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# CONTRACTORS, APPROXIMATE IDENTITIES AND FACTORIZATION IN BANACH ALGEBRAS

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**The concept of a contractor has been introduced as a tool for solving equations in Banach spaces. In this way various existence theorems for solutions of equations have been obtained as well as convergence theorems for a broad class of iterative procedures. Moreover, the contractor method yields unified approach to a large variety of iterative processes different in nature. The contractor idea can also be exploited in Banach algebras.**

**A contractor is rather weaker than an approximate identity. Since every approximate identity is a contractor, the following seems to be a natural question: When is a contractor an approximate identity? The answer to this question is investigated in the present paper.**

Concerning the approximate identity in a Banach algebra $A$ it is shown that if a subset $U$ of $A$ is a bounded weak left approximate identity, then $U$ is a bounded left approximate identity. This important fact makes it possible to prove the well-known factorization theorems for Banach algebras under weaker conditions of existence of a bounded weak approximate identity.

## 2. Approximate identities.

Let $A$ be a Banach algebra.

DEFINITION 2.1. A subset $U \subset B \subseteqq A$ is called a left weak (or simple) approximate identity for the set $B$ if for arbitrary $b \in B$ and $\varepsilon > 0$ there exists an element $u \in U$ such that

$$\|ub - b\| < \varepsilon . \tag{2.1}$$

DEFINITION 2.2. A subset $U \subset B \subseteqq A$ is called a left approximate identity for $B$ if for every arbitrary finite subset of elements $b_i \in B$ $(i = 1, 2, \cdots, n)$ and arbitrary $\varepsilon > 0$ there exists an element $u \in U$ such that

$$\|ub_i - b_i\| < \varepsilon \qquad \text{for} \quad i = 1, 2, \cdots, n . \tag{2.2}$$

A (weak) approximate identity $U$ is called bounded if there is a constant $d$ such that $\|u\| \leqq d$ for all $u \in U$.

LEMMA 2.1. *If $U$ is a bounded subset of $B \subseteqq A$ such that for every pair of elements $b_i \in B$ $(i = 1, 2)$ and arbitrary $\varepsilon > 0$ there exists* *an element* $u \in U$ *satisfying* (2.2) *with* $n = 2$, *then* $U$ *is a left approximate identity for* $B$.

*Proof.* The proof will be given by the finite induction. Given arbitrary $b_i \in B$ $(i = 1, 2, \cdots, n + 1)$ and $\varepsilon > 0$. For $\varepsilon_0 > 0$ let $u_0 \in B$ be chosen so as to satisfy

$$\|u_0 b_i - b_i\| < \varepsilon_0 \quad \text{for} \quad i = 1, 2, \cdots, n \quad \text{and} \quad \|u_0\| \leqq d . \tag{2.3}$$

For the pair $u_0, b_{n+1} \in B$ and $\varepsilon_0 > 0$ there is an element $u \in U$ such that

$$\|uu_0 - u_0\| < \varepsilon_0 \quad \text{and} \quad \|ub_{n+1} - b_{n+1}\| < \varepsilon_0, \ \|u\| \leqq d . \tag{2.4}$$

After such a choice we have $\|ub_i - b_i\| \leqq \|ub_i - uu_0 b_i\| + \|uu_0 b_i - u_0 b_i\| + \|u_0 b_i - b_i\| \leqq d\varepsilon_0 + M\varepsilon_0 + \varepsilon_0 < \varepsilon$ for $i = 1, 2, \cdots, n$ and $\|ub_{n+1} - b_{n+1}\| < \varepsilon_0 < \varepsilon$, by (2.3) and (2.4), where $M = \max(\|b_i\| : i = 1, 2, \cdots, n)$ and $\varepsilon_0 < (d + M + 1)^{-1}\varepsilon$.

