COMMON FIXED POINTS OF TWO MAPPINGS

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Let $S$, $T$ be functions on a nonempty complete metric space $(X, d)$. The main result of this paper is the following. $S$ or $T$ has a fixed point if there exist decreasing functions $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ of $(0, \infty)$ into $[0, 1)$ such that (a) $\sum_{i=1}^{5} \alpha_i < 1$; (b) $\alpha_1 = \alpha_2$ or $\alpha_3 = \alpha_4$, (c) $\lim_{t \to 0} (\alpha_1 + \alpha_2) < 1$ and $\lim_{t \to 1} (\alpha_3 + \alpha_4) < 1$ and (d) for any distinct $x, y$ in $X$,

$$d(S(x), T(y)) \leq \alpha_1 d(x, S(x)) + \alpha_2 d(y, T(y)) + \alpha_3 d(x, T(y)) + \alpha_4 d(y, S(x)) + \alpha_5 d(x, y),$$

where $a_i = \alpha_i(d(x, y))$. A number of related results are obtained.

1. Introduction. Let $(X, d)$ be a nonempty complete metric space and let $S$, $T$ be mappings of $X$ into itself which are not necessarily continuous nor commuting. Suppose that there are nonnegative real numbers $a_1, a_2, a_3, a_4, a_5$ such that

(a) $a_1 + a_2 + a_3 + a_4 + a_5 < 1$,

(b) $a_1 = a_2$ or $a_3 = a_4$,

and for any $x, y$ in $X$,

(c) $d(S(x), T(y)) \leq a_1 d(x, S(x)) + a_2 d(y, T(y)) + a_3 d(x, T(y)) + a_4 d(y, S(x)) + a_5 d(x, y)$.

It is proved in this paper that each of $S, T$ has a unique fixed point and these two fixed points coincide. Among others, a generalization is obtained by replacing $a_1, a_2, a_3, a_4, a_5$ with nonnegative real-valued functions on $(0, \infty)$. This result generalizes the Banach contraction mapping theorem and some results of G. Hardy and T. Rogers [5], R. Kannan [7], E. Rakotch [8], S. Reich [9], P. Srivastava, and V. K. Gupta [10]. It also gives a different proof for these special cases. Note that even if $X = [0, 1]$ and if $T_1, T_2$ are commuting continuous functions of $X$ into itself, $T_1, T_2$ need not have a common fixed point [1], [2], and [6].

2. Basic results.

THEOREM 1. Let $S$, $T$ be mappings of a complete metric space $(X, d)$ into itself. Suppose that there exist nonnegative real numbers $a_1, a_2, a_3, a_4, a_5$ which satisfy (a), (b), and (c). Then each of $S, T$
has a unique fixed point and these two fixed points coincide.

**Proof.** Let \( x_0 \in X \). Define
\[
x_{2n+1} = S(x_{2n}), \quad x_{2n+2} = T(x_{2n+1}), \quad n = 0, 1, 2, \ldots.
\]
From (c),
\[
d(x_1, x_2) = d(S(x_0), T(x_1)) \leq (a_1 + a_3)d(x_0, x_1) + a_3d(x_1, x_2) \leq (a_1 + a_3)d(x_0, x_1) + a_3d(x_0, x_1) + d(x_1, x_2).
\]
So
\[
(1) \quad d(x_1, x_2) \leq \frac{a_1 + a_3}{1 - a_2 - a_3}d(x_0, x_1).
\]
Similarly,
\[
(2) \quad d(x_2, x_3) \leq \frac{a_2 + a_4 + a_5}{1 - a_1 - a_4}d(x_1, x_2).
\]
Let
\[
r = \frac{a_1 + a_3 + a_5}{1 - a_2 - a_3}, \quad s = \frac{a_2 + a_4 + a_5}{1 - a_1 - a_4}.
\]
Repeating the above argument, we obtain, for each \( n = 0, 1, 2, \ldots \),
\[
(3) \quad d(x_{2n+1}, x_{2n+2}) \leq rd(x_{2n+1}, x_{2n}),
\]
\[
(4) \quad d(x_{2n+3}, x_{2n+4}) \leq sd(x_{2n+3}, x_{2n+1}).
\]
By (3), (4), and induction, we have, for each \( n = 0, 1, 2, \ldots \),
\[
(5) \quad d(x_{2n+1}, x_{2n+2}) \leq r(s)^n d(x_0, x_1),
\]
\[
(6) \quad d(x_{2n+3}, x_{2n+4}) \leq (rs)^{n+1} d(x_0, x_1).
\]
Since \( rs < 1 \) and
\[
\sum_{n=0}^{\infty} d(x_n, x_{n+1}) \leq (1 + r) \sum_{n=0}^{\infty} (rs)^n d(x_0, x_1),
\]
\( \{x_n\} \) is Cauchy. By completeness of \((X, d)\), \( \{x_n\} \) converges to some point \( x \) in \( X \). We shall now prove that \( x \) is a fixed point of \( S \) and \( T \). Let \( n \) be given. Then
\[
(7) \quad d(x, S(x)) \leq d(x, x_{2n+2}) + d(S(x), x_{2n+2}) = d(x, x_{2n+2}) + d(S(x), T(x_{2n+1})).
\]
By (c),
\begin{align*}
d(S(x), T(x_{2n+i})) \leq & \alpha_i d(x, S(x)) + \alpha_2 d(x_{2n+1}, x_{2n+2}) + \alpha_3 d(x, x_{2n+2}) \\
& + \alpha_4 d(x_{2n+1}, S(x)) + \alpha_5 d(x, x_{2n+1}) .
\end{align*}

