COMMON FIXED POINTS OF TWO MAPPINGS

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Let $S, T$ be functions on a nonempty complete metric space $(X, d)$. The main result of this paper is the following. $S$ or $T$ has a fixed point if there exist decreasing functions $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ of $(0, \infty)$ into $[0, 1)$ such that (a) $\sum_{i=1}^{5} \alpha_i < 1$; (b) $\alpha_1 = \alpha_2$ or $\alpha_3 = \alpha_4$; (c) $\lim_{i \to 0} (\alpha_1 + \alpha_2) < 1$ and $\lim_{i \to 0} (\alpha_3 + \alpha_4) < 1$; and (d) for any distinct $x, y$ in $X$,

$$d(S(x), T(y)) \leq \alpha_1 d(x, S(x)) + \alpha_2 d(y, T(y)) + \alpha_3 d(x, T(y)) + \alpha_4 d(y, S(x)) + \alpha_5 d(x, y),$$

where $\alpha_i = a_i(d(x, y))$. A number of related results are obtained.

1. Introduction. Let $(X, d)$ be a nonempty complete metric space and let $S, T$ be mappings of $X$ into itself which are not necessarily continuous nor commuting. Suppose that there are nonnegative real numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ such that

(a) $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1$,

(b) $\alpha_1 = \alpha_2$ or $\alpha_3 = \alpha_4$,

and for any $x, y$ in $X$,

(c) $d(S(x), T(y)) \leq \alpha_1 d(x, S(x)) + \alpha_2 d(y, T(y)) + \alpha_3 d(x, T(y)) + \alpha_4 d(y, S(x)) + \alpha_5 d(x, y)$.

It is proved in this paper that each of $S, T$ has a unique fixed point and these two fixed points coincide. Among others, a generalization is obtained by replacing $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ with nonnegative real-valued functions on $(0, \infty)$. This result generalizes the Banach contraction mapping theorem and some results of G. Hardy and T. Rogers [5], R. Kannan [7], E. Rakotch [8], S. Reich [9], P. Srivastava, and V. K. Gupta [10]. It also gives a different proof for these special cases. Note that even if $X = [0, 1]$ and if $T_1, T_2$ are commuting continuous functions of $X$ into itself, $T_1, T_2$ need not have a common fixed point [1], [2], and [6].

2. Basic results.

**Theorem 1.** Let $S, T$ be mappings of a complete metric space $(X, d)$ into itself. Suppose that there exist nonnegative real numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ which satisfy (a), (b), and (c). Then each of $S, T$
has a unique fixed point and these two fixed points coincide.

Proof. Let \( x_0 \in X \). Define

\[
  x_{2n+1} = S(x_{2n}), \quad x_{2n+2} = T(x_{2n+1}), \quad n = 0, 1, 2, \ldots
\]

From (c),

\[
  d(x, x_2) = d(S(x_0), T(x_1)) \\
  \leq (a_1 + a_5)d(x_0, x_1) + a_2d(x_1, x_2) + a_3d(x_2, x_3) \\
  \leq (a_1 + a_5)d(x_0, x_1) + a_2d(x_1, x_2) + a_3\left(d(x_2, x_1) + d(x_1, x_2)\right).
\]

So

\[
  (1) \quad d(x_1, x_2) \leq \frac{a_1 + a_5 + a_3}{1 - a_2 - a_5}d(x_0, x_1).
\]

Similarly,

\[
  (2) \quad d(x_2, x_3) \leq \frac{a_2 + a_4 + a_5}{1 - a_1 - a_4}d(x_1, x_2).
\]

Let

\[
  r = \frac{a_1 + a_5 + a_3}{1 - a_2 - a_5}, \quad s = \frac{a_2 + a_4 + a_5}{1 - a_1 - a_4}.
\]

Repeating the above argument, we obtain, for each \( n = 0, 1, 2, \ldots \),

\[
  (3) \quad d(x_{2n+1}, x_{2n+2}) \leq rd(x_{2n+1}, x_{2n}), \\
  (4) \quad d(x_{2n+2}, x_{2n+3}) \leq sd(x_{2n+2}, x_{2n+1}).
\]

By (3), (4), and induction, we have, for each \( n = 0, 1, 2, \ldots \),

\[
  (5) \quad d(x_{2n+1}, x_{2n+2}) \leq rs^nd(x_0, x_1), \\
  (6) \quad d(x_{2n+2}, x_{2n+3}) \leq rs^{n+1}d(x_0, x_1).
\]

Since \( rs < 1 \) and

\[
  \sum_{n=0}^{\infty} d(x_n, x_{n+1}) \leq (1 + r) \sum_{n=0}^{\infty} (rs)^n d(x_0, x_1),
\]