LEMMA 2.2. *If the subset* $U$ *of* $A$ *is a bounded weak left approximate identity for* $B \subseteqq A$, *then* $U \circ U = [a \in A \mid a = u \circ v;\ u, v \in U]$, *where* $u \circ v = u + v - uv$, *has the following property: for every pair of elements* $a, b \in B$ *and* $\varepsilon > 0$ *there exists* $u \in U \circ U$ *such that*

$$\|ua - a\| < \varepsilon \quad \textit{and} \quad \|ub - b\| < \varepsilon .$$

*Proof.* Given an arbitrary pair of elements $a, b \in B$ and $\varepsilon > 0$, let $v \in U$ be chosen so as to satisfy

$$\|a - va\| < (1 + d)^{-1}\varepsilon, \ \|v\| \leqq d . \tag{2.5}$$

For $b - vb$ and $\varepsilon > 0$ there exists $w \in U$ such that

$$\|(b - vb) - w(b - vb)\| < \varepsilon, \ \|w\| \leqq d .$$

Hence we obtain

$$\|b - ub\| = \|b - (w + v - wv)b\| < \varepsilon$$

and $\|a - ua\| = \|(a - va) - w(a - va)\| < (1 + d)^{-1}\varepsilon + d(1 + d)^{-1}\varepsilon = \varepsilon$, by (2.5), where $u = w + v - wv \in U \circ U$.

LEMMA 2.3. *If* $U \subset A$ *is a bounded weak left approximate identity for* $A$, *then* $U$ *is a bounded left approximate identity for* $A$.

*Proof.* In virtue of Lemma 2.2 the set $U \circ U$ satisfies the assumption of Lemma 2.1 and it can be replaced by $U$.

REMARK 2.1. A partial result concerning this problem has been obtained by Reiter [10], §7, p. 30, Lemma 1.

LEMMA 2.4. *If $U$ is a left bounded approximate identity for itself, then $U$ is the same for the Banach algebra generated by $U$ and in particular for $P$.*

*Proof.* The proof follows from the argument used at the end of the proof of Theorem 2.1.

THEOREM 2.1. *Let $U$ be a bounded subset of the Banach algebra $A$ satisfying the following conditions:*

(a) *For every $u \in U \cup U \circ U$ and $\varepsilon > 0$ there exists an element $v \in U$ such that $\|u - vu\| < \varepsilon$.*

(b) *For every element of the form $u - vu$ with $u, v \in U$ there exists an element $w \in U$ such that*

$$\|(u - vu) - w(u - vu)\| < \varepsilon .$$

*Then $U$ is a bounded left approximate identity for the Banach algebra generated by $U$ as well as for the right ideal generated by $U$. If $U$ is commutative, then Condition* (b) *can be dropped.*

*Proof.* Let $a, b \in U$ and $\varepsilon > 0$ be arbitrary. In virtue of Condition (a) there exists $v \in U$ such that $\|a - va\| < (1 + d)^{-1}\varepsilon$, where $d$ is the bound for $U$. Using (b) for $b - vb$ we can choose $w \in U$ such that

$$\|(b - vb) - w(b - vb)\| < \varepsilon .$$

Thus, we obtain

$$\|a - ua\| < \varepsilon \quad \text{and} \quad \|b - ub\| < \varepsilon ,$$

where $u = w + v - wv \in U \circ U$. Suppose that $b \in U \circ U$. Then for $\varepsilon_0 > 0$ there exists $u_0 \in U$ such that $\|b - u_0 b\| < \varepsilon_0$. For $\varepsilon > 0$ let $u \in U \circ U$ be chosen so as to satisfy

$$\|a - ua\| < \varepsilon \quad \text{and} \quad \|u_0 - uu_0\| < \varepsilon .$$

Hence, we obtain

$$\|ub - b\| \leqq \|ub - uu_0 b\| + \|uu_0 b - u_0 b\| + \|u_0 b - b\| \leqq d\varepsilon_0 + \|b\|\varepsilon_0 + \varepsilon_0 < \varepsilon$$

for proper choice of $\varepsilon_0$. If $a, b \in U \circ U$, then for $\varepsilon_0 > 0$ choose $u_1, u_2 \in U$ such that

$$\|a - u_1 a\| < \varepsilon_0 \quad \text{and} \quad \|b - u_2 b\| < \varepsilon_0 .$$

Then we find $u \in U \circ U$ such that

$$\|u_1 - uu_1\| < \varepsilon_0 \quad \text{and} \quad \|u_2 - uu_2\| < \varepsilon_0 .$$

After such a choice we have

$$\|ua - a\| \leqq \|ua - uu_1a\| + \|uu_1a - u_1a\| + \|u_1a - a\| \leqq d\varepsilon_0 + \|a\|\varepsilon_0 + \varepsilon_0 < \varepsilon$$

for proper $\varepsilon_0$, and similarly

$$\|ub - b\| \leqq \|ub - uu_2b\| + \|uu_2b - u_2b\| + \|u_2b - b\| \leqq d\varepsilon_0 + \|b\|\varepsilon_0 + \varepsilon_0 < \varepsilon$$

for proper $\varepsilon_0$. Thus, by Lemma 2.1, $U \circ U$ is a bounded left approximate identity for $U \cup U \circ U$ and so is $U$.