Combining (7) and (8) and letting \( n \) tend to infinity, we obtain
\[ d(x, S(x)) \leq (\alpha_1 + \alpha_2) d(x, S(x)) . \]

Since \( \alpha_1 + \alpha_4 < 1, S(x) = x \). Similarly \( T(x) = x \). Let \( y \) be a fixed point of \( T \). Then from \( d(x, y) = d(S(x), T(y)) \) and (c), we obtain
\[ d(x, y) \leq (\alpha_3 + \alpha_4 + \alpha_5) d(x, y) . \]

Since \( \alpha_3 + \alpha_4 + \alpha_5 < 1 \), \( d(x, y) = 0 \). So \( T \) has a unique fixed point. Similarly, \( S \) has a unique fixed point.

When \( \alpha_3 = \alpha_4 = \alpha_5 = 0 \), \( S = T \) and \( T \) is continuous (or even \( x \to d(x, T(x)) \) is lower semicontinuous) on \( X \), Theorem 1 can be obtained by an earlier result of the author [11, Theorem 1].

From the proof of Theorem 1, we know that \( S, T \) still have a common fixed point if conditions (a), (b) are replaced by the following conditions:
\begin{align*}
(9) \quad (\alpha_1 + \alpha_3 + \alpha_5)(\alpha_2 + \alpha_4 + \alpha_5) < & (1 - \alpha_2 - \alpha_3)(1 - \alpha_1 - \alpha_4) , \\
(10) \quad \alpha_1 + \alpha_4 < & 1 .
\end{align*}

If in addition,
\begin{equation}
(11) \quad \alpha_3 + \alpha_4 + \alpha_5 < 1 ,
\end{equation}
then the common fixed point of \( S, T \) is the unique fixed point of \( S \) (and \( T \)). Note that conditions (a) and (b) imply (9), but (a) alone does not. Indeed, for any \( \alpha_1, \alpha_2, \alpha_3 \) in \([0, \infty)\) with \( \alpha_1 \neq \alpha_2 \) and \( \alpha_1 + \alpha_2 + \alpha_3 < 1 \), we can always find \( \alpha_3, \alpha_4 \) in \([0, \infty)\) such that (a) holds but (9) does not. This can be seen by considering the affine function \( f \):
\[ f(x, y) = (1 - \alpha_2 - x)(1 - \alpha_1 - y) - (\alpha_1 + x + \alpha_3)(\alpha_2 + y + \alpha_3) \]
defined on the compact convex set
\[ K = \{(x, y) \in [0, 1] \times [0, 1]: \alpha_1 + \alpha_2 + x + y + \alpha_3 \leq 1\} . \]

\( f \) takes its minimum value at one of the extreme points of \( K \). With some computation, we conclude that
\[ \min f(K) = - |\alpha_1 - \alpha_2| (1 - \alpha_1 - \alpha_2 - \alpha_3) . \]

Since \( \alpha_1 + \alpha_2 + \alpha_5 > 1 \), \( \min f(K) < 0 \) if and only if \( \alpha_1 \neq \alpha_2 \). Thus if \( \alpha_1 \neq \alpha_2 \), then by continuity of \( f \), there exists a point \((\alpha_3, \alpha_4)\) in
such that \( f(a_3, a_4) < 0 \).

**COROLLARY 1.** R. Kannan [7, Theorem 1]. Let \( S \) be a mapping of a complete metric space \((X, d)\) into itself. Suppose that there exists a number \( r \) in \([0, 1/2)\) such that

\[
d(S(x), S(y)) \leq r(d(x + S(x)) + d(y, S(y)))
\]

for all \( x, y \) in \( X \). Then \( S \) has a unique fixed point.