\( \{x_n\} \) is Cauchy. By completeness of \((X, d)\), \( \{x_n\} \) converges to some point \( x \) in \( X \). We shall now prove that \( x \) is a fixed point of \( S \) and \( T \). Let \( n \) be given. Then

\[
  d(x, S(x)) \leq d(x, x_{2n+2}) + d(S(x), x_{2n+2}) \\
  = d(x, x_{2n+2}) + d(S(x), T(x_{2n+1})).
\]

By (e),
Combining (7) and (8) and letting \( n \) tend to infinity, we obtain

\[
d(x, S(x)) \leq (a_1 + a_4) d(x, S(x)) .
\]

Since \( a_1 + a_4 < 1 \), \( S(x) = x \). Similarly \( T(x) = x \). Let \( y \) be a fixed point of \( T \). Then from \( d(x, y) = d(S(x), T(y)) \) and (c), we obtain

\[
d(x, y) \leq (a_3 + a_4 + a_5) d(x, y) .
\]

Since \( a_3 + a_4 + a_5 < 1 \), \( d(x, y) = 0 \). So \( T \) has a unique fixed point. Similarly, \( S \) has a unique fixed point.

When \( a_3 = a_4 = a_5 = 0 \), \( S = T \) and \( T \) is continuous (or even \( x \to d(x, T(x)) \) is lower semicontinuous) on \( X \), Theorem 1 can be obtained by an earlier result of the author [11, Theorem 1].

From the proof of Theorem 1, we know that \( S, T \) still have a common fixed point if conditions (a), (b) are replaced by the following conditions:

\[
(a_3 + a_4 + a_5)(a_2 + a_4 + a_5) < (1 - a_2 - a_5)(1 - a_1 - a_4) ,
\]

(9) \( a_1 + a_4 < 1 \).

(10)

If in addition,

\[
a_3 + a_4 + a_5 < 1 ,
\]

then the common fixed point of \( S, T \) is the unique fixed point of \( S \) (and \( T \)). Note that conditions (a) and (b) imply (9), but (a) alone does not. Indeed, for any \( a_1, a_2, a_5 \) in \( [0, \infty) \) with \( a_1 \neq a_2 \) and \( a_1 + a_2 + a_5 < 1 \), we can always find \( a_3, a_4 \) in \( [0, \infty) \) such that (a) holds but (9) does not. This can be seen by considering the affine function \( f' \):

\[
f(x, y) = (1 - a_2 - x)(1 - a_1 - y) - (a_1 + x + a_2)(a_2 + y + a_5)
\]

defined on the compact convex set

\[
K = \{(x, y) \in [0, 1] \times [0, 1]: a_1 + a_2 + a_3 + a_4 + a_5 \leq 1\} .
\]

\( f \) takes its minimum value at one of the extreme points of \( K \). With some computation, we conclude that

\[
\min f(K) = -|a_1 - a_2|(1 - a_1 - a_2 - a_5) .
\]

Since \( a_1 + a_2 + a_5 > 1 \), \( \min f(K) < 0 \) if and only if \( a_1 \neq a_2 \). Thus if \( a_1 \neq a_2 \), then by continuity of \( f \), there exists a point \((a_3, a_4)\) in
such that \( f(a, a) < 0 \).

**Corollary 1.** R. Kannan [7, Theorem 1]. Let \( S \) be a mapping of a complete metric space \((X, d)\) into itself. Suppose that there exists a number \( r \) in \([0, 1/2)\) such that
\[
d(S(x), S(y)) \leq r(d(x + S(x)) + d(y, S(y)))
\]
for all \( x, y \) in \( X \). Then \( S \) has a unique fixed point.

**Corollary 2.** P. Srivastava and V. K. Gupta [10, Theorem 1]. Let \( S, T \) be mappings of a complete metric space \((X, d)\) into itself. Suppose that there exists nonnegative real numbers \( a_1, a_2 \) such that
\[
\begin{align*}
(a) & \quad a_1 + a_2 < 1 \\
(b) & \quad d(S(x), T(y)) \leq a_1d(x, S(x)) + a_2d(y, T(y))
\end{align*}
\]
for all \( x, y \) in \( X \).

Then \( S, T \) have a unique common fixed point.

Srivastava and Gupta stated the above result in a more general form with \( S, T \) replaced by \( S^p, T^q \) for some positive integers \( p, q \). Since the unique fixed point of \( S^p \) (similarly \( T^q \)) is the unique fixed point of \( S \), this result is equivalent to Corollary 2.

For Corollaries 1 and 2, we have the following related result.

**Proposition.** Let \( S, T \) be self-maps of a nonempty complete metric space \((X, d)\). Suppose that there exist nonnegative real numbers \( a_1, a_2 \) such that \( a_1 + a_2 < 1 \) and
\[
\begin{align*}
(*) & \quad d(S(x), T(y)) \leq a_1d(x, S(x)) + a_2d(y, T(y)) , \quad x, y \in X.
\end{align*}
\]
Then either \((*)\) is true when all of its \( S \) are replaced by \( T \) or \((*)\) is true when all of its \( T \) are replaced by \( S \).