Now let $a = \sum_{i=1}^{n} u_i a_i$ and $b = \sum_{j=1}^{m} v_j b_j$, where $u_i, v_j \in U$ and $a_i, b_j \in A$ for $i = 1, \cdots, n; j = 1, \cdots, m$. For $\varepsilon_0 > 0$ choose $u \in U$ such that $\|u_i - uu_i\| < \varepsilon_0$ and $\|v_j - uv_j\| < \varepsilon_0$ for $i = 1, \cdots, n$ and $j = 1, \cdots, m$. Then

$$\|a - ua\| \leqq \left\| \sum_{i=1}^{n} (u_i - uu_i)a_i \right\| < \varepsilon_0 \sum_{i=1}^{n} \|a_i\| < \varepsilon$$

for sufficiently small $\varepsilon_0$. The same holds for $b$, that is $\|b - ub\| < \varepsilon$. The assertion of the theorem follows now from Lemma 2.1. If $U$ is commutative, then (b) follows from (a). For let $a = u - vu$, $u, v \in U$. Then $\|a - wa\| = \|(u - wu) - (u - wu)v\| < \varepsilon$ if $w \in U$ is such that $\|u - wu\| < (1 + d)^{-1}\varepsilon$.

For the set $U \subset A$ let us define an infinite sequence of sets $\{P_n\}$ as follows. Put $P_1 = U$, $P_2 = U \circ U$. Then $P_n = U \circ P_{n-1} = U \circ U \circ \cdots \circ U$ ($n$ times) is the set of all elements $p$ of the form $p = u + v - uv$, where $u \in U$ and $v \in P_{n-1}$. Let $P$ be the union of all sets $P_n$, that is $P = P_1 \cup P_2 \cup \cdots$.

## 3. Contractors.

Definition 3.1 (see [2]). A subset $U$ of a Banach algebra $A$ is called a left contractor for $A$ if there is a positive constant $q < 1$ with the following property.

For every $a \in A$ there exists an element $u \in U$ (depending on $a$) such that

$$\|a - ua\| \leqq q\|a\| . \tag{3.1}$$

A contractor $U$ is said to be bounded if $U$ is bounded by some constant $d$.

LEMMA 3.1. *Let $U$ be a left contractor for $A$. Then for arbitrary $a \in A$ there exists an infinite sequence $\{a_n\} \subset P$ such that*

$$\|a - a_n a\| \leqq q^n \|a\| \quad \textit{and} \quad a_n \in P_n . \tag{3.2}$$

*Proof.* By (3.1), let $u_1 \in U$ be chosen so as to satisfy the inequality

$$\|a - u_1 a\| \leqq q\|a\| . \tag{3.3}$$

Now let $u_2 \in U$ be such that

$$\|(a - u_1 a) - u_2(a - u_1 a)\| \leqq q\|a - u_1 a\| . \tag{3.4}$$

Hence, we obtain from (3.3) and (3.4)

$$\|a - a_2 a\| \leqq q^2\|a\| , \quad \text{where} \quad a_2 = u_2 + u_1 - u_2 u_1 = u_2 \circ u_1 \in P_2 .$$

We repeat this procedure replacing in (3.4) $u_1$ by $a_2$ and $u_2$ by $u_3$. Thus, we have $a_3 = u_3 \circ a_2 \in P_3$. After $n$ iteration steps we obtain (3.2).

DEFINITION 3.2. A subset $U \subset A$ is called a strong left contractor for $A$ if there exists a positive $q < 1$ with the following property: for every arbitrary finite set of $a_i \in A$, $i = 1, 2, \cdots, n$, there is an element $u \in U$ such that

$$\|a_i - u a_i\| \leqq q\|a_i\| , \qquad i = 1, 2, \cdots, n . \tag{3.5}$$

A left contractor $U$ is said to be quasi-strong if for arbitrary pair $a_i \in A (i = 1, 2,)$ of there exists an element $u \in U$ satisfying (3.5) with $n=2$.

