IDEALIZERS AND NONSINGULAR RINGS

KENNETH R. GOODEARL
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This paper deals with the relationship between a ring $T$ and the idealizer $R$ of a right ideal $M$ of $T$. [The ring $R$ is the largest subring of $T$ which contains $M$ as a two-sided ideal.] Assuming $M$ to be a finite intersection of maximal right ideals of $T$, the properties of $T$ and $R$ are shown to be very similar. The main theorem of the first section shows that under these hypotheses the right global dimensions of $T$ and $R$ almost always coincide. In the second section, where $T$ is assumed to be a nonsingular ring, the major theorem asserts that the singular submodule of every $R$-module is a direct summand if and only if the corresponding property holds for $T$-modules.

We assume throughout the paper that all rings are associative with identity, and that all modules are unitary. Unless otherwise noted, all modules are right modules.

1. Idealizers. This section is concerned with idealizers in arbitrary rings, and is based on the work of J. C. Robson in [7].

Given a ring $T$ and a right ideal $M$ of $T$, the idealizer of $M$ in $T$ is the set $R = \{t \in T | tM \subseteq M\}$, which is easily seen to be the largest subring of $T$ which contains $M$ as a two-sided ideal. The aim of this investigation is to discover properties of $T$ which carry over to $R$ (and vice versa).

We shall mainly consider the case when $M$ is a finite intersection of maximal right ideals of $T$; following [7], we say in this case that $M$ is a semimaximal right ideal of $T$. Equivalently, $M$ is a semimaximal right ideal of $T$ if $T/M$ is a semisimple right $T$-module, i.e., a module which is a sum of simple submodules. In accordance with this terminology, we use the term “semisimple ring” to refer to a ring which is semisimple as a module over itself, rather than a ring whose Jacobson radical is zero.

The concept of the idealizer of $M$ is of course not needed if $M$ is already a two-sided ideal of $T$, i.e., if $TM = M$. When $M$ is maximal, the only other possibility is $TM = T$, and in general this condition seems to be required for some proofs. Fortunately, [7, Proposition 1.7] allows us to assume it without loss of generality: Assuming that $M$ is a semimaximal right ideal of $T$, then there is another semimaximal right ideal $M'$, containing $M$, such that $TM' = T$ and the idealizers of $M$ and $M'$ coincide.
Thus we assume throughout this section that $M$ is a semimaximal right ideal of $T$ satisfying $TM = T$.

**Proposition 1.** [Robson] (a) $R/M$ is a semisimple ring.
(b) $T/R$ is a semisimple right $R$-module.
(c) $T$ is a finitely generated projective right $R$-module.
(d) The natural map $T \otimes_R T \to T$ is an isomorphism.

*Proof.* (b), (c), and (d) are contained in Corollary 1.5 and Lemma 2.1 of [7], while (a) follows from the observation [7, Proposition 1.1] that $R/M$ is isomorphic to the endomorphism ring of the right $T$-module $T/M$.

A simple consequence of (d) is that for any modules $A_T$ and $B_T$, the natural map $A \otimes_R B \to A \otimes_R B$ is an isomorphism, from which we infer that the following maps are also isomorphisms: $A \otimes_R T \to A$, $T \otimes_R B \to B$, $A \to A \otimes_R T$, $B \to T \otimes_R B$. Then for any modules $A_T$ and $C_T$ we conclude using the isomorphisms $A \to A \otimes_R T$ and $C \to C \otimes_R T$ that $\text{Hom}_T(A, C) = \text{Hom}_T(A, C)$. Given these observations and the projectivity of $T_R$, a straightforward induction establishes the following results:

**Proposition 2.** (a) $\text{Tor}_n^R(A, B) \cong \text{Tor}_n^T(A, B)$ for all $A_T$, $B_T$ and all $n > 0$.
(b) $\text{Ext}_n^R(A, C) \cong \text{Ext}_n^T(A, C)$ for all $A_T$, $C_T$ and all $n > 0$.

These results suggest comparing the global dimensions of $R$ and $T$, which is done in [7, Theorem 2.9] for the case when $T$ is right noetherian: Provided that $R \neq T$, then

$$\text{r. gl. dim.}(R) = \max\{1, \text{r. gl. dim.}(T)\}.$$  

In Theorem 5 we shall remove the noetherian restriction on this theorem, but first two intermediate results are needed.

The key to the next two propositions is a consideration of the module $JT/J$, where $J$ is a right ideal of $R$. There is an epimorphism $f: F \to JT/J$ for some direct sum $F$ of copies of $T/R$, and we see from Proposition 1 that $F$ is a semisimple right $R$-module, hence $\ker f$ must be a summand of $F$. Thus $JT/J$ is isomorphic to a summand of a direct sum of copies of $T/R$. For the proof of Theorem 10, we must notice that this same conclusion follows when $J$ is an $R$-submodule of a right $T$-module.

**Proposition 3.** $T$ is a flat left $R$-module.
Proof. The natural maps $R \otimes_R T \to T \otimes_R T \to T$ and $T \otimes_R T \to T$ are both isomorphisms; hence $R \otimes_R T \to T \otimes_R T$ is an isomorphism. Inasmuch as $T_R$ is projective, it follows that $\text{Tor}^R_1(T/R, T) = 0$. Now given any right ideal $J$ of $R$, $JT/J$ is isomorphic to a summand of a direct sum of copies of $T/R$, from which we infer that $\text{Tor}^R_1(JT/J, T) = 0$. According to Proposition 2 we also have $\text{Tor}^R_1(T/JT, T) = 0$, whence $\text{Tor}^R_1(T/J, T) = 0$. Thus $J \otimes_R T \to T \otimes_R T$ is injective, hence $J \otimes_R T \to R \otimes_R T$ must be injective.