The following example proves that our result is actually more general than that of Srivastava and Gupta.

**Example.** Let \( X = \{1, 2, 3\} \). Let \( d \) be the metric for \( X \) determined by
\[
d(1, 2) = 1, \quad d(2, 3) = \frac{4}{7}, \quad d(1, 3) = \frac{5}{7}.
\]
Let $S, T$ be the function on $X$ such that
\[ S(1) = S(2) = S(3) = 1; \]
\[ T(1) = T(3) = 1, \quad T(2) = 3. \]
Let $a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 5/7, a_5 = 0$. Then the conditions of Theorem 1 are satisfied. However, no nonnegative real numbers $a_1, a_2, a_3, a_4$ can be chosen such that $a_1 + a_2 + a_3 + a_4 < 1$ and for $x, y \in X$,
\[ d(S(x), T(y)) \leq a_1 d(x, S(x)) + a_2 d(y, T(y)) + a_3 d(x, T(y)) + a_4 d(x, y). \]
For if there exist such $a_1, a_2, a_3, a_4$, then
\[ d(S(3), T(2)) \leq a_1 d(3, S(3)) + a_2 d(2, T(2)) + a_3 d(3, T(2)) + a_4 d(3, 2). \]
So
\[ \frac{5}{7} \leq \frac{5a_1}{7} + \frac{4a_2}{7} + \frac{4a_3}{7} \leq \frac{5}{7} (a_1 + a_2 + a_3) < \frac{5}{7}, \]
a contradiction.

**Corollary 3.** G. Hardy and T. Rogers [5, Theorem 1]. Let $S$ be a mapping of a nonempty complete metric space $(X, d)$ into itself. Suppose that there exist nonnegative real numbers $a_1, a_2, a_3, a_4, a_5$ such that
(a) \[ a_1 + a_2 + a_3 + a_4 + a_5 < 1 \]
and
(b) \[ d(S(x), S(y)) \leq a_1 d(x, S(x)) + a_2 d(y, S(y)) + a_3 d(x, S(y)) + a_4 d(y, S(x)) + a_5 d(x, y) \]
for all $x, y \in X$.
Then $S$ has a unique fixed point.

Note that in the above case, we may without loss of generality assume that $a_1 = a_2, a_3 = a_4$ (replace $a_1, a_2, a_3, a_4, a_5$ respectively by $a_1 + a_2, a_1 + a_3, a_1 + a_4, a_1 + a_5$, if necessary). So the above result follows from Theorem 1. The above example shows that there is no such symmetry $(a_1 = a_2, a_3 = a_4)$ for the general case. Indeed, we cannot even assume $a_3 = a_4$. For if $a_3 = a_4$, then for the above example, we have
\[
\frac{5}{7} = d(S(3), T(3)) \leq \frac{5}{7} a_1 + \frac{4}{7} a_2 + a_4 + \frac{4}{7} a_5.
\]
\[
= \frac{5}{7} a_1 + \frac{4}{7} a_2 + \frac{1}{2} a_3 + \frac{1}{2} a_4 + \frac{4}{7} a_5
\]
\[
< \frac{5}{7} (a_1 + a_2 + a_3 + a_4 + a_5) < \frac{5}{7},
\]
a contradiction.

2. Extensions and some related results. The following result generalizes Theorem 1. Its proof is different from the one we gave for Theorem 1.

**Theorem 2.** Let \( S, T \) be functions on a nonempty complete metric space \((X, d)\). Suppose that there exist decreasing functions \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \) of \((0, \infty)\) into \([0,1)\) such that

(a) \( \sum_{i=1}^{5} \alpha_i < 1 \);
(b) \( \alpha_1 = \alpha_2 \) or \( \alpha_3 = \alpha_4 \);
(c) \( \lim_{t \to 0^+} (\alpha_2 + \alpha_3) < 1 \) and \( \lim_{t \to 0^+} (\alpha_1 + \alpha_4) < 1 \);
(d) for any distinct \( x, y \) in \( X \),
\[
d(S(x), T(y)) \leq a_1 d(x, S(x)) + a_2 d(y, T(y)) + a_3 d(x, T(y)) + a_4 d(y, S(x)) + a_5 d(x, y),
\]
where \( \alpha_i = \alpha_i(d(x, y)) \).

Then at least one of \( S, T \) has a fixed point. If both \( S \) and \( T \) have fixed points, then each of \( S, T \) has a unique fixed point and these two fixed points coincide.

