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# COMMON FIXED POINTS OF TWO MAPPINGS

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**Let $S$, $T$ be functions on a nonempty complete metric space $(X, d)$. The main result of this paper is the following. $S$ or $T$ has a fixed point if there exist decreasing functions $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ of $(0, \infty)$ into $[0, 1)$ such that (a) $\sum_{i=1}^{5} \alpha_i < 1$; (b) $\alpha_1 = \alpha_2$ or $\alpha_3 = \alpha_4$, (c) $\lim_{t\downarrow 0}(\alpha_1+\alpha_2)<1$ and $\lim_{t\downarrow 0}(\alpha_3+\alpha_4)<1$ and (d) for any distinct $x$, $y$ in $X$,**

$$d(S(x), T(y)) \leqq a_1 d(x, S(x)) + a_2 d(y, T(y)) + a_3 d(x, T(y)) + a_4 d(y, S(x)) + a_5 d(x, y)\ ,$$

**where $a_i = \alpha_i(d(x, y))$. A number of related results are obtained.**

1. **Introduction.** Let $(X, d)$ be a nonempty complete metric space and let $S$, $T$ be mappings of $X$ into itself which are not necessarily continuous nor commuting. Suppose that there are nonnegative real numbers $a_1, a_2, a_3, a_4, a_5$ such that

$$\text{(a)} \qquad a_1 + a_2 + a_3 + a_4 + a_5 < 1\ ,$$

$$\text{(b)} \qquad a_1 = a_2 \quad \text{or} \quad a_3 = a_4\ ,$$

and for any $x$, $y$ in $X$,

$$\text{(c)} \qquad d(S(x), T(y)) \leqq a_1 d(x, S(x)) + a_2 d(y, T(y)) + a_3 d(x, T(y)) + a_4 d(y, S(x)) + a_5 d(x, y)\ .$$

It is proved in this paper that each of $S$, $T$ has a unique fixed point and these two fixed points coincide. Among others, a generalization is obtained by replacing $a_1, a_2, a_3, a_4, a_5$ with nonnegative real-valued functions on $(0, \infty)$. This result generalizes the Banach contraction mapping theorem and some results of G. Hardy and T. Rogers [5], R. Kannan [7], E. Rakotch [8], S. Reich [9], P. Srivastava, and V. K. Gupta [10]. It also gives a different proof for these special cases. Note that even if $X = [0, 1]$ and if $T_1$, $T_2$ are commuting continuous functions of $X$ into itself, $T_1$, $T_2$ need not have a common fixed point [1], [2], and [6].

2. **Basic results.**

THEOREM 1. *Let $S$, $T$ be mappings of a complete metric space $(X, d)$ into itself. Suppose that there exist nonnegative real numbers $a_1, a_2, a_3, a_4, a_5$ which satisfy* (a), (b), *and* (c). *Then each of $S$, $T$*

*has a unique fixed point and these two fixed points coincide.*

*Proof.* Let $x_0 \in X$. Define

$$x_{2n+1} = S(x_{2n}),\ x_{2n+2} = T(x_{2n+1})\ , \qquad n = 0, 1, 2, \cdots .$$

From (c),

$$\begin{aligned} d(x_1, x_2) &= d(S(x_0),\ T(x_1)) \\ &\leqq (a_1 + a_5)d(x_0, x_1) + a_2 d(x_1, x_2) + a_3 d(x_0, x_2) \\ &\leqq (a_1 + a_5)d(x_0, x_1) + a_2 d(x_1, x_2) + a_3(d(x_0, x_1) + d(x_1, x_2)) . \end{aligned}$$

So

$$(1) \qquad d(x_1, x_2) \leqq \frac{a_1 + a_3 + a_5}{1 - a_2 - a_3} d(x_0, x_1) .$$

Similarly,

$$(2) \qquad d(x_2, x_3) \leqq \frac{a_2 + a_4 + a_5}{1 - a_1 - a_4} d(x_1, x_2) .$$

Let

$$r = \frac{a_1 + a_3 + a_5}{1 - a_2 - a_3}\ , \qquad s = \frac{a_2 + a_4 + a_5}{1 - a_1 - a_4} .$$

Repeating the above argument, we obtain, for each $n = 0, 1, 2, \cdots$,

$$(3) \qquad d(x_{2n+1}, x_{2n+2}) \leqq r d(x_{2n+1}, x_{2n})\ ,$$

$$(4) \qquad d(x_{2n+3}, x_{2n+2}) \leqq s d(x_{2n+2}, x_{2n+1}) .$$

By (3), (4), and induction, we have, for each $n = 0, 1, 2, \cdots$,

$$(5) \qquad d(x_{2n+1}, x_{2n+2}) \leqq r(rs)^n d(x_0, x_1)\ ,$$

$$(6) \qquad d(x_{2n+2}, x_{2n+3}) \leqq (rs)^{n+1} d(x_0, x_1) .$$

Since $rs < 1$ and

$$\sum_{n=0}^{\infty} d(x_n, x_{n+1}) \leqq (1 + r) \sum_{n=0}^{\infty} (rs)^n d(x_0, x_1)\ ,$$