LEMMA 3.2. *Let $U \subset A$ be a left quasi-strong contractor for $A$. Then for every arbitrary pair of $a, b \in A$ there exists an infinite sequence $\{c_n\} \subset P$ such that*

$$\|a - c_n a\| \leqq q^n\|a\|, \ \|b - c_n b\| \leqq q^n\|b\| , \tag{3.6}$$

*where $c_n \in P_n$, $n = 1, 2, \cdots$.*

*Proof.* The proof is similar to that of Lemma 3.1.

A similar lemma holds for strong contractors.

LEMMA 3.3. *Let $U \subset A$ be a left strong contractor for $A$. Then for every arbitrary finite set of elements $a_i \in A$, $i = 1, 2, \cdots, m$, there exists and infinite sequence $\{c_n\} \subset P$ such that*

$$\|a_i - c_n a_i\| \leqq q^n\|a_i\| \qquad \text{for} \quad i = 1, 2, \cdots, m ,$$

*where $c_n \in P_n$, $n = 1, 2, \cdots$.*

LEMMA 3.4. *Suppose that $U$ is a left bounded contractor for $A$ satisfying the condition $(d + 1)q < 1$. Then $U \circ U$ is a left bounded quasi-strong contractor for $A$.*

*Proof.* Let $\bar{q} = (d + 1)q < 1$ and let $a, b \in A$ be arbitrary. Then choose $v \in U$ so as to satisfy

$$||a - va|| \leqq q||a||, \ ||v|| \leqq d .$$

For $b - vb$ let $w \in U$ be such that

$$||(b - vb) - w(b - vb)|| \leqq q||b - vb|| .$$

Put $u = w + v - wv \in U \circ U$. Then

$$||a - ua|| = ||(a - va) - w(a - va)|| \leqq q||a|| + dq||a|| = \bar{q}||a||$$

and

$$||b - ub|| = ||(b-vb) - w(b - vb)|| \leqq q||b - vb|| \leqq q||b|| + dq||b|| = \bar{q}||b|| .$$

Thus, $U \circ U$ is a bounded left quasi-strong contractor for $A$ with contractor constant $\bar{q} < 1$.

THEOREM 3.1. *A left bounded contractor $U$ for $A$ is a left bounded approximate identity for $A$ iff $U$ is a left approximate identity for itself.*

*Proof.* Let $a, b \in A$ and $\varepsilon > 0$ be arbitrary. Using Lemma 3.1, we construct a sequence $\{a_n\} \subset P$ for $a \in A$ and $\{b_n\} \subset P$ for $b \in A$ such that

$$(3.7) \qquad ||a - a_n a|| \leqq q^n||a|| \quad \text{and} \quad ||b - b_n b|| \leqq q^n||b|| ,$$

where $a_n, b_n \in P_n$, $n = 1, 2, \cdots$. In virtue of Lemma 2.4, for $a_n, b_n \in P_n \subset P$ and $\varepsilon_0 > 0$ we can choose $u \in U$ so as to satisfy $||a_n - ua_n|| < \varepsilon_0$ and $||b_n - ub_n|| < \varepsilon_0$. Then we obtain, by (3.7), $||ua - a|| \leqq ||ua - ua_n a|| + ||ua_n a - a_n a|| + ||a_n a - a|| < dq^n||a|| + \varepsilon_0||a|| + q^n||a|| < \varepsilon$ for sufficiently large $n$ and proper choice of $\varepsilon_0$. A similar estimate holds for $b$:

$$||ub - b|| \leqq dq^n||b|| + \varepsilon_0||b|| + q^n||b|| < \varepsilon$$

for sufficiently large $n$ and proper choice of $\varepsilon_0$. The proof of necessity is obvious.

THEOREM 3.2. *Let $U$ be a bounded left contractor for $A$. If $U$ satisfies the hypotheses of Theorem 2.1, then $U$ is a left bounded approximate identity for $A$.*

*Proof.* The proof is the same as that of Theorem 3.1. The only difference is replacing there Lemma 2.4 by Theorem 2.1.