**Proof.** Let \( x_0 \in X \). Define for each \( n = 0, 1, 2, \ldots \),
\[
x_{2n+1} = S(x_{2n}) , \quad x_{2n+2} = T(x_{2n+1}) , \quad b_n = d(x_n, x_{n+1}).
\]
We may assume that \( b_n > 0 \) for each \( n \), for otherwise some \( x_n \) is a fixed point of \( S \) or \( T \). Let
\[
r(t) = \frac{\alpha_1(t) + \alpha_2(t) + \alpha_5(t)}{1 - \alpha_4(t) - \alpha_5(t)} , \quad t > 0,
\]
\[
s(t) = \frac{\alpha_3(t) + \alpha_4(t) + \alpha_5(t)}{1 - \alpha_1(t) - \alpha_5(t)} , \quad t > 0.
\]
Then \( r, s \) are decreasing. From (a) and (c), the limits
\[
r_0 = \lim_{t \to 0^+} r(t) , \quad s_0 = \lim_{t \to 0^+} s(t)
\]
are nonnegative real numbers. Let
COMMON FIXED POINTS OF TWO MAPPINGS

$$f(t) = r(t)s(t), \quad t > 0.$$  

Then $f$ is decreasing and $f(t) < 1$ for each $t > 0$. As in the proof of Theorem 1, we have for each $n = 0, 1, 2, \ldots$,

$$(12) \quad b_{2n+1} \leq r(b_{2n})b_{2n},$$

$$(13) \quad b_{2n+2} \leq s(b_{2n+1})b_{2n+1}.$$  

Let $n$ be given. Then

$$(14) \quad b_{2n+3} \leq r(b_{2n+2})s(b_{2n+1})b_{2n+1},$$

$$(15) \quad b_{2n+4} \leq s(b_{2n+3})r(b_{2n})b_{2n}.$$  

Since $r$, $s$ are decreasing,

$$(16) \quad b_{2n+5} \leq f(\min \{b_{2n+2}, b_{2n+3}\})b_{2n+1},$$

$$(17) \quad b_{2n+6} \leq f(\min \{b_{2n+3}, b_{2n}\})b_{2n}.$$  

Since $f(t) < 1$ for each $t > 0$, $\{b_{n+1}\}$, $\{b_n\}$ are decreasing sequences. So $\{b_{2n+1}\}$, $\{b_{2n}\}$ converge respectively to some points $c_1$, $c_2$. We shall prove that $c_1 = 0$, $c_2 = 0$. From (12) and (13),

$$c_1 \leq r_c_2, \quad c_2 \leq s_c_1.$$  

So either both $c_1$, $c_2$ are zero or both $c_1$, $c_2$ are not zero. Suppose to the contrary that $c_1 \neq 0$, $c_2 \neq 0$. Then from (16) and (17),

$$(18) \quad b_{n+2} \leq f(\min \{c_1, c_2\})b_n, \quad n = 0, 1, 2, \ldots.$$  

By induction,

$$(19) \quad b_{2n} \leq f(\min \{c_1, c_2\})^n b_0 \quad n = 0, 1, 2, \ldots.$$  

So $c_2 = 0$, a contradiction. Therefore, $c_1 = c_2 = 0$. This proves that $\{b_n\}$ converges to 0.

Now we shall prove that $\{x_n\}$ is Cauchy. Suppose not. Then there exist $\varepsilon \in (0, \infty)$ and sequences $\{p(n)\}$, $\{q(n)\}$ such that for each $n \geq 0$,

$$(20) \quad p(n) > q(n) > n,$$

$$(21) \quad d(x_{p(n)}, x_{q(n)}) \geq \varepsilon,$$

and (by the well-ordering principle),

$$(22) \quad d(x_{p(n)-1}, x_{q(n)}) < \varepsilon.$$  

Let $n \geq 0$ be given, $c_n = d(x_{p(n)}, x_{q(n)})$. Then
\(\epsilon \leq c_n\)
\[\leq d(x_{p(n)}, x_{p(n)-1}) + d(x_{p(n)-1}, x_{q(n)}) < b_{p(n)-1} + \epsilon.\]

From \(c_i = c_i = 0\), we conclude that \(\{c_n\}\) converges to \(\epsilon\) from the right. Let

\[I_1 = \{n: p(n), q(n) \text{ are odd}\},\]
\[I_2 = \{n: p(n) \text{ is odd, } q(n) \text{ is even}\},\]
\[I_3 = \{n: p(n) \text{ is even, } q(n) \text{ is odd}\},\]
\[I_4 = \{n: p(n), q(n) \text{ are even}\}.