**COROLLARY 2.** P. Srivastava and V. K. Gupta [10, Theorem 1]. Let \( S, T \) be mappings of a complete metric space \((X, d)\) into itself. Suppose that there exists nonnegative real numbers \( a_1, a_2 \) such that

(a) \[ a_1 + a_2 < 1 \]

and

(b) \[
d(S(x), T(y)) \leq a_1 d(x, S(x)) + a_2 d(y, T(y))
\]

for all \( x, y \) in \( X \).

Then \( S, T \) have a unique common fixed point.

Srivastava and Gupta stated the above result in a more general form with \( S, T \) replaced by \( S^p, T^q \) for some positive integers \( p, q \). Since the unique fixed point of \( S^p \) (similarly \( T^q \)) is the unique fixed point of \( S \), this result is equivalent to Corollary 2.

For Corollaries 1 and 2, we have the following related result.

**PROPOSITION.** Let \( S, T \) be self-maps of a nonempty complete metric space \((X, d)\). Suppose that there exist nonnegative real numbers \( a_1, a_2 \) such that \( a_1 + a_2 < 1 \) and

(*) \[ d(S(x), T(y)) \leq a_1 d(x, S(x)) + a_2 d(y, T(y)), \quad x, y \in X. \]

Then either (*) is true when all of its \( S \) are replaced by \( T \) or (*) is true when all of its \( T \) are replaced by \( S \).

The following example proves that our result is actually more general than that of Srivastava and Gupta.

**EXAMPLE.** Let \( X = \{1, 2, 3\} \). Let \( d \) be the metric for \( X \) determined by

\[
d(1, 2) = 1, \quad d(2, 3) = \frac{4}{7}, \quad d(1, 3) = \frac{5}{7}.
\]
Let $S, T$ be the function on $X$ such that

\[
S(1) = S(2) = S(3) = 1; \\
T(1) = T(3) = 1, \quad T(2) = 3.
\]

Let $a_1 = 0$, $a_2 = 0$, $a_3 = 0$, $a_4 = 5/7$, $a_5 = 0$. Then the conditions of Theorem 1 are satisfied. However, no nonnegative real numbers $a_1, a_2, a_3, a_5$ can be chosen such that $a_1 + a_2 + a_3 + a_5 < 1$ and for $x, y \in X$,

\[
d(S(x), T(y)) \leq a_1d(x, S(x)) + a_2d(y, T(y)) + a_3d(x, T(y)) + a_5d(x, y).
\]

For if there exist such $a_1, a_2, a_3, a_5$, then

\[
d(S(3), T(2)) \leq a_1d(3, S(3)) + a_2d(2, T(2)) + a_3d(3, T(2)) + a_5d(3, 2).
\]

So

\[
\frac{5}{7} \leq \frac{5a_1}{7} + \frac{4a_2}{7} + \frac{4a_3}{7} \leq \frac{5}{7} (a_1 + a_2 + a_5) < \frac{5}{7},
\]

a contradiction.

**COROLLARY 3.** G. Hardy and T. Rogers [5, Theorem 1]. Let $S$ be a mapping of a nonempty complete metric space $(X, d)$ into itself. Suppose that there exist nonnegative real numbers $a_1, a_2, a_3, a_4, a_5$ such that

(a) \quad $a_1 + a_2 + a_3 + a_4 + a_5 < 1$

and

(b) \quad $d(S(x), S(y)) \leq a_1d(x, S(x)) + a_2d(y, S(y)) + a_3d(x, S(y))$

\[+ a_4d(y, S(x)) + a_5d(x, y)\]

for all $x, y \in X$.

Then $S$ has a unique fixed point.

Note that in the above case, we may without loss of generality assume that $a_1 = a_2, a_3 = a_4$ (replace $a_1, a_2, a_3, a_4, a_5$ respectively by $
\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, \frac{a_3 + a_4}{2}, \frac{a_3 + a_4}{2}, a_5$

if necessary). So the above result follows from Theorem 1. The above example shows that there is no such symmetry ($a_1 = a_2, a_3 = a_4$) for the general case. Indeed, we cannot even assume $a_3 = a_4$. For if $a_3 = a_4$, then for the above example, we have
\[ \frac{5}{7} = d(S(3), T(3)) \leq \frac{5}{7} a_1 + \frac{4}{7} a_2 + a_4 + \frac{4}{7} a_5 . \]
\[
= \frac{5}{7} a_1 + \frac{4}{7} a_2 + \frac{1}{2} a_3 + \frac{1}{2} a_4 + \frac{4}{7} a_5
\]
\[
< \frac{5}{7} (a_1 + a_2 + a_3 + a_4 + a_5) < \frac{5}{7} ,
\]
a contradiction.