THEOREM 3.3. *Let $U$ be a left bounded contractor for $A$ satisfying* *the condition* $(d+1)^3 q < 1$. *If* $U$ *is a weak left approximate identity for* $U \circ U$, *then* $U$ *is a bounded left approximate identity for* $A$.

*Proof.* Let $q$ and $\varepsilon_0 > 0$ be such that

$$(d+1)^3 q < ((d+1)^3 + 2\varepsilon_0)q \leqq \bar{q} < 1 \, .$$

By Lemma 3.4, $U \circ U$ is a quasi-strong contractor for A with contractor constant $(d+1)q$. Hence, for arbitrary $a, b \in A$ and $u_1 \in U$ there exists an element $w \in U \circ U$ such that $||(a - u_1 a) - w(a - u_1 a)|| \leqq (d+1)q||a - u_1 a||$ and $||(b - u_1 b) - w(b - u_1 b)|| \leqq (d+1)q||b - u_1 b||$. By assumption, there exists $v \in U$ such that $||w - vw|| < \varepsilon_0 q(d+1)^{-1}$. Therefore,

$$\begin{aligned} ||v(a - u_1 a) - (a - u_1 a)|| &\leqq ||v(a - u_1 a) - vw(a - u_1 a)|| \\ &\quad + ||vw(a - u_1 a) - w(a - u_1 a)|| \\ &\quad + ||w(a - u_1 a) - (a - u_1 a)|| \\ &< (d(d+1)q + \varepsilon_0 (d+1)^{-1} q \\ &\quad + (d+1)q)||a - u_1 a|| \, . \end{aligned}$$

Hence,

$$||a - (v \circ u_1)a|| \leqq ((d+1)^2 q + \varepsilon_0 (d+1)^{-1}_{q})||a - u_1 a|| \, . \tag{3.8}$$

Using the assumption again we can find $u_2 \in U$ such that

$$||(v \circ u_1) - u_2(v \circ u_1)|| < ||a||^{-1}\varepsilon_0 q||a - u_1 a|| \, . \tag{3.9}$$

Hence, we have, by (3.8) and (3.9), $||u_2 a - a|| \leqq ||u_2 a - u_2(v \circ u_1)a|| + ||u_2(v \circ u_1)a - (v \circ u_1)a|| + ||(v \circ u_1)a - a|| \leqq [d((d+1)^2 q + \varepsilon_0(d+1)^{-1}q + q\varepsilon_0 + ((d+1)^2 q + \varepsilon_0(d+1)^{-1}_{q})]||a - u_1 a||$. Thus, we obtain,

$$||a - u_2 a|| \leqq \bar{q}||a - u_1 a||, \, u_2 \in U \, . \tag{3.10}$$

Similarly, we get

$$||b - u_2 b|| \leqq q||b - u_1 b|| \, . \tag{3.11}$$

Since $u_1 \in U$ was arbitrary, by the same argument, for $u_2 \in U$ there exists $u_3 \in U$ satisfying Conditions (3.10) and (3.11) with $u_2$ and $u_3$ replacing $u_1$ and $u_2$ respectively. After $n - 1$ iteration steps we obtain $||a - u_n a|| \leqq \bar{q}^n||a - u_1 a|| < \varepsilon$ and $||b - u_n b|| \leqq \bar{q}^n||b - u_1 b|| < \varepsilon$, $u_n \in U$, for sufficiently large $n$. Since $a$, $b$ and $\varepsilon > 0$ are arbitrary, it follows from Lemma 2.1 that $U$ is a bounded left approximate identity for $A$.

Using the same technique one can prove the following

PROPOSITION 3.1. *A left bounded quasi-strong contractor* $U$ *for* $A$ *is a left approximate identity for* $A$ *iff* $U$ *is a left weak approxi-*

*mate identity for an infinite subsequence of* $\{P_n\}$.