Then at least one of \(I_1, I_2, I_3, I_4\) is infinite. Suppose first that \(I_1\) is infinite. Let

\[d_n = d(x_{p(n)-1}, x_{q(n)}) , \quad n = 0, 1, 2, \ldots.\]

Since \(\{c_n\}\) converges to \(\epsilon\) and \(\{b_n\}\) converges to 0, we conclude from (22) that \(\{d_n\}\) converges to \(\epsilon\) from the left. Thus

\[J_1 = \{n \in I_1: x_{p(n)-1} = x_{q(n)}\}\]

is infinite. Let \(n \in J_1, u_n = d(x_{p(n)-1}, x_{q(n)+1})\). Then

\[c_n = d(x_{p(n)}), x_{q(n)} \leq d(x_{p(n)}, x_{q(n)+1}) + d(x_{q(n)+1}, x_{q(n)}) \leq d(S(x_{p(n)-1}), T(x_{q(n)})) + b_{q(n)} .\]

From (d),

\[d(S(x_{p(n)-1}), T(x_{q(n)})) \leq \alpha_1(d_n)b_{p(n)-1} + \alpha_2(d_n)b_{q(n)} + \alpha_3(d_n)u_n + \alpha_4(d_n)c_n + \alpha_5(d_n)d_n .\]

From (24) and (25),

\[c_n \leq \alpha_1(d_n)b_{p(n)-1} + \alpha_2(d_n)b_{q(n)} + \alpha_3(d_n)u_n + \alpha_4(d_n)c_n + \alpha_5(d_n)d_n + b_{q(n)} .\]

Without loss of generality, we may assume that each \(\alpha_i\) is continuous from the left, for we can replace the \(\alpha_i\)'s by

\[\beta_i(t) = \lim_{s \uparrow t} \alpha_i(s) , \quad t > 0 , \quad i = 1, 2, 3, 4, 5\]

and conditions (a), (b), (c), and (d) still hold. Thus

\[\lim_{n \to \infty} \alpha_i(d_n) = \alpha_i(\epsilon) , \quad i = 1, 2, 3, 4, 5 .\]

So from (26),

\[\epsilon \leq (\alpha_2(\epsilon) + \alpha_4(\epsilon) + \alpha_5(\epsilon))\epsilon < \epsilon ,\]
a contradiction. Now suppose that $I_2$ is infinite. By a similar argument, $J_2 = \{n \in I_2: x_{p(n)-1} \neq x_{q(n)-1}\}$ is infinite. Let $n \in J_2$,

$$v_n = d(x_{p(n)-1}, x_{q(n)-1}), \quad w_n = d(x_{p(n)}, x_{q(n)-1}).$$

Then

$$e_n = d(S(x_{p(n)-1}), T(x_{q(n)-1}))$$

1. If $I_2$ is infinite,

$$\leq \alpha_1(v_n)d_{p(n)-1} + \alpha_2(v_n)d_{q(n)-1} + \alpha_3(v_n)v_n + \alpha_4(v_n)w_n + \alpha_5(v_n)v_n.$$

Since $\{v_n\}$ converges to $\varepsilon$ (not necessarily from the left or right), we obtain the same contradiction from (27). The other two cases are similar to the above two except the roles of $S, T$ interchange. Hence $\{x_n\}$ is Cauchy. By completeness, $\{x_n\}$ converges to a point $x$ in $X$. Since $b_n > 0$ for each $n$, $J = \{n: x \neq x_{2n+1}\}$ or $K = \{n: x \neq x_{2n}\}$ is infinite. Suppose that $K$ is infinite. Let $n \in K$,

$$l_n = d(x, x_{2n}), \quad h_n = d(x, x_{2n+1}).$$

Then

$$d(x, T(x)) \leq d(x, x_{2n+1}) + d(x_{2n+1}, T(x))$$

1. If $I_2$ is infinite,

$$= h_n + d(S(x_{2n}), T(x))$$

$$\leq h_n + \alpha_1(l_n)b_{2n} + \alpha_2(l_n)d(x, T(x)) + \alpha_3(l_n)d(x_{2n}, T(x))$$

$$+ \alpha_4(l_n)h_n + \alpha_5(l_n)l_n$$

$$\leq h_n + \alpha_1(l_n)b_{2n} + \alpha_2(l_n)d(x, T(x)) + \alpha_3(l_n)[l_n + d(x, T(x))]$$

$$+ \alpha_4(l_n)h_n + \alpha_5(l_n)l_n.$$

So

$$d(x, T(x)) \leq \frac{1 + \alpha_1(l_n)}{1 - \alpha_3(l_n) - \alpha_5(l_n)} h_n + \frac{\alpha_1(l_n) + \alpha_3(l_n)}{1 - \alpha_3(l_n) - \alpha_5(l_n)} l_n$$

(28)

$$+ \frac{\alpha_4(l_n)}{1 - \alpha_3(l_n) - \alpha_5(l_n)} b_{2n}.$$

From (a) and (c), the sequences

$$\frac{1 + \alpha_1(l_n)}{1 - \alpha_3(l_n) - \alpha_5(l_n)}, \quad \frac{\alpha_1(l_n) + \alpha_3(l_n)}{1 - \alpha_3(l_n) - \alpha_5(l_n)}, \quad \frac{\alpha_4(l_n)}{1 - \alpha_3(l_n) - \alpha_5(l_n)}$$

are bounded. So from (28), $T(x) = x$. Similarly, $S(x) = x$ if $J$ is infinite. Hence $S$ or $T$ has a fixed point.