$\{x_n\}$ is Cauchy. By completeness of $(X, d)$, $\{x_n\}$ converges to some point $x$ in $X$. We shall now prove that $x$ is a fixed point of $S$ and $T$. Let $n$ be given. Then

$$(7) \qquad \begin{aligned} d(x, S(x)) &\leqq d(x, x_{2n+2}) + d(S(x), x_{2n+2}) \\ &= d(x, x_{2n+2}) + d(S(x),\ T(x_{2n+1})) . \end{aligned}$$

By (c),

$$(8)\qquad \begin{aligned} d(S(x),\, T(x_{2n+1})) \leqq a_1 d(x,\, S(x)) + a_2 d(x_{2n+1},\, x_{2n+2}) + a_3 d(x,\, x_{2n+2}) \\ + a_4 d(x_{2n+1},\, S(x)) + a_5 d(x,\, x_{2n+1}) \,. \end{aligned}$$

Combining (7) and (8) and letting $n$ tend to infinity, we obtain

$$d(x,\, S(x)) \leqq (a_1 + a_4) d(x,\, S(x)) \,.$$

Since $a_1 + a_4 < 1$, $S(x) = x$. Similarly $T(x) = x$. Let $y$ be a fixed point of $T$. Then from $d(x, y) = d(S(x), T(y))$ and (c), we obtain

$$d(x,\, y) \leqq (a_3 + a_4 + a_5) d(x,\, y) \,.$$

Since $a_3 + a_4 + a_5 < 1$, $d(x, y) = 0$. So $T$ has a unique fixed point. Similarly, $S$ has a unque fixed point.

When $a_3 = a_4 = a_5 = 0$, $S = T$ and $T$ is continuous (or even $x \to d(x, T(x))$ is lower semicontinuous) on $X$, Theorem 1 can be obtained by an earlier result of the author [11, Theorem 1].

From the proof of Theorem 1, we know that $S$, $T$ still have a common fixed point if conditions (a), (b) are replaced by the following conditions:

$$(9)\qquad (a_1 + a_3 + a_5)(a_2 + a_4 + a_5) < (1 - a_2 - a_3)(1 - a_1 - a_4) \,,$$

$$(10)\qquad a_1 + a_4 < 1 \,.$$

If in addition,

$$(11)\qquad a_3 + a_4 + a_5 < 1 \,,$$

then the common fixed point of $S$, $T$ is the unique fixed point of $S$ (and $T$). Note that conditions (a) and (b) imply (9), but (a) alone does not. Indeed, for any $a_1$, $a_2$, $a_5$ in $[0, \infty)$ with $a_1 \neq a_2$ and $a_1 + a_2 + a_5 < 1$, we can always find $a_3$, $a_4$ in $[0, \infty)$ such that (a) holds but (9) does not. This can be seen by considering the affine function $f$:

$$f(x,\, y) = (1 - a_2 - x)(1 - a_1 - y) - (a_1 + x + a_5)(a_2 + y + a_5)$$

defined on the compact convex set

$$K = \{(x,\, y) \in [0,\, 1] \times [0,\, 1] \colon a_1 + a_2 + x + y + a_5 \leqq 1\} \,.$$

$f$ takes its minimum value at one of the extreme points of $K$. With some computation, we conclude that

$$\min f(K) = -\,|a_1 - a_2|\,(1 - a_1 - a_2 - a_5) \,.$$

Since $a_1 + a_2 + a_5 > 1$, $\min f(K) < 0$ if and only if $a_1 \neq a_2$. Thus if $a_1 \neq a_2$, then by continuity of $f$, there exists a point $(a_3, a_4)$ in

$$K\backslash\{(x, y) \in K:\ a_1 + a_2 + x + y + a_5 = 1\}$$

such that $f(a_3, a_4) < 0$.

COROLLARY 1. R. Kannan [7, Theorem 1]. *Let $S$ be a mapping of a complete metric space $(X, d)$ into itself. Suppose that there exists a number $r$ in $[0, 1/2)$ such that*

$$d(S(x), S(y)) \leqq r(d(x + S(x)) + d(y, S(y)))$$

*for all $x$, $y$ in $X$. Then $S$ has a unique fixed point.*

COROLLARY 2. P. Srivastava and V. K. Gupta [10, Theorem 1]. *Let $S$, $T$ be mappings of a complete metric space $(X, d)$ into itself. Suppose that there exists nonnegative real numbers $a_1$, $a_2$ such that*

(a) $$a_1 + a_2 < 1$$

and

(b) $$d(S(x), T(y)) \leqq a_1 d(x, S(x)) + a_2 d(y, T(y)) \quad \text{for all } x, y \text{ in } X.$$

*Then $S$, $T$ have a unique common fixed point.*

Srivastava and Gupta stated the above result in a more general form with $S$, $T$ replaced by $S^p$, $T^q$ for some positive integers $p$, $q$. Since the unique fixed point of $S^p$ (similarly $T^q$) is the unique fixed point of $S$, this result is equivalent to Corollary 2.