*Proof.* Let $a, b \in A$ and $\varepsilon > 0$ be arbitrary. Using Lemma 3.2 for the pair $a$, $b$ we construct an infinite sequence $\{c_n\}$ satisfying (3.6). Now let us choose $u \in U$ so as to satisfy $\|uc_n - c_n\| < \varepsilon_0$ for infinitely many $n$. Then we obtain for $a \in A$

$$\begin{aligned}\|ua - a\| &\leqq \|ua - uc_na\| + \|uc_na - c_na\| + \|c_na - a\| \\ &< dq^n\|a\| + \varepsilon_0\|a\| + q^n\|a\| < \varepsilon\end{aligned}$$

for some sufficiently large $n$ and proper choice of $\varepsilon_0$. A similar estimate holds for $b$:

$$\|ub - b\| < dq^n\|b\| + \varepsilon_0\|b\| + q^n\|b\| < \varepsilon$$

for some sufficiently large $n$ and proper choice of $\varepsilon_0$. The proof of necessity is obvious.

As an immediate corollary to Proposition 3.1 we obtain the following

PROPOSITION 3.2. *A left bounded weak approximate identity $U$ for $U$ is a left approximate identity for $A$ iff $U$ is a left quasi-strong contractor for $A$.*

PROPOSITION 3.3. *Suppose that $U$ is a left bounded contractor for $A$ satisfying the condition $(d+1)q < 1$. Then $U$ is a left bounded weak approximate identity for $A$ iff $U$ is the same for $U \circ U$.*

*Proof.* Let $\bar{q}$ and $\varepsilon_0$ be such that

$$(d + 1)q < (d + 1 + \varepsilon_0)q \leqq \bar{q} < 1 . \tag{3.12}$$

For arbitrary $a \in A$ let $u_1 \in U$ be such that $\|u_1a - a\| \leqq q\|a\|$. Then choose $v_1 \in U$ so as to satisfy

$$\|v_1(u_1a - a) - (u_1a - a)\| \leqq q\|u_1a - a\| ,$$

or equivalently, $\|a_2a - a\| \leqq q\|u_1a - a\|$ with $a_2 = v_1 \circ u_1 \in U \circ U$. By assumption, for $a_2$ there is an element $u_2 \in U$ such that

$$\|u_2a_2 - a_2\| < \varepsilon_0\|a_2a - a\| \cdot \|a\|^{-1} .$$

Hence,

$$\begin{aligned}\|u_2a - a\| &\leqq \|u_2a_2 - u_2a_2a\| + \|u_2a_2a - a_2a\| + \|a_2a - a\| \\ &< dq\|u_1a - a\| + \varepsilon_0\|a_2a - a\| + q\|u_1a - a\| \\ &\leqq (d + 1 + \varepsilon_0)q\|u_1a - a\| \\ &\leqq \bar{q}\|u_1a - a\| ,\end{aligned}$$

by (3.12). Thus, for arbitrary $u_1 \in U$ there exists an element $u_2 \in U$ such that

$$||u_2 a - a|| \leqq \bar{q}||u_1 a - a|| .$$

After $n$ iteration steps we obtain

$$||u_n a - a|| \leqq \bar{q}^n ||u_1 a - a|| < \varepsilon (u_n \in U)$$

if $n$ is sufficiently large.

## 4. Factorization theorems.

Let $A$ be a Banach algebra and let $X$ be a Banach space. Suppose that there is a composition mapping of $A \times X$ with values $a \cdot x$ in $X$. $X$ is called a left Banach $A$-module (see [8], II (32.14)), if this mapping has the following properties:

(i) $(a + b) \cdot x = a \cdot x + b \cdot x$ and $a \cdot (x + y) = a \cdot x + a \cdot y$;
(ii) $(ta) \cdot x = t(a \cdot x) = a \cdot (tx)$;
(iii) $(ab) \cdot x = a \cdot (b \cdot x)$;
(iv) $||a \cdot x|| \leqq C||a|| \cdot ||x||$

for all $a, b \in A$; $x, y \in X$; real or complex $t$, where $C$ is a constant $\geqq 1$. Denote by $A_e$ the Banach algebra obtained from $A$ by adjoining a unit $e$, and with the customary norm $||a + te|| = ||a|| + |t|$. Properties (i)–(iv) hold for the extended operation $(a + te) \cdot x = a \cdot x + tx$.

The well-known factorization theorems for Banach algebras and their extension to Banach $A$-modules are usually proved under the hypothesis that the Banach algebra $A$ has a bounded (left) approximate identity. Since, by Lemma 2.3 the existence of a bounded weak left approximate identity implies the existence of a bounded left approximate identity, all factorization theorems in question remain true under the weaker assumption of the existence of a bounded weak left approximate identity for $A$. However, a short proof of the basic factorization theorem can be given without proving the existence of a bounded left approximate identity for $A$. This proof is based on Lemma 2.2 and on the argument used in the proof of Theorem 2 in [3].