The following result follows easily from Theorem 2.

**Theorem 3.** With the conditions of Theorem 2, if further,

$$d(S(x), T(x)) \leq \alpha [d(x, S(x)) + d(x, T(x))], \quad x \in X$$
for some $\alpha \in [0, 1)$, then each of $S$, $T$ has a unique fixed point and these two fixed points coincide.

We remark that the conditions of Theorem 1 imply the conditions of Theorem 3. Also, G. Hardy and T. Rogers [5, Theorem 2] gave a different proof for the case $S = T$. Their proof cannot be modified for the general case. To see that the conclusion of Theorem 2 is best possible, we note that if $X = \{0, 1\}$ with the usual distance and if $S$, $T$ are two distinct functions of $X$ onto $X$, then $S$, $T$ satisfy the conditions of Theorem 2 (and Theorem 3 with $\alpha = 1$), but one has two fixed points and the other has none.

**THEOREM 4.** Let $(X, d)$ be a nonempty compact metric space. Let $S$, $T$ be functions of $X$ into itself. Suppose that $S$ or $T$ is continuous. Suppose further that there exist nonnegative real-valued decreasing functions $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ on $(0, \infty)$ such that

(a) $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 \leq 1$,
(b) $\alpha_1 = \alpha_2$ and $\alpha_3 = \alpha_4$,
(c) for any distinct $x, y$ in $X$,

$$d(S(x), T(y)) < \alpha_1 d(x, S(x)) + \alpha_2 d(y, T(y)) + \alpha_3 d(x, T(y)) + \alpha_4 d(y, S(x)) + \alpha_5 d(x, y),$$

where $\alpha_i = \alpha_i(d(x, y))$.

Then $S$ or $T$ has a fixed point. If both $S$ and $T$ have fixed points, then each of $S$ and $T$ has a unique fixed point and these two fixed points coincide.

**Proof.** By symmetry, we may assume that $S$ is continuous. Let $f$ be the function on $X$ such that

$$f(x) = d(x, S(x)),$$  \hspace{1cm}  $x \in X$.

Then $f$ is continuous (we merely need the fact that $f$ is lower semi-continuous) on $X$. So $f$ takes its minimum value at some $x_0$ in $X$. We claim that $x_0$ is a fixed point of $S$ or $S(x_0)$ is a fixed point of $T$. Suppose not. Let

$$x_1 = S(x_0), \quad x_2 = T(x_1), \quad x_3 = S(x_2),$$

$$b_0 = d(x_0, x_1), \quad b_1 = d(x_2, x_3), \quad b_2 = d(x_2, x_3).$$

Then $b_0 > 0$, $b_1 > 0$. From (c), we can prove that

$$1 - \alpha_4(b_0) - \alpha_5(b_0))b_1 < (\alpha_1(b_0) + \alpha_4(b_0) + \alpha_5(b_0))b_0.$$

Let
\[ p(t) = 1 - \alpha_2(t) - \alpha_4(t), \quad q(t) = \alpha_3(t) + \alpha_5(t) + \alpha_6(t), \quad t > 0. \]

From (a) and (b), \( p(b_0) > 0. \) So

\[ b_1 < \frac{q(b_0)}{p(b_0)} b_0. \]  

Similarly,

\[ b_2 < \frac{v(b_1)}{u(b_1)} b_1, \]

where

\[ u(t) = 1 - \alpha_1(t) - \alpha_2(t), \quad v(t) = \alpha_3(t) + \alpha_4(t) + \alpha_5(t), \quad t > 0. \]

From (30) and (31),

\[ b_2 < \frac{v(b_1) q(b_0)}{u(b_1) p(b_0)} b_0. \]

It suffices to prove that \( (v(b_1) q(b_0)) / (u(b_1) p(b_0)) < 1, \) for then, \( b_2 < b_0, \) a contradiction to the minimality of \( b_0. \) Let \( b = \min \{ b_0, b_1 \}. \) Then

\[ v(b_1) q(b_0) - u(b_1) p(b_0) \leq v(b) q(b) - u(b) p(b) < 0 \]

if \( \alpha_1 = \alpha_2 \) and \( \alpha_3 = \alpha_4. \) So \( S \) or \( T \) has a fixed point. Now suppose that \( x \) is a fixed point of \( S \) and \( y \) is a fixed point of \( T. \) Then \( x = y, \) otherwise, from (c),

\[ d(x, y) = d(S(x), T(y)) < d(x, y), \]

a contradiction.

The following result is stated without proof.

