 's. So there is a subsequence  $\{S_{n(k)}\}$  of  $\{S_n\}$  such that each  $S_{n(k)}$  has a fixed point, say  $x_k$ . By compactness, we may (by taking a subsequence) assume that  $\{x_k\}$  converges to some  $x$  in  $X$ . We shall prove that  $x$  is a fixed point of  $S$  or  $T$ . If  $x_k \neq x$  for only finitely many of  $k$ 's, then

$$\begin{aligned} S(x) &= \lim_{k \rightarrow \infty} S_{n(k)}(x) \\ &= \lim_{k \rightarrow \infty} S_{n(k)}(x_k) \\ &= \lim_{k \rightarrow \infty} x_k \\ &= x . \end{aligned}$$

So we may assume that  $x_k \neq x$  for infinitely many of  $k$ 's. By taking a subsequence, we may assume that  $x_k \neq x$  for each  $k$ . Let  $k \geq 1$  and  $b_k = d(x, x_k)$ . Then

$$(33) \quad \begin{aligned} d(x, T(x)) &\leq d(x, x_k) + d(x_k, T_{n(k)}(x)) + d(T_{n(k)}(x), T(x)) \\ &= d(x, x_k) + d(S_{n(k)}(x_k), T_{n(k)}(x)) + d(T_{n(k)}(x), T(x)) . \end{aligned}$$

From (c),

$$(34) \quad \begin{aligned} d(S_{n(k)}(x_k), T_{n(k)}(x)) &< \alpha_2^k(b_k)d(x, T_{n(k)}(x)) + \alpha_3^k(b_k)d(x_k, T_{n(k)}(x)) \\ &\quad + \alpha_4^k(b_k)d(x, x_k) + \alpha_5^k(b_k)b_k . \end{aligned}$$

Combining (33) and (34) and letting  $k$  tend to the infinity, we have

$$(35) \quad \begin{aligned} d(x, T(x)) &\leq \limsup_{k \rightarrow \infty} (\alpha_2^k(b_k) + \alpha_3^k(b_k))d(x, T(x)) \\ &\leq \limsup_{k \rightarrow \infty} \lim_{t \downarrow 0} (\alpha_2^k(t) + \alpha_3^k(t))d(x, T(x)) . \end{aligned}$$

From (b),  $\alpha_2^k(t) + \alpha_3^k(t) \leq 1/2$  for each  $t > 0$ ,  $k = 1, 2, \dots$ . So

$$(36) \quad \limsup_{k \rightarrow \infty} \lim_{t \downarrow 0} (\alpha_2^k(t) + \alpha_3^k(t)) \leq \frac{1}{2}.$$

From (35) and (36), we conclude that  $T(x) = x$ .

From the proof, we know that the same conclusion holds if in Theorem 6, we replace (b) by the following weaker conditions:

$$\alpha_1^n = \alpha_2^n \quad \text{or} \quad \alpha_3^n = \alpha_4^n,$$

$$\limsup_{k \rightarrow \infty} \lim_{t \downarrow 0} (\alpha_2^k(t) + \alpha_3^k(t)) < 1,$$

and

$$\limsup_{k \rightarrow \infty} \lim_{t \downarrow 0} (\alpha_1^k(t) + \alpha_4^k(t)) < 1.$$

We note that, unlike Theorem 5,  $S$ ,  $T$  in Theorem 6 need not satisfy the condition required for the pairs  $(S_n, T_n)$ .

**THEOREM 7.** *Let  $(X, d)$  be a nonempty compact metric space. Let  $\{S_n\}$  be a sequence of functions of  $X$  into itself which converges pointwise to some function  $S$  on  $X$ . Suppose that for each  $n$ , there exist decreasing functions  $\alpha_1^n, \alpha_2^n, \alpha_3^n, \alpha_4^n, \alpha_5^n$  of  $(0, \infty)$  into  $[0, \infty)$  such that*

$$(a) \quad \alpha_1^n + \alpha_2^n + \alpha_3^n + \alpha_4^n + \alpha_5^n \leq 1,$$

$$(b) \quad \text{for any distinct } x, y \text{ in } X,$$

$$d(S_n(x), S_n(y)) < a_1 d(x, S_n(x)) + a_2 d(y, S_n(y)) + a_3 d(x, S_n(y)) \\ + a_4 d(y, S_n(x)) + a_5 d(x, y),$$

where

$$a_i = \alpha_i(d(x, y)).$$

*Then  $S$  has a fixed point. Indeed, every cluster point of the sequence of fixed points of  $S_n$  is a fixed point of  $S$ .*

The above result follows from Theorem 6 by averaging two applications of condition (b).

We shall now give a simple example to show that the conclusion of Theorem 7 is best possible. Let  $X$  be a star-shaped [4] compact subset of a normed linear space  $B$ . Then there exists a point  $z$  in  $X$  such that for any  $y$  in  $X$ , the line segment

$$\{tz + (1 - t)y: t \in [0, 1]\}$$

is contained in  $X$ . For each  $n$ , let

$$S_n(x) = \frac{1}{n}z + \left(1 - \frac{1}{n}\right)x, \quad x \in X.$$

Then  $\{S_n\}$  is a sequence of mappings of  $X$  into  $X$  which satisfy the conditions of Theorem 7.  $\{S_n\}$  converges pointwise to the identity function  $S$  on  $X$ . Every point of  $X$  is a fixed point of  $S$ . So unlike Theorem 5, it is too much to ask that  $S$  in Theorem 7 has a unique fixed point.

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