2. Extensions and some related results. The following result generalizes Theorem 1. Its proof is different from the one we gave for Theorem 1.

**Theorem 2.** Let \( S, T \) be functions on a nonempty complete metric space \((X, d)\). Suppose that there exist decreasing functions \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \) of \((0, \infty) \) into \([0,1)\) such that

(a) \( \sum_{i=1}^{5} \alpha_i < 1; \)
(b) \( \alpha_1 = \alpha_2 \) or \( \alpha_3 = \alpha_4; \)
(c) \( \lim_{t \to 0} (\alpha_2 + \alpha_3) < 1 \) and \( \lim_{t \to 0} (\alpha_1 + \alpha_4) < 1; \)
(d) for any distinct \( x, y \) in \( X, \)

\[
d(S(x), T(y)) \leq a_1 d(x, S(x)) + a_2 d(y, T(y)) + a_3 d(x, T(y)) + a_4 d(y, S(x)) + a_5 d(x, y),
\]

where \( a_i = \alpha_i(d(x, y)). \)

Then at least one of \( S, T \) has a fixed point. If both \( S \) and \( T \) have fixed points, then each of \( S, T \) has a unique fixed point and these two fixed points coincide.

**Proof.** Let \( x_0 \in X. \) Define for each \( n = 0, 1, 2, \cdots, \)

\( x_{2n+1} = S(x_{2n}), \quad x_{2n+2} = T(x_{2n+1}), \quad b_n = d(x_n, x_{n+1}). \)

We may assume that \( b_n > 0 \) for each \( n, \) for otherwise some \( x_n \) is a fixed point of \( S \) or \( T. \) Let

\[
r(t) = \frac{\alpha_1(t) + \alpha_2(t) + \alpha_3(t)}{1 - \alpha_1(t) - \alpha_3(t)} , \quad t > 0, \]
\[
s(t) = \frac{\alpha_2(t) + \alpha_4(t) + \alpha_5(t)}{1 - \alpha_2(t) - \alpha_4(t)} , \quad t > 0.
\]

Then \( r, s \) are decreasing. From (a) and (c), the limits

\[
r_0 = \lim_{t \to 0} r(t) , \quad s_0 = \lim_{t \to 0} s(t)
\]

are nonnegative real numbers. Let
\[ f(t) = r(t)s(t), \quad t > 0. \]

Then \( f \) is decreasing and \( f(t) < 1 \) for each \( t > 0 \). As in the proof of Theorem 1, we have for each \( n = 0, 1, 2, \ldots \),

\[
\begin{align*}
(12) & \quad b_{2n+1} \leq r(b_{2n})b_{2n}, \\
(13) & \quad b_{2n+2} \leq s(b_{2n+1})b_{2n+1}.
\end{align*}
\]

Let \( n \) be given. Then

\[
\begin{align*}
(14) & \quad b_{2n+3} \leq r(b_{2n+2})s(b_{2n+1})b_{2n}, \\
(15) & \quad b_{2n+2} \leq s(b_{2n+1})r(b_{2n})b_{2n}.
\end{align*}
\]

Since \( r, s \) are decreasing,

\[
\begin{align*}
(16) & \quad b_{2n+3} \leq f(\min\{b_{2n+2}, b_{2n+1}\})b_{2n+1}, \\
(17) & \quad b_{2n+2} \leq f(\min\{b_{2n+1}, b_{2n}\})b_{2n}.
\end{align*}
\]

Since \( f(t) < 1 \) for each \( t > 0 \), \{\( b_{2n+1} \), \( b_{2n} \)\} are decreasing sequences. So \{\( b_{2n+1} \), \( b_{2n} \)\} converge respectively to some points \( c_1 \), \( c_2 \). We shall prove that \( c_1 = 0 \), \( c_2 = 0 \). From (12) and (13),

\[
\begin{align*}
& c_1 \leq r_0c_2, \quad c_2 \leq s_0c_1.
\end{align*}
\]

So either both \( c_1 \), \( c_2 \) are zero or both \( c_1 \), \( c_2 \) are not zero. Suppose to the contrary that \( c_1 \neq 0 \), \( c_2 \neq 0 \). Then from (16) and (17),

\[
\begin{align*}
(18) & \quad b_{n+2} \leq f(\min\{c_1, c_2\})b_n, \quad n = 0, 1, 2, \ldots.
\end{align*}
\]

By induction,

\[
\begin{align*}
(19) & \quad b_{2n} \leq (f(\min\{c_1, c_2\}))^nb_0 \quad n = 0, 1, 2, \ldots.
\end{align*}
\]

So \( c_2 = 0 \), a contradiction. Therefore, \( c_1 = c_2 = 0 \). This proves that \( \{b_n\} \) converges to 0.