**Theorem 5.** Let \( (X, d) \) be complete metric space. Let \( \{S_n\}, \{T_n\} \) be sequence of functions of \( X \) into \( X \) which converge pointwise to \( S, T \) respectively. Suppose that the pairs \( (S_n, T_n) \) satisfy the conditions of Theorem 3 with the same \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5. \) Then \( S, T \) have a unique common fixed point \( x \) and \( x \) is the limit of the sequence \( \{x_n\} \) of the fixed points \( x_n \) of \( S_n. \)

**Theorem 6.** Let \( (X, d) \) be a nonempty compact metric space. Let \( \{S_n\}, \{T_n\} \) be sequences of functions of \( X \) into itself which converge pointwise to the functions \( S, T \) on \( X \) respectively. Suppose that for each \( n, \) there exist decreasing functions \( \alpha_1^*, \alpha_2^*, \alpha_3^*, \alpha_4^*, \alpha_5^* \) of \((0, \infty)\) into \([0, \infty)\) such that
(a) \(\alpha_i^* + \alpha_i^* + \alpha_i^* + \alpha_i^* \leq 1\),
(b) \(\alpha_i^* = \alpha_i^* \) and \(\alpha_i^* = \alpha_i^* \),
(c) for any distinct \(x, y \in X\),
\[
d(S_n(x), T_n(y)) < \alpha_i^* d(x, S_n(x)) + \alpha_i^* d(y, T_n(y)) + \alpha_i^* d(x, y) + \alpha_i^* d(y, S_n(x)) + \alpha_i^* d(x, y),
\]
where
\[
\alpha_i^* = \alpha_i^*(d(x, y)).
\]
Then \(S\) or \(T\) has a fixed point. Indeed, every cluster point of a sequence \(\{x_n\}\) of fixed points \(x_n\) of \(S_n\) or \(T_n\) is a fixed point of \(S\) or \(T\).

**Proof.** By Theorem 4, for each \(n\), either \(S_n\) or \(T_n\) has a fixed point. By symmetry, we may assume that \(S_n\) has a fixed point for infinitely many of \(n\)'s. So there is a subsequence \(\{S_{n(k)}\}\) of \(\{S_n\}\) such that each \(S_{n(k)}\) has a fixed point, say \(x_k\). By compactness, we may (by taking a subsequence) assume that \(\{x_k\}\) converges to some \(x\) in \(X\). We shall prove that \(x\) is a fixed point of \(S\) or \(T\). If \(x_k \neq x\) for only finitely many of \(k\)'s, then
\[
S(x) = \lim_{k \to \infty} S_{n(k)}(x) = \lim_{k \to \infty} x_k = x.
\]
So we may assume that \(x_k \neq x\) for infinitely many of \(k\)'s. By taking a subsequence, we may assume that \(x_k \neq x\) for each \(k\). Let \(k \geq 1\) and \(b_k = d(x, x_k)\). Then
\[
d(x, T(x)) \leq d(x, x_k) + d(x_k, T_m(x)) + d(T_m(x), T(x)) = d(x, x_k) + d(S_{n(k)}(x_k), T_{n(k)}(x)) + d(T_{n(k)}(x), T(x)).
\]
From (c),
\[
d(S_{n(k)}(x_k), T_{n(k)}(x)) < \alpha_i^* d(x, S_{n(k)}(x)) + \alpha_i^* d(y, T_{n(k)}(x)) + \alpha_i^* d(x, T_{n(k)}(x)) + \alpha_i^* d(y, S_{n(k)}(x)) + \alpha_i^* d(x, y) + \alpha_i^* d(y, S_{n(k)}(x)) + \alpha_i^* d(x, y).
\]
Combining (33) and (34) and letting \(k\) tend to the infinity, we have
\[
d(x, T(x)) \leq \limsup_{k \to \infty} (\alpha_i^*(b_k) + \alpha_i^*(b_k))d(x, T(x)) \leq \limsup_{k \to \infty} \sup_{t > 0} (\alpha_i^*(t) + \alpha_i^*(t))d(x, T(x)).
\]
From (b), \(\alpha_i^*(t) + \alpha_i^*(t) \leq 1/2\) for each \(t > 0\), \(k = 1, 2, \ldots\). So
From (35) and (36), we conclude that $T(x) = x$.

From the proof, we know that the same conclusion holds if in Theorem 6, we replace (b) by the following weaker conditions:

$$\alpha_i^* = \alpha_i^* \quad \text{or} \quad \alpha_i^* = \alpha_i^* ,$$

and

$$\limsup_{h \to \infty} \lim_{t \to 0} (\alpha_i^*(t) + \alpha_i^*(t)) < 1 .$$

We note that, unlike Theorem 5, $S, T$ in Theorem 6 need not satisfy the condition required for the pairs $(S_n, T_n)$.