For Corollaries 1 and 2, we have the following related result.

PROPOSITION. *Let $S$, $T$ be self-maps of a nonempty complete metric space $(X, d)$. Suppose that there exist nonnegative real numbers $a_1$, $a_2$ such that $a_1 + a_2 < 1$ and*

(*) $$d(S(x), T(y)) \leqq a_1 d(x, S(x)) + a_2 d(y, T(y)), \quad x, y \in X.$$

*Then either (*) is true when all of its $S$ are replaced by $T$ or (*) is true when all of its $T$ are replaced by $S$.*

The following example proves that our result is actually more general than that of Srivastava and Gupta.

EXAMPLE. Let $X = \{1, 2, 3\}$. Let $d$ be the metric for $X$ determined by

$$d(1, 2) = 1,\ d(2, 3) = \frac{4}{7}, \quad d(1, 3) = \frac{5}{7}.$$

Let $S$, $T$ be the function on $X$ such that

$$S(1) = S(2) = S(3) = 1;$$

$$T(1) = T(3) = 1\,, \qquad T(2) = 3\,.$$

Let $a_1 = 0$, $a_2 = 0$, $a_3 = 0$, $a_4 = 5/7$, $a_5 = 0$. Then the conditions of Theorem 1 are satisfied. However, no nonnegative real numbers $a_1$, $a_2$, $a_3$, $a_5$ can be chosen such that $a_1 + a_2 + a_3 + a_5 < 1$ and for $x, y \in X$,

$$d(S(x), T(y)) \leqq a_1 d(x, S(x)) + a_2 d(y, T(y)) + a_3 d(x, T(y)) + a_5 d(x, y)\,.$$

For if there exist such $a_1$, $a_2$, $a_3$, $a_5$, then

$$d(S(3), T(2)) \leqq a_1 d(3, S(3)) + a_2 d(2, T(2)) + a_3 d(3, T(2)) + a_5 d(3, 2)\,.$$

So

$$\frac{5}{7} \leqq \frac{5a_1}{7} + \frac{4a_2}{7} + \frac{4a_5}{7} \leqq \frac{5}{7}(a_1 + a_2 + a_5) < \frac{5}{7}\,,$$

a contradiction.

COROLLARY 3. G. Hardy and T. Rogers [5, Theorem 1]. *Let $S$ be a mapping of a nonempty complete metric space $(X, d)$ into itself. Suppose that there exist nonnegative real numbers $a_1$, $a_2$, $a_3$, $a_4$, $a_5$ such that*

(a) $$a_1 + a_2 + a_3 + a_4 + a_5 < 1$$

*and*

(b) $$d(S(x), S(y)) \leqq a_1 d(x, S(x) + a_2 d(y, S(y)) + a_3 d(x, S(y)) + a_4 d(y, S(x)) + a_5 d(x, y)$$

*for all $x$, $y$ in $X$.*

*Then $S$ has a unique fixed point.*

Note that in the above case, we may without loss of generality assume that $a_1 = a_2$, $a_3 = a_4$ (replace $a_1$, $a_2$, $a_3$, $a_4$, $a_5$ respectively by

$$\frac{a_1 + a_2}{2}\,, \; \frac{a_1 + a_2}{2}\,, \; \frac{a_3 + a_4}{2}\,, \; \frac{a_3 + a_4}{2}\,, \; a_5$$

if necessary). So the above result follows from Theorem 1. The above example shows that there is no such symmetry ($a_1 = a_2$, $a_3 = a_4$) for the general case. Indeed, we cannot even assume $a_3 = a_4$. For if $a_3 = a_4$, then for the above example, we have

$$\frac{5}{7} = d(S(3), T(3)) \leqq \frac{5}{7} a_1 + \frac{4}{7} a_2 + a_4 + \frac{4}{7} a_5 .$$
$$= \frac{5}{7} a_1 + \frac{4}{7} a_2 + \frac{1}{2} a_3 + \frac{1}{2} a_4 + \frac{4}{7} a_5$$
$$< \frac{5}{7} (a_1 + a_2 + a_3 + a_4 + a_5) < \frac{5}{7} ,$$

a contradiction.