Now we shall prove that \( \{x_n\} \) is Cauchy. Suppose not. Then there exist \( \varepsilon \in (0, \infty) \) and sequences \( \{p(n)\}, \{q(n)\} \) such that for each \( n \geq 0 \),

\[
\begin{align*}
(20) & \quad p(n) > q(n) > n, \\
(21) & \quad d(x_{p(n)}, x_{q(n)}) \geq \varepsilon,
\end{align*}
\]

and (by the well-ordering principle),

\[
\begin{align*}
(22) & \quad d(x_{p(n)-1}, x_{q(n)}) < \varepsilon.
\end{align*}
\]

Let \( n \geq 0 \) be given, \( e_n = d(x_{p(n)}, x_{q(n)}) \). Then
\[ \varepsilon \leq c_n \leq d(x_{p(n)}, x_{p(n)-1}) + d(x_{p(n)-1}, x_{q(n)}) < b_{p(n)-1} + \varepsilon. \]

From \( c_1 = c_2 = 0 \), we conclude that \( \{c_n\} \) converges to \( \varepsilon \) from the right. Let

\[ I_1 = \{n: p(n), q(n) \text{ are odd}\}, \]
\[ I_2 = \{n: p(n) \text{ is odd, } q(n) \text{ is even}\}, \]
\[ I_3 = \{n: p(n) \text{ is even, } q(n) \text{ is odd}\}, \]
\[ I_4 = \{n: p(n), q(n) \text{ are even}\}. \]

Then at least one of \( I_1, I_2, I_3, I_4 \) is infinite. Suppose first that \( I_1 \) is infinite. Let

\[ d_n = d(x_{p(n)}, x_{q(n)}), \quad n = 0, 1, 2, \ldots. \]

Since \( \{c_n\} \) converges to \( \varepsilon \) and \( \{b_n\} \) converges to 0, we conclude from (22) that \( \{d_n\} \) converges to \( \varepsilon \) from the left. Thus

\[ J_1 = \{n \in I_1: x_{p(n)-1} \neq x_{q(n)}\} \]

is infinite. Let \( n \in J_1, u_n = d(x_{p(n)-1}, x_{q(n)+1}). \) Then

\[ c_n = d(x_{p(n)}, x_{q(n)}) \leq d(x_{p(n)}, x_{q(n)+1}) + d(x_{q(n)+1}, x_{q(n)}) \]
\[ \leq d(S(x_{p(n)-1}), T(x_{q(n)})) + b_{q(n)}. \]

From (d),

\[ d(S(x_{p(n)-1}), T(x_{q(n)})) \leq \alpha_1(d_n)b_{p(n)-1} + \alpha_2(d_n)b_{q(n)} + \alpha_3(d_n)u_n \]
\[ + \alpha_4(d_n)c_n + \alpha_5(d_n)d_n. \]

From (24) and (25),

\[ c_n \leq \alpha_1(d_n)b_{p(n)-1} + \alpha_2(d_n)b_{q(n)} + \alpha_3(d_n)u_n + \alpha_4(d_n)c_n \]
\[ + \alpha_5(d_n)d_n + b_{q(n)}. \]

Without loss of generality, we may assume that each \( \alpha_i \) is continuous from the left, for we can replace the \( \alpha_i \)'s by

\[ \beta_i(t) = \lim_{s \downarrow t} \alpha_i(s), \quad t > 0, \]
\[ i = 1, 2, 3, 4, 5 \]

and conditions (a), (b), (c), and (d) still hold. Thus

\[ \lim_{n \to \infty} \alpha_i(d_n) = \alpha_i(\varepsilon), \quad i = 1, 2, 3, 4, 5. \]

So from (26),

\[ \varepsilon \leq (\alpha_1(\varepsilon) + \alpha_4(\varepsilon) + \alpha_5(\varepsilon))\varepsilon < \varepsilon, \]
a contradiction. Now suppose that $I_2$ is infinite. By a similar argument, $J_2 = \{ n \in I_2 : x_{p(2n-1)} \neq x_{q(2n-1)} \}$ is infinite. Let $n \in J_2$,

$$v_n = d(x_{p(2n-1)}, x_{q(2n-1)}), \quad w_n = d(x_{p(2n)}, x_{q(2n-1)}).$$

Then

$$c_n = d(S(x_{p(n-1)}), T(x_{q(n-1)})) \leq \alpha_1(v_n)b_{p(n-1)} + \alpha_2(v_n)b_{q(n-1)} + \alpha_3(v_n)d_n + \alpha_4(v_n)w_n + \alpha_5(v_n)v_n.$$  