**Theorem 7.** Let $(X, d)$ be a nonempty compact metric space. Let $\{S_n\}$ be a sequence of functions of $X$ into itself which converges pointwise to some function $S$ on $X$. Suppose that for each $n$, there exist decreasing functions $\alpha_i^*, \alpha_i^*, \alpha_i^*, \alpha_i^*$ of $(0, \infty)$ into $[0, \infty)$ such that

(a) $\alpha_i^* + \alpha_i^* + \alpha_i^* + \alpha_i^* \leq 1$,

(b) for any distinct $x, y$ in $X$,

$$d(S_n(x), S_n(y)) < a_1d(x, S_n(x)) + a_2d(y, S_n(y)) + a_3d(x, S_n(y)) + a_4d(y, S_n(x)) + a_5d(x, y) ,$$

where

$$a_i = \alpha_i(d(x, y)) .$$

Then $S$ has a fixed point. Indeed, every cluster point of the sequence of fixed points of $S_n$ is a fixed point of $S$.

The above result follows from Theorem 6 by averaging two applications of condition (b).

We shall now give a simple example to show that the conclusion of Theorem 7 is best possible. Let $X$ be a star-shaped [4] compact subset of a normed linear space $B$. Then there exists a point $z$ in $X$ such that for any $y$ in $X$, the line segment

$$\{tz + (1-t)y: t \in [0, 1]\}$$

is contained in $X$. For each $n$, let

$$S_n(x) = \frac{1}{n}z + \left(1 - \frac{1}{n}\right)x , \quad x \in X .$$
Then \( \{S_n\} \) is a sequence of mappings of \( X \) into \( X \) which satisfy the conditions of Theorem 7. \( \{S_n\} \) converges pointwise to the identity function \( S \) on \( X \). Every point of \( X \) is a fixed point of \( S \). So unlike Theorem 5, it is too much to ask that \( S \) in Theorem 7 has a unique fixed point.

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Jan Aarts and David John Lutzer, *Pseudo-completeness and the product of Baire spaces* ................................................................. 1
Gordon Owen Berg, *Metric characterizations of Euclidean spaces* ................................. 11
Ajit Kaur Chilana, *The space of bounded sequences with the mixed topology* ............... 29
Philip Throop Church and James Timourian, *Differentiable open maps of*  
\((p + 1)\)-manifold to \(p\)-manifold ......................................................... 35
P. D. T. A. Elliott, *On additive functions whose limiting distributions possess a finite mean and variance* ......................................... 47
M. Solveig Espelie, *Multiplicative and extreme positive operators* ................................. 57
Jacques A. Ferland, *Domains of negativity and application to generalized convexity on a real topological vector space* .............................. 67
Michael Benton Freeman and Reese Harvey, *A compact set that is locally holomorphically convex but not holomorphically convex* .................. 77
Roe William Goodman, *Positive-definite distributions and intertwining operators* ........ 83
Elliot Charles Gootman, *The type of some \(C^*\) and \(W^*\)-algebras associated with transformation groups* ........................................... 93
David Charles Haddad, *Angular limits of locally finitely valent holomorphic functions* .......................... 107
William Buhmann Johnson, *On quasi-complements* .................................................. 113
William M. Kantor, *On \(2\)-transitive collineation groups of finite projective spaces* .......................... 119
Joachim Lambek and Gerhard O. Michler, *Completions and classical localizations of right Noetherian rings* ........................................... 133
Kenneth Lamar Lange, *Borel sets of probability measures* ....................................... 141
David Lowell Lovelady, *Product integrals for an ordinary differential equation in a Banach space* .......................................................... 163
Jorge Martinez, *A hom-functor for lattice-ordered groups* ........................................ 169
W. K. Mason, *Weakly almost periodic homeomorphisms of the two sphere* ................. 185
Anthony G. Mucci, *Limits for martingale-like sequences* .......................................... 197
Eugene Michael Norris, *Relationally induced semigroups* ......................................... 203
Arthur E. Olson, *A comparison of \(c\)-density and \(k\)-density* ................................ 209
Donald Steven Passman, *On the semisimplicity of group rings of linear groups. II* ........ 215
Charles Radin, *Ergodicity in von Neumann algebras* ............................................... 235
P. Rosenthal, *On the singularities of the function generated by the Bergman operator of the second kind* .............................................. 241
Arthur Argyle Sagle and J. R. Schumi, *Multiplications on homogeneous spaces, nonassociative algebras and connections* ............................. 247
Leo Sario and Cecilia Wang, *Existence of Dirichlet finite biharmonic functions on the Poincaré 3-ball* ......................................................... 267
Ramachandran Subramanian, *On a generalization of martingales due to Blake* ............ 275
Bui An Ton, *On strongly nonlinear elliptic variational inequalities* ............................ 279
Seth Warner, *A topological characterization of complete, discretely valued fields* ........... 293
Chi Song Wong, *Common fixed points of two mappings* ......................................... 299