**2. Extensions and some ralated results.** The following result generalizes Theorem 1. Its proof is different from the one we gave for Theorem 1.

THEOREM 2. *Let $S$, $T$ be functions on a nonempty complete metric space $(X, d)$. Suppose that there exist decreasing functions $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_4$, $\alpha_5$ of* $(0, \infty)$ *into* $[0, 1)$ *such that*

(a) $\sum_{i=1}^5 \alpha_i < 1$;

(b) $\alpha_1 = \alpha_2$ *or* $\alpha_3 = \alpha_4$;

(c) $\lim_{t\downarrow 0}(\alpha_2 + \alpha_3) < 1$ and $\lim_{t\downarrow 0}(\alpha_1 + \alpha_4) < 1$;

(d) *for any distinct $x$, $y$ in $X$,*

$$d(S(x), T(y)) \leqq a_1 d(x, S(x)) + a_2 d(y, T(y)) + a_3 d(x, T(y)) + a_4 d(y, S(x)) + a_5 d(x, y) ,$$

*where $a_i = \alpha_i(d(x, y))$.*

*Then at least one of $S$, $T$ has a fixed point. If both $S$ and $T$ have fixed points, then each of $S$, $T$ has a unique fixed point and these two fixed points coincide.*

*Proof.* Let $x_0 \in X$. Define for each $n = 0, 1, 2, \cdots$,

$$x_{2n+1} = S(x_{2n}) , \quad x_{2n+2} = T(x_{2n+1}) , \quad b_n = d(x_n, x_{n+1}) .$$

We may assume that $b_n > 0$ for each $n$, for otherwise some $x_n$ is a fixed point of $S$ or $T$. Let

$$r(t) = \frac{\alpha_1(t) + \alpha_3(t) + \alpha_5(t)}{1 - \alpha_2(t) - \alpha_3(t)} , \qquad t > 0 ,$$

$$s(t) = \frac{\alpha_2(t) + \alpha_4(t) + \alpha_5(t)}{1 - \alpha_1(t) - \alpha_4(t)} , \qquad t > 0 .$$

Then $r$, $s$ are decreasing. From (a) and (c), the limits

$$r_0 = \lim_{t\downarrow 0} r(t) , \quad s_0 = \lim_{t\downarrow 0} s(t)$$

are nonnegative real numbers. Let

$$f(t) = r(t)s(t)\ , \qquad t > 0\ .$$

Then $f$ is decreasing and $f(t) < 1$ for each $t > 0$. As in the proof of Theorem 1, we have for each $n = 0, 1, 2, \cdots$,

$$b_{2n+1} \leqq r(b_{2n})b_{2n}\ , \tag{12}$$

$$b_{2n+2} \leqq s(b_{2n+1})b_{2n+1}\ . \tag{13}$$

Let $n$ be given. Then

$$b_{2n+3} \leqq r(b_{2n+2})\, s(b_{2n+1})b_{2n+1}\ , \tag{14}$$

$$b_{2n+2} \leqq s(b_{2n+1})\, r(b_{2n})b_{2n}\ . \tag{15}$$

Since $r$, $s$ are decreasing,

$$b_{2n+3} \leqq f(\min\{b_{2n+2}, b_{2n+1}\})b_{2n+1}\ , \tag{16}$$

$$b_{2n+2} \leqq f(\min\{b_{2n+1}, b_{2n}\})b_{2n}\ . \tag{17}$$

Since $f(t) < 1$ for each $t > 0$, $\{b_{2n+1}\}$, $\{b_{2n}\}$ are decreasing sequences. So $\{b_{2n+1}\}$, $\{b_{2n}\}$ converge respectively to some points $c_1$, $c_2$. We shall prove that $c_1 = 0$, $c_2 = 0$. From (12) and (13),

$$c_1 \leqq r_0 c_2\ , \qquad c_2 \leqq s_0 c_1\ .$$

So either both $c_1$, $c_2$ are zero or both $c_1$, $c_2$ are not zero. Suppose to the contrary that $c_1 \neq 0$, $c_2 \neq 0$. Then from (16) and (17),

$$b_{n+2} \leqq f(\min\{c_1, c_2\})b_n\ , \qquad n = 0, 1, 2, \cdots\ . \tag{18}$$

By induction,

$$b_{2n} \leqq (f(\min\{c_1, c_2\}))^n b_0 \qquad n = 0, 1, 2, \cdots\ . \tag{19}$$

So $c_2 = 0$, a contradiction. Therefore, $c_1 = c_2 = 0$. This proves that $\{b_n\}$ converges to 0.