Since $\{v_n\}$ converges to $\epsilon$ (not necessarily from the left or right), we obtain the same contradiction from (27). The other two cases are similar to the above two except the roles of $S$, $T$ interchange. Hence $\{x_n\}$ is Cauchy. By completeness, $\{x_n\}$ converges to a point $x$ in $X$. Since $b_n > 0$ for each $n$, $J = \{ n: x \neq x_{2n+1} \}$ or $K = \{ n: x \neq x_{2n} \}$ is infinite. Suppose that $K$ is infinite. Let $n \in K$,

$$l_n = d(x, x_{2n}) \quad \text{and} \quad h_n = d(x, x_{2n+1}).$$

Then

$$d(x, T(x)) \leq d(x, x_{2n+1}) + d(x_{2n+1}, T(x))$$

$$= h_n + d(S(x_{2n}), T(x))$$

$$\leq h_n + \alpha_1(l_n)b_{2n} + \alpha_2(l_n)d(x, T(x)) + \alpha_3(l_n)d(x_{2n}, T(x))$$

$$+ \alpha_4(l_n)h_n + \alpha_5(l_n)l_n$$

$$\leq h_n + \alpha_1(l_n)b_{2n} + \alpha_2(l_n)d(x, T(x)) + \alpha_3(l_n)[l_n + d(x, T(x))]$$

$$+ \alpha_4(l_n)h_n + \alpha_5(l_n)l_n.$$  

So

$$d(x, T(x)) \leq \frac{1 + \alpha_4(l_n)}{1 - \alpha_2(l_n) - \alpha_3(l_n)} h_n + \frac{\alpha_3(l_n) + \alpha_5(l_n)}{1 - \alpha_2(l_n) - \alpha_3(l_n)} l_n$$

$$+ \frac{\alpha_4(l_n)}{1 - \alpha_2(l_n) - \alpha_3(l_n)} b_{2n}.$$  

(28)

From (a) and (c), the sequences

$$\frac{1 + \alpha_4(l_n)}{1 - \alpha_2(l_n) - \alpha_3(l_n)}, \quad \frac{\alpha_3(l_n) + \alpha_5(l_n)}{1 - \alpha_2(l_n) - \alpha_3(l_n)}, \quad \frac{\alpha_4(l_n)}{1 - \alpha_2(l_n) - \alpha_3(l_n)}$$

are bounded. So from (28), $T(x) = x$. Similarly, $S(x) = x$ if $J$ is infinite. Hence $S$ or $T$ has a fixed point.

The following result follows easily from Theorem 2.

**Theorem 3.** With the conditions of Theorem 2, if further,

$$d(S(x), T(x)) \leq \alpha [d(x, S(x)) + d(x, T(x))]$$

$$x \in X.$$
for some $\alpha \in [0, 1)$, then each of $S$, $T$ has a unique fixed point and these two fixed points coincide.

We remark that the conditions of Theorem 1 imply the conditions of Theorem 3. Also, G. Hardy and T. Rogers [5, Theorem 2] gave a different proof for the case $S = T$. Their proof cannot be modified for the general case. To see that the conclusion of Theorem 2 is best possible, we note that if $X = \{0, 1\}$ with the usual distance and if $S$, $T$ are two distinct functions of $X$ onto $X$, then $S$, $T$ satisfy the conditions of Theorem 2 (and Theorem 3 with $\alpha = 1$), but one has two fixed points and the other has none.

**Theorem 4.** Let $(X, d)$ be a nonempty compact metric space. Let $S$, $T$ be functions of $X$ into itself. Suppose that $S$ or $T$ is continuous. Suppose further that there exist nonnegative real-valued decreasing functions $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ on $(0, \infty)$ such that

(a) $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 \leq 1$,

(b) $\alpha_1 = \alpha_2$ and $\alpha_3 = \alpha_4$,

(c) for any distinct $x, y$ in $X$,

\[
d(S(x), T(y)) < \alpha_1 d(x, S(x)) + \alpha_2 d(y, T(y)) + \alpha_3 d(x, T(y)) + \alpha_4 d(y, S(x)) + \alpha_5 d(x, y),
\]

where $a_i = \alpha_i(d(x, y))$.

Then $S$ or $T$ has a fixed point. If both $S$ and $T$ have fixed points, then each of $S$ and $T$ has a unique fixed point and these two fixed points coincide.