Let $U$ be a bounded weak left approximate identity for $A$. Put $W = U \circ U$ and denote by $d$ the bound for $W$.

THEOREM 4.1. *Let $A$ be a Banach algebra having a bounded weak left approximate identity $U$. If $X$ is a left Banach $A$-module, then $A \cdot X$ is a closed linear subspace of $X$. For arbitrary $z \in A \cdot X$ and $r > 0$ there exist an element $a \in A$ and an element $x \in X$ such that $z = a \cdot x$, $||z - x|| \leqq r$, where $x$ is in the closure of $A \cdot z$.*

*Proof.* It is easy to see that if $z$ is in the closure of $A \cdot X$, then for arbitrary $a \in A$ and $\varepsilon > 0$ there exists $u \in W$ such that

$$(4.1) \qquad \|ua - a\| < \varepsilon \quad \text{and} \quad \|u\cdot z - z\| < \varepsilon .$$

In fact, for $\varepsilon_0 > 0$ there exist $b \in A$ and $y \in X$ such that $\|b \cdot y - z\| < \varepsilon_0$. Since $U$ is a weak bounded left approximate identity for $A$, by Lemma 2.2, for $\varepsilon_0 > 0$ there exists $u \in W$ such that $\|ua - a\| < \varepsilon_0$ and $\|ub - b\| < \varepsilon_0$. Hence, we obtain

$$\begin{aligned} \|u \cdot z - z\| &\leqq \|u \cdot z - ub \cdot y\| + \|ub \cdot y - b \cdot y\| + \|b \cdot y - z\| \\ &\leqq dC\|z - b \cdot y\| + \varepsilon_0 C\|y\| + \varepsilon_0 \\ &< (dC + C\|y\| + 1)\varepsilon_0 < \varepsilon \end{aligned}$$

for sufficiently small $\varepsilon_0$. Now put $a_0 = e$, $a_1 = (2d + 1)^{-1}(u_1 + 2de)a_0 = a_1' + qe$, where $a_1' \in A$, $u_1 \in W$, $a_{n+1} = (2d + 1)^{-1}(u_{n+1} + 2de)a_n$; $n = 1, 2, \cdots$. We have $a_n = a_n' + q^n e$, where $a_n' \in A$, $q = 2d(2d+1)^{-1}$; $a_n^{-1} \in A_e$; $a_{n+1} - a_n = (2d + 1)^{-1}(u_{n+1}a_n - a_n) = (2d + 1)^{-1}(u_{n+1}a_n' - a_n') + (2d + 1)^{-1}q^n(u_{n+1} - e)$; $a_{n+1}^{-1} - a_n^{-1} = a_n^{-1}(2d+1)(u_{n+1} + 2de)^{-1} - a_n^{-1} = a_{n+1}^{-1}(e - (2d + 1)^{-1}(u_{n+1} + 2de)) = (2d + 1)^{-1}a_{n+1}^{-1}(e - u_{n+1})$. Let $x_n = a_n^{-1} \cdot z$. Then we obtain

$$\|x_{n+1} - x_n\| \leqq C(2d + 1)^{-1}\|a_{n+1}^{-1}\|\,\|z - u_{n+1} \cdot z\| .$$

Since $\|a_n^{-1}\| \leqq (2 + d^{-1})^n$, let us choose $u_{n+1}$ so as to satisfy (4.1) with $a = a_n'$ and $\varepsilon = \varepsilon_n = C^{-1}(2d + 1)(2 + d^{-1})^{-1-n_2-1-n_r}$. Hence, we have $\|u_{n+1}a_n' - a_n'\| < \varepsilon_n$ and $\|x_{n+1} - x_n\| \leqq 2^{-1-n_r}$. It follows that the sequences $\{a_n\}$ and $\{x_n\}$ converge toward $a \in A$ and $x \in X$, respectively. Evidently, $z = a \cdot x$ and $\|z - x\| \leqq \sum_{n=0}^{\infty}\|x_{n+1} - x_n\| \leqq r$. By (4.1), $z$ is in the closure of $A \cdot z$ and so are $x_n = a^{-1}z$ and, consequently, $x$. Thus, $A \cdot x$ is closed and its linearity follows from the following observation. For arbitrary $a, b \in A$; $x, y \in X$ and $\varepsilon > 0$ let $u \in W$ be such that $\|ua - a\| < C^{-1}(\|x\| + \|y\|)^{-1}$ and $\|ub - b\| < C^{-1}(\|x\| + \|y\|)^{-1}\varepsilon$. Then we have