Now we shall prove that $\{x_n\}$ is Cauchy. Suppose not. Then there exist $\varepsilon \in (0, \infty)$ and sequences $\{p(n)\}$, $\{q(n)\}$ such that for each $n \geqq 0$,

$$p(n) > q(n) > n\ , \tag{20}$$

$$d(x_{p(n)}, x_{q(n)}) \geqq \varepsilon\ , \tag{21}$$

and (by the well-ordering principle),

$$d(x_{p(n)-1}, x_{q(n)}) < \varepsilon\ . \tag{22}$$

Let $n \geqq 0$ be given, $c_n = d(x_{p(n)}, x_{q(n)})$. Then

$$\begin{aligned}\varepsilon &\leqq c_n \\ &\leqq d(x_{p(n)}, x_{p(n)-1}) + d(x_{p(n)-1}, x_{q(n)}) < b_{p(n)-1} + \varepsilon .\end{aligned} \tag{23}$$

From $c_1 = c_2 = 0$, we conclude that $\{c_n\}$ converges to $\varepsilon$ from the right. Let

$$I_1 = \{n\colon\ p(n), q(n) \text{ are odd}\} ,$$

$$I_2 = \{n\colon\ p(n) \text{ is odd, } q(n) \text{ is even}\} .$$

$$I_3 = \{n\colon\ p(n) \text{ is even, } q(n) \text{ is odd}\} ,$$

$$I_4 = \{n\colon\ p(n), q(n) \text{ are even}\} .$$

Then at least one of $I_1$, $I_2$, $I_3$, $I_4$ is infinite. Suppose first that $I_1$ is infinite. Let

$$d_n = d(x_{p(n)-1}, x_{q(n)}) , \qquad n = 0, 1, 2, \cdots .$$

Since $\{c_n\}$ converges to $\varepsilon$ and $\{b_n\}$ converges to 0, we conclude from (22) that $\{d_n\}$ converges to $\varepsilon$ from the left. Thus

$$J_1 = \{n \in I_1\colon\ x_{p(n)-1} \neq x_{q(n)}\}$$

is infinite. Let $n \in J_1$, $u_n = d(x_{p(n)-1}, x_{q(n)+1})$. Then

$$\begin{aligned}c_n = d(x_{p(n)}, x_{q(n)}) &\leqq d(x_{p(n)}, x_{q(n)+1}) + d(x_{q(n)+1}, x_{q(n)}) \\ &\leqq d(S(x_{p(n)-1}), T(x_{q(n)})) + b_{q(n)} .\end{aligned} \tag{24}$$

From (*d*),

$$\begin{aligned}d(S(x_{p(n)-1}), T(x_{q(n)})) \leqq\ & \alpha_1(d_n)b_{p(n)-1} + \alpha_2(d_n)b_{q(n)} + \alpha_3(d_n)u_n \\ & + \alpha_4(d_n)c_n + \alpha_5(d_n)d_n .\end{aligned} \tag{25}$$

From (24) and (25),

$$\begin{aligned}c_n \leqq\ & \alpha_1(d_n)b_{p(n)-1} + \alpha_2(d_n)b_{q(n)} + \alpha_3(d_n)u_n + \alpha_4(d_n)c_n \\ & + \alpha_5(d_n)d_n + b_{q(n)} .\end{aligned} \tag{26}$$

Without loss of generality, we may assume that each $\alpha_i$ is continuous from the left, for we can replace the $\alpha_i$'s by

$$\beta_i(t) = \lim_{s \uparrow t} \alpha_i(s) , \quad t > 0 , \qquad i = 1, 2, 3, 4, 5$$

and conditions (a), (b), (c), and (d) still hold. Thus

$$\lim_{n\to\infty} \alpha_i(d_n) = \alpha_i(\varepsilon) , \qquad i = 1, 2, 3, 4, 5 .$$

So from (26),

$$\varepsilon \leqq (\alpha_3(\varepsilon) + \alpha_4(\varepsilon) + \alpha_5(\varepsilon))\varepsilon < \varepsilon ,$$

a contradiction. Now suppose that $I_2$ is infinite. By a similar argument, $J_2 = \{n \in I_2: x_{p(n)-1} \neq x_{q(n)-1}\}$ is infinite. Let $n \in J_2$,

$$v_n = d(x_{p(n)-1}, x_{q(n)-1}) , \quad w_n = d(x_{p(n)}, x_{q(n)-1}) .$$

Then

$$\begin{aligned} c_n &= d(S(x_{p(n)-1}), T(x_{q(n)-1})) \\ &\leqq \alpha_1(v_n)b_{p(n)-1} + \alpha_2(v_n)b_{q(n)-1} + \alpha_3(v_n)d_n + \alpha_4(v_n)w_n + \alpha_5(v_n)v_n . \end{aligned} \tag{27}$$