**Proof.** By symmetry, we may assume that $S$ is continuous. Let $f$ be the function on $X$ such that

\[
f(x) = d(x, S(x)), \quad x \in X.
\]

Then $f$ is continuous (we merely need the fact that $f$ is lower semicontinuous) on $X$. So $f$ takes its minimum value at some $x_0$ in $X$. We claim that $x_0$ is a fixed point of $S$ or $S(x_0)$ is a fixed point of $T$. Suppose not. Let

\[
x_1 = S(x_0), \quad x_2 = T(x_1), \quad x_3 = S(x_2),
\]

\[
b_0 = d(x_0, x_1), \quad b_1 = d(x_2, x_3), \quad b_2 = d(x_2, x_3).
\]

Then $b_0 > 0$, $b_1 > 0$. From (c), we can prove that

\[
(1 - \alpha_2(b_0) - \alpha_3(b_0))b_1 < (\alpha_1(b_0) + \alpha_3(b_0) + \alpha_5(b_0))b_0.
\]

Let
\[ p(t) = 1 - \alpha_2(t) - \alpha_3(t), \quad q(t) = \alpha_1(t) + \alpha_4(t) + \alpha_5(t), \quad t > 0. \]

From (a) and (b), \( p(b_0) > 0 \). So

\[ \frac{q(b_1)}{p(b_0)} < b_1. \quad (30) \]

Similarly,

\[ \frac{v(b_2)}{u(b_1)} < b_2. \quad (31) \]

where

\[ u(t) = 1 - \alpha_1(t) - \alpha_3(t), \quad v(t) = \alpha_1(t) + \alpha_4(t) + \alpha_5(t), \quad t > 0. \]

From (30) and (31),

\[ \frac{v(b_2)q(b_0)}{u(b_1)p(b_0)} < b_2. \quad (32) \]

It suffices to prove that \( v(b_2)q(b_0)/u(b_1)p(b_0) < 1 \), for then, \( b_2 < b_0 \), a contradiction to the minimality of \( b_0 \). Let \( b = \min \{b_1, b_2\} \). Then

\[ v(b_2)q(b_0) - u(b_1)p(b_0) \leq v(b)q(b) - u(b)p(b) < 0 \]

if \( \alpha_1 = \alpha_2 \) and \( \alpha_3 = \alpha_4 \). So \( S \) or \( T \) has a fixed point. Now suppose that \( x \) is a fixed point of \( S \) and \( y \) is a fixed point of \( T \). Then \( x = y \), otherwise, from (c),

\[ d(x, y) = d(S(x), T(y)) < d(x, y), \]

a contradiction.

The following result is stated without proof.

**Theorem 5.** Let \( (X, d) \) be complete metric space. Let \( \{S_n\}, \{T_n\} \) be sequence of functions of \( X \) into \( X \) which converge pointwise to \( S, T \) respectively. Suppose that the pairs \( (S_n, T_n) \) satisfy the conditions of Theorem 3 with the same \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \). Then \( S, T \) have a unique common fixed point \( x \) and \( x \) is the limit of the sequence \( \{x_n\} \) of the fixed points \( x_n \) of \( S_n \).

**Theorem 6.** Let \( (X, d) \) be a nonempty compact metric space. Let \( \{S_n\}, \{T_n\} \) be sequences of functions of \( X \) into itself which converge pointwise to the functions \( S, T \) on \( X \) respectively. Suppose that for each \( n \), there exist decreasing functions \( \alpha_1^*, \alpha_2^*, \alpha_3^*, \alpha_4^*, \alpha_5^* \) of \((0, \infty)\) into \([0, \infty)\) such that
(a) \(\alpha_1^n + \alpha_2^n + \alpha_3^n + \alpha_4^n + \alpha_5^n \leq 1\),
(b) \(\alpha_1^n = \alpha_2^n \) and \(\alpha_3^n = \alpha_4^n\),
(c) for any distinct \(x, y\) in \(X\),
\[
d(S_n(x), T_n(y)) < \alpha_1^n d(x, S_n(x)) + \alpha_2^n d(y, T_n(y)) + \alpha_3^n d(x, T_n(y)) + \alpha_4^n d(y, S_n(x)) + \alpha_5^n d(x, y),
\]
where
\[
\alpha_5^n = \alpha_6^n(d(x, y)).
\]
Then \(S\) or \(T\) has a fixed point. Indeed, every cluster point of a sequence \(\{x_n\}\) of fixed points \(x_n\) of \(S_n\) or \(T_n\) is a fixed point of \(S\) or \(T\).