$$\begin{aligned} \|a \cdot x + b \cdot y - u(a \cdot x + b \cdot y)\| &= \|(a - ua) \cdot x + (b - ub) \cdot y\| \\ &< C\|a - uax\|\,\|x\| + C\|b - ub\|\,\|y\| \\ &< \varepsilon . \end{aligned}$$

That is $a \cdot x + b \cdot y$ is in the closure of $A \cdot X$.

REMARK 4.1. Theorem 4.1 generalizes the factorization theorems of Cohen [4], Hewitt [7], Curtis and Figa-Talamanca [6] [see also Koosis [9], Collins and Summers [5], Hewitt and Ross [8]: (32.22), (32.23), (32.26)].

In terms of contractors Theorem 4.1 can be formulated as

THEOREM 4.2. *Suppose that the Banach algebra $A$ has a left bounded* (*by $d$*) *contractor $U$ satisfying one of the following conditions*:

(a) *$U$ is a left approximate identity for itself.*

(b) *U satisfies the hypotheses of Theorem* 2.1.

(c) $(d+1)q < 1$ *and U is a weak left approximate identity for* $U \circ U$.

*Then all assertions of Theorem* 4.1 *hold.*

Notice that in Case (c) Proposition 3.3 is used.

A corollary to Theorem 4.1 is the following generalization of the well-known theorem [see [8], II (32.23)].

THEOREM 4.3. *Let A be a Banach algebra with a weak bounded left approximate identity U. Let* $\zeta = \{z_n\}$ *be a convergent sequence of elements of* $A \cdot X$, *and suppose that* $r > 0$. *Then there exists an element* $a \in A$ *and a convergent sequence* $\xi = \{x_n\}$ *of elements of* $A \cdot X$ *such that*:

$z_n = a \cdot x_n$ *and* $\|z_n - x_n\| \leqq r$ *for* $n = 1, 2, \cdots$, *where* $x_n$ *is in the closure of* $A \cdot z_n$.

*Proof.* Let $\mathscr{X}$ be the Banach space of all convergent sequences $\xi = \{x_n\}$ of elements of the closed linear subspace $A \cdot X$ of $X$ with the norm $\|\xi\| = \sup(\|x_n\|: n = 1, 2, \cdots)$. Consider the left Banach $A$-module $\mathscr{X}$ with $a \cdot \xi = \{a \cdot x_n\} \in \mathscr{X}$. For $\xi \in \mathscr{X}$ put $\xi_m = \{x_n\} \in \mathscr{X}$ with $x_n = x_m$ for $n \geqq m$. By Theorem 4.1 it is sufficient to show that every $\xi \in \mathscr{X}$ is in the closure of $A \cdot \mathscr{X}$. But $\xi_m \to \xi$ as $m \to \infty$. Therefore, let $\xi_m = \{a_n \cdot x_n\} \in \mathscr{X}$ with $a_n \cdot x_n = a_m \cdot x_m$ for $n \geqq m$. By Lemma 2.3 for $\varepsilon_0 > 0$ there exists $u \in A$ such that

$$\|ua_i - a_i\| < \varepsilon_0 \qquad \text{for} \quad i = 1, \cdots, m .$$

Hence, we have

$$\|ua_i \cdot x_i - a_i \cdot x_i\| < C\varepsilon_0 \|x_i\| < \varepsilon$$

for sufficiently small $\varepsilon_0$ and, consequently, $\|u \cdot \xi_m - \xi_m\| < \varepsilon$, where $\varepsilon > 0$ is arbitrary.

REMARK 4.1. In Theorem 4.3 convergent sequences can be replaced be sequences convergent toward zero. Then $\mathscr{X}$ will be the space of all sequences of $A \cdot X$ convergent toward zero.

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