Since $\{v_n\}$ converges to $\varepsilon$ (not necessarily from the left or right), we obtain the same contradiction from (27). The other two cases are similar to the above two except the roles of $S$, $T$ interchange. Hence $\{x_n\}$ is Cauchy. By completeness, $\{x_n\}$ converges to a point $x$ in $X$. Since $b_n > 0$ for each $n$, $J = \{n: x \neq x_{2n+1}\}$ or $K = \{n: x \neq x_{2n}\}$ is infinite. Suppose that $K$ is infinite. Let $n \in K$,

$$l_n = d(x, x_{2n}) , \quad h_n = d(x, x_{2n+1}) .$$

Then

$$\begin{aligned} d(x, T(x)) &\leqq d(x, x_{2n+1}) + d(x_{2n+1}, T(x)) \\ &= h_n + d(S(x_{2n}), T(x)) \\ &\leqq h_n + \alpha_1(l_n)b_{2n} + \alpha_2(l_n)d(x, T(x)) + \alpha_3(l_n)d(x_{2n}, T(x)) \\ &\quad + \alpha_4(l_n)h_n + \alpha_5(l_n)l_n \\ &\leqq h_n + \alpha_1(l_n)b_{2n} + \alpha_2(l_n)d(x, T(x)) + \alpha_3(l_n)[l_n + d(x, T(x))] \\ &\quad + \alpha_4(l_n)h_n + \alpha_5(l_n)l_n . \end{aligned}$$

So

$$\begin{aligned} d(x, T(x)) \leqq \frac{1 + \alpha_4(l_n)}{1 - \alpha_2(l_n) - \alpha_3(l_n)} h_n &+ \frac{\alpha_3(l_n) + \alpha_5(l_n)}{1 - \alpha_2(l_n) - \alpha_3(l_n)} l_n \\ &+ \frac{\alpha_1(l_n)}{1 - \alpha_2(l_n) - \alpha_3(l_n)} b_{2n} . \end{aligned} \tag{28}$$

From (a) and (c), the sequences

$$\frac{1 + \alpha_4(l_n)}{1 - \alpha_2(l_n) - \alpha_3(l_n)} , \quad \frac{\alpha_3(l_n) + \alpha_5(l_n)}{1 - \alpha_2(l_n) - \alpha_3(l_n)} , \quad \frac{\alpha_1(l_n)}{1 - \alpha_2(l_n) - \alpha_3(l_n)}$$

are bounded. So from (28), $T(x) = x$. Similarly, $S(x) = x$ if $J$ is infinite. Hence $S$ or $T$ has a fixed point.

The following result follows easily from Theorem 2.

THEOREM 3. *With the conditions of Theorem 2, if further,*

$$d(S(x), T(x)) \leqq \alpha [d(x, S(x)) + d(x, T(x))] , \quad x \in X$$

*for some* $\alpha \in [0, 1)$, *then each of* $S$, $T$ *has a unique fixed point and these two fixed points coincide.*

We remark that the conditions of Theorem 1 imply the conditions of Theorem 3. Also, G. Hardy and T. Rogers [5, Theorem 2] gave a different proof for the case $S = T$. Their proof cannot be modified for the general case. To see that the conclusion of Theorem 2 is best possible, we note that if $X = \{0, 1\}$ with the usual distance and if $S$, $T$ are two distinct functions of $X$ onto $X$, then $S$, $T$ satisfy the conditions of Theorem 2 (and Theorem 3 with $\alpha = 1$), but one has two fixed points and the other has none.

THEOREM 4. *Let* $(X, d)$ *be a nonempty compact metric space. Let* $S$, $T$ *be functions of* $X$ *into itself. Suppose that* $S$ *or* $T$ *is continuous. Suppose further that there exist nonnegative real-valued decreasing functions* $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ on $(0, \infty)$ *such that*

(a) $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 \leqq 1$,

(b) $\alpha_1 = \alpha_2$ *and* $\alpha_3 = \alpha_4$,

(c) *for any distinct* $x$, $y$ *in* $X$,

$$d(S(x), T(y)) < a_1 d(x, S(x)) + a_2 d(y, T(y)) + a_3 d(x, T(y)) + a_4 d(y, S(x)) + a_5 d(x, y) ,$$

*where* $a_i = \alpha_i(d(x, y))$.