**Proof.** By Theorem 4, for each \(n\), either \(S_n\) or \(T_n\) has a fixed point. By symmetry, we may assume that \(S_n\) has a fixed point for infinitely many of \(n\)'s. So there is a subsequence \(\{S_{n(k)}\}\) of \(\{S_n\}\) such that each \(S_{n(k)}\) has a fixed point, say \(x_k\). By compactness, we may (by taking a subsequence) assume that \(\{x_k\}\) converges to some \(x\) in \(X\). We shall prove that \(x\) is a fixed point of \(S\) or \(T\). If \(x_k \neq x\) for only finitely many of \(k\)'s, then
\[
S(x) = \lim_{k \to \infty} S_{n(k)}(x)
= \lim_{k \to \infty} S_{n(k)}(x_k)
= \lim_{k \to \infty} x_k
= x.
\]
So we may assume that \(x_k \neq x\) for infinitely many of \(k\)'s. By taking a subsequence, we may assume that \(x_k \neq x\) for each \(k\). Let \(k \geq 1\) and \(b_k = d(x, x_k)\). Then
\[
d(x, T(x)) \leq d(x, x_k) + d(x_k, T_{n(k)}(x)) + d(T_{n(k)}(x), T(x))
= d(x, x_k) + d(S_{n(k)}(x_k), T_{n(k)}(x)) + d(T_{n(k)}(x), T(x)).
\]
From (c),
\[
d(S_{n(k)}(x_k), T_{n(k)}(x_k)) < \alpha_6^k(b_k) d(x, T_{n(k)}(x)) + \alpha_5^k(b_k) d(x_k, T_{n(k)}(x)) + \alpha_5^k(b_k) d(x, x_k) + \alpha_5^k(b_k) b_k.
\]
Combining (33) and (34) and letting \(k\) tend to the infinity, we have
\[
d(x, T(x)) \leq \lim_{k \to \infty} \sup_{t \geq 0} (\alpha_6^k(b_k) + \alpha_5^k(b_k)) d(x, T(x))
\leq \lim_{k \to \infty} \lim_{t \to 0} (\alpha_6^k(t) + \alpha_5^k(t)) d(x, T(x))
\]
From (b), \(\alpha_4^k(t) + \alpha_5^k(t) \leq 1/2\) for each \(t > 0\), \(k = 1, 2, \ldots\). So
(36) \[ \limsup_{k \to \infty} \lim_{t \downarrow 0} (\alpha^*(t) + \alpha_0^*(t)) \leq \frac{1}{2}. \]

From (35) and (36), we conclude that \( T(x) = x \).

From the proof, we know that the same conclusion holds if in Theorem 6, we replace (b) by the following weaker conditions:

\[ \alpha^n_i = \alpha^n_2 \quad \text{or} \quad \alpha^n_i = \alpha^n_3, \]

\[ \limsup_{k \to \infty} \lim_{t \downarrow 0} (\alpha^*_k(t) + \alpha^*_0(t)) < 1, \]

and

\[ \limsup_{k \to \infty} \lim_{t \downarrow 0} (\alpha^*_k(t) + \alpha^*_0(t)) < 1. \]

We note that, unlike Theorem 5, \( S, T \) in Theorem 6 need not satisfy the condition required for the pairs \( (S_n, T_n) \).

**Theorem 7.** Let \((X, d)\) be a nonempty compact metric space. Let \( \{S_n\} \) be a sequence of functions of \( X \) into itself which converges pointwise to some function \( S \) on \( X \). Suppose that for each \( n \), there exist decreasing functions \( \alpha^n_i, \alpha^n_2, \alpha^n_3, \alpha^n_4, \) of \((0, \infty)\) into \([0, \infty)\) such that

(a) \( \alpha^n_1 + \alpha^n_2 + \alpha^n_3 + \alpha^n_4 \leq 1 \),

(b) for any distinct \( x, y \) in \( X \),

\[ d(S_n(x), S_n(y)) < a_1d(x, S_n(x)) + a_2d(y, S_n(y)) + a_3d(x, y) + a_4d(x, y), \]

where

\[ a_i = \alpha_i(d(x, y)). \]

Then \( S \) has a fixed point. Indeed, every cluster point of the sequence of fixed points of \( S_n \) is a fixed point of \( S \).

The above result follows from Theorem 6 by averaging two applications of condition (b).

We shall now give a simple example to show that the conclusion of Theorem 7 is best possible. Let \( X \) be a star-shaped [4] compact subset of a normed linear space \( B \). Then there exists a point \( z \) in \( X \) such that for any \( y \) in \( X \), the line segment

\[ \{tz + (1 - t)y : t \in [0, 1]\} \]

is contained in \( X \). For each \( n \), let

\[ S_n(x) = \frac{1}{n}z + \left(1 - \frac{1}{n}\right)x, \quad x \in X. \]
Then \( \{S_n\} \) is a sequence of mappings of \( X \) into \( X \) which satisfy the conditions of Theorem 7. \( \{S_n\} \) converges pointwise to the identity function \( S \) on \( X \). Every point of \( X \) is a fixed point of \( S \). So unlike Theorem 5, it is too much to ask that \( S \) in Theorem 7 has a unique fixed point.

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