*Then* $S$ *or* $T$ *has a fixed point. If both* $S$ *and* $T$ *have fixed points, then each of* $S$ *and* $T$ *has a unique fixed point and these two fixed points coincide.*

*Proof*. By symmetry, we may assume that $S$ is continuous. Let $f$ be the function on $X$ such that

$$f(x) = d(x, S(x)) , \quad x \in X .$$

Then $f$ is continuous (we merely need the fact that $f$ is lower semi-continuous) on $X$. So $f$ takes its minimum value at some $x_0$ in $X$. We claim that $x_0$ is a fixed point of $S$ or $S(x_0)$ is a fixed point of $T$. Suppose not. Let

$$x_1 = S(x_0) , \quad x_2 = T(x_1) , \quad x_3 = S(x_2) ,$$
$$b_0 = d(x_0, x_1), \quad b_1 = d(x_2, x_3), \quad b_2 = d(x_2, x_3) .$$

Then $b_0 > 0$, $b_1 > 0$. From (c), we can prove that

$$(1 - \alpha_2(b_0) - \alpha_3(b_0))b_1 < (\alpha_1(b_0) + \alpha_3(b_0) + \alpha_5(b_0))b_0 . \tag{29}$$

Let

$$p(t) = 1 - \alpha_2(t) - \alpha_3(t)\ , \quad q(t) = \alpha_1(t) + \alpha_3(t) + \alpha_5(t)\ , \quad t > 0\ .$$

From (a) and (b), $p(b_0) > 0$. So

$$b_1 < \frac{q(b_0)}{p(b_0)}\, b_0\ . \tag{30}$$

Similarly,

$$b_2 < \frac{v(b_1)}{u(b_1)}\, b_1\ , \tag{31}$$

where

$$u(t) = 1 - \alpha_1(t) - \alpha_4(t), v(t) = \alpha_2(t) + \alpha_4(t) + \alpha_5(t)\ , \quad t > 0\ .$$

From (30) and (31),

$$b_2 < \frac{v(b_1)}{u(b_1)} \frac{q(b_0)}{p(b_0)}\, b_0\ . \tag{32}$$

It suffices to prove that $(v(b_1)q(b_0)/u(b_1)p(b_0)) < 1$, for then, $b_2 < b_0$, a contradiction to the minimality of $b_0$. Let $b = \min\{b_0, b_1\}$. Then

$$v(b_1)q(b_0) - u(b_1)p(b_0) \leqq v(b)q(b) - u(b)p(b) < 0$$

if $\alpha_1 = \alpha_2$ and $\alpha_3 = \alpha_4$. So $S$ or $T$ has a fixed point. Now suppose that $x$ is a fixed point of $S$ and $y$ is a fixed point of $T$. Then $x = y$, otherwise, from (c),

$$d(x, y) = d(S(x), T(y)) < d(x, y)\ ,$$

a contradiction.

The following result is stated without proof.

THEOREM 5. *Let $(X, d)$ be complete metric space. Let $\{S_n\}$, $\{T_n\}$ be sequence of functions of $X$ into $X$ which converge pointwise to $S$, $T$ respectively. Suppose that the pairs $(S_n, T_n)$ satisfy the conditions of Theorem 3 with the same $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$. Then $S$, $T$ have a unique common fixed point $x$ and $x$ is the limit of the sequence $\{x_n\}$ of the fixed points $x_n$ of $S_n$.*

THEOREM 6. *Let $(X, d)$ be a nonempty compact metric space. Let $\{S_n\}$, $\{T_n\}$ be sequences of functions of $X$ into itself which converge pointwise to the functions $S$, $T$ on $X$ respectively. Suppose that for each $n$, there exist decreasing functions $\alpha_1^n, \alpha_2^n, \alpha_3^n, \alpha_4^n, \alpha_5^n$ of $(0, \infty)$ into $[0, \infty)$ such that*

(a) $\alpha_1^n + \alpha_2^n + \alpha_3^n + \alpha_4^n + \alpha_5^n \leqq 1$,
(b) $\alpha_1^n = \alpha_2^n$ *and* $\alpha_3^n = \alpha_4^n$,
(c) *for any distinct* $x$, $y$ *in* $X$,

$$d(S_n(x), T_n(y)) < a_1^n d(x, S_n(x)) + a_2^n d(y, T_n(y)) + a_3^n d(x, T_n(y)) + a_4^n d(y, S_n(x)) + a_5^n d(x, y) ,$$

*where*

$$a_i^n = \alpha_i^n(d(x, y)) .$$

*Then* $S$ *or* $T$ *has a fixed point. Indeed, every cluster point of a sequence* $\{x_n\}$ *of fixed points* $x_n$ *of* $S_n$ *or* $T_n$ *is a fixed point of* $S$ *or* $T$.

*Proof*. By Theorem 4, for each $n$, either $S_n$ or $T_n$ has a fixed point. By symmetry, we may assume that $S_n$ has a fixed point for infinitely many of $n$'s. So there is a subsequence $\{S_{n(k)}\}$ of $\{S_n\}$ such that each $S_{n(k)}$ has a fixed point, say $x_k$. By compactness, we may (by taking a subsequence) assume that $\{x_k\}$ converges to some $x$ in $X$. We shall prove that $x$ is a fixed point of $S$ or $T$. If $x_k \neq x$ for only finitely many of $k$'s, then

$$\begin{aligned} S(x) &= \lim_{k\to\infty} S_{n(k)}(x) \\ &= \lim_{k\to\infty} S_{n(k)}(x_k) \\ &= \lim_{k\to\infty} x_k \\ &= x . \end{aligned}$$

So we may assume that $x_k \neq x$ for infinitely many of $k$'s. By taking a subsequence, we may assume that $x_k \neq x$ for each $k$. Let $k \geqq 1$ and $b_k = d(x, x_k)$. Then

$$\begin{aligned} d(x, T(x)) &\leqq d(x, x_k) + d(x_k, T_{n(k)}(x)) + d(T_{n(k)}(x), T(x)) \\ &= d(x, x_k) + d(S_{n(k)}(x_k), T_{n(k)}(x)) + d(T_{n(k)}(x), T(x)) . \end{aligned} \tag{33}$$

From (c),

$$\begin{aligned} d(S_{n(k)}(x_k), T_{n(k)}(x)) &< \alpha_2^k(b_k)d(x, T_{n(k)}(x)) + \alpha_3^k(b_k)d(x_k, T_{n(k)}(x)) \\ &\quad + \alpha_4^k(b_k)d(x, x_k) + \alpha_5^k(b_k)b_k . \end{aligned} \tag{34}$$

Combining (33) and (34) and letting $k$ tend to the infinity, we have

$$\begin{aligned} d(x, T(x)) &\leqq \limsup_{k\to\infty} (\alpha_2^k(b_k) + \alpha_3^k(b_k))d(x, T(x)) \\ &\leqq \limsup_{k\to\infty} \lim_{t\downarrow 0} (\alpha_2^k(t) + \alpha_3^k(t))d(x, T(x)) . \end{aligned} \tag{35}$$

From (b), $\alpha_2^k(t) + \alpha_3^k(t) \leqq 1/2$ for each $t > 0$, $k = 1, 2, \cdots$. So

$$\limsup_{k\to\infty} \lim_{t\downarrow 0} (\alpha_2^k(t) + \alpha_3^k(t)) \leqq \frac{1}{2} . \tag{36}$$

From (35) and (36), we conclude that $T(x) = x$.

From the proof, we know that the same conclusion holds if in Theorem 6, we replace (b) by the following weaker conditions:

$$\alpha_1^n = \alpha_2^n \quad \text{or} \quad \alpha_3^n = \alpha_4^n ,$$

$$\limsup_{k\to\infty} \lim_{t\downarrow 0} (\alpha_2^k(t) + \alpha_3^k(t)) < 1 ,$$

and

$$\limsup_{k\to\infty} \lim_{t\downarrow 0} (\alpha_1^n(t) + \alpha_4^n(t)) < 1 .$$

We note that, unlike Theorem 5, $S$, $T$ in Theorem 6 need not satisfy the condition required for the pairs $(S_n, T_n)$.

THEOREM 7. *Let $(X, d)$ be a nonempty compact metric space. Let $\{S_n\}$ be a sequence of functions of $X$ into itself which converges pointwise to some function $S$ on $X$. Suppose that for each $n$, there exist decreasing functions $\alpha_1^n$, $\alpha_2^n$, $\alpha_3^n$, $\alpha_4^n$, $\alpha_5^n$ of $(0, \infty)$ into $[0, \infty)$ such that*

(a) $\alpha_1^n + \alpha_2^n + \alpha_3^n + \alpha_4^n + \alpha_5^n \leqq 1$,

(b) *for any distinct $x$, $y$ in $X$,*

$$d(S_n(x), S_n(y)) < a_1 d(x, S_n(x)) + a_2 d(y, S_n(y)) + a_3 d(x, S_n(y)) + a_4 d(y, S_n(x)) + a_5 d(x, y) ,$$

*where*

$$a_i = \alpha_i(d(x, y)) .$$

*Then $S$ has a fixed point. Indeed, every cluster point of the sequence of fixed points of $S_n$ is a fixed point of $S$.*

The above result follows from Theorem 6 by averaging two applications of condition (b).

We shall now give a simple example to show that the conclusion of Theorem 7 is best possible. Let $X$ be a star-shaped [4] compact subset of a normed linear space $B$. Then there exists a point $z$ in $X$ such that for any $y$ in $X$, the line segment

$$\{tz + (1 - t)y\colon\ t \in [0, 1]\}$$

is contained in $X$. For each $n$, let

$$S_n(x) = \frac{1}{n} z + \left(1 - \frac{1}{n}\right) x , \qquad x \in X .$$

Then $\{S_n\}$ is a sequence of mappings of $X$ into $X$ which satisfy the conditions of Theorem 7. $\{S_n\}$ converges pointwise to the identity function $S$ on $X$. Every point of $X$ is a fixed point of $S$. So unlike Theorem 5, it is too much to ask that $S$ in Theorem 7 has a unique fixed point.

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