We shall use the notation $pd_R(A)$ to stand for the projective dimension of an $R$-module $A$.

**Proposition 4.** If $J$ is any right ideal of $R$, then $pd_R(J) = pd_T(JT)$.

**Proof.** Since $R_T$ is flat, the tensor product of $T$ with any projective resolution of $R$ yields a projective resolution of $(J \otimes_R T)_R$, thus $pd_T(J \otimes_R T) \leq pd_R(J)$. The flatness of $R_T$ also implies that $J \otimes_R T \cong JT$; hence we get $pd_T(JT) \leq pd_R(J)$.

In view of the projectivity of $T_R$ and $R_T$, $pd_R(T/R) \leq 1$. Inasmuch as $JT/J$ is isomorphic to a summand of a direct sum of copies of $T/R$, we obtain $pd_R(JT/J) \leq 1$. Examining the long exact sequence of Ext, we infer from this that $pd_R(J) \leq pd_R(JT)$. Recalling again that $T_R$ is projective, we see that any projective resolution of $(JT)_R$ is also a projective resolution of $(JT)_T$, from which we conclude that $pd_R(JT) \leq pd_T(JT)$. Thus $pd_R(J) \leq pd_T(JT)$.

[After the preparation of this paper, Professor Robson informed the author that he too had obtained the following theorem, which appears in [8, Theorem 2.8].]

**Theorem 5.** If $R \neq T$, then $\text{r.gl. dim.}(R) = \max \{1, \text{r.gl. dim.}(T)\}$.

**Proof.** If $\text{r.gl. dim.}(R) > 0$, then from Proposition 4 we obtain $\text{r.gl. dim.}(R) = 1 + \sup \{pd_R(J) \mid J \leq R_R\} \leq 1 + \sup \{pd_T(K) \mid K \leq T_T\} = \max \{1, \text{r.gl. dim.}(T)\}$. On the other hand, it is immediate from Proposition 2 that $\text{r.gl. dim.}(T) \leq \text{r.gl. dim.}(R)$. Thus it only remains to prove that $\text{r.gl. dim.}(R) \geq 1$.

In view of the assumption $R \neq T$, we see that $M$ cannot be a two-sided ideal of $T$; hence $1 \notin M$ and $M < R$. Inasmuch as $TM = T$, it follows that the map $R \otimes_R (R/M) \to T \otimes_R (R/M)$ is not injective, from which we conclude that $R(R/M)$ is not flat. Thus $\text{GWD}(R) > 0$; hence $\text{r.gl. dim.}(R) > 0$.

For weak dimension, the proofs of Proposition 4 and Theorem 5
can be used, mutatis mutandis, to prove the following theorem:

**Theorem 6.** If $R \not= T$, then $G\Psi\Omega(R) = \max \{1, GWD(T)\}$.

2. Nonsingular rings. In this section we shall assume that $T$ is a nonsingular ring and then investigate the relationship between singular and nonsingular modules over $T$ and $R$. First we recall the relevant definitions: Letting $\mathcal{S}(T)$ denote the collection of essential right ideals of $T$, then the singular submodule of a right $T$-module $A$ is the set $Z_r(A) = \{x \in A | xI = 0 \text{ for some } I \in \mathcal{S}(T)\}$. We say that $A$ is singular [nonsingular] provided $Z_r(A) = A [Z_r(A) = 0]$. The singular submodule of $T_r$ is a two-sided ideal of $T$, called the right singular ideal of $T$ and denoted $Z_r(T)$; $T$ is a right nonsingular ring if $Z_r(T) = 0$. Analogous definitions and notations hold for $R$ and its modules.

Throughout this section, we assume that $T$ is a right nonsingular ring and that $M$ is an essential right ideal of $T$, and we investigate the idealizer $R$ of $M$. For all but the next two propositions, we make the additional assumptions that $M$ is a semimaximal right ideal of $T$ and that $TM = T$.

**Proposition 7.** (a) $\mathcal{S}(T) = \{K \leq T_r | K \cap R \in \mathcal{S}(R)\}$.
(b) $\mathcal{S}(R) = \{J \leq R_r | JM \in \mathcal{S}(T)\}$.
(c) $Z_r(A) = Z_R(A)$ for all $A_r$.
(d) $Z_r(T) = Z_r(T) = 0$.

**Proof.** (a) Suppose that $K \in \mathcal{S}(T)$ and $A \leq R_r$ such that $A \cap (K \cap R) = 0$. Then $AM \cap K = 0$, whence $AM = 0$ [because $AM$ is a right ideal of $T$ and $K \in \mathcal{S}(T)$]. Thus $A \leq Z_r(T) = 0$ and so $K \cap R \in \mathcal{S}(R)$.

Now let $K \leq T_r$ and assume that $K \cap R \in \mathcal{S}(R)$. If $A \leq T_r$ and $A \cap K = 0$, then from $(A \cap R) \cap (K \cap R) = 0$ we obtain $A \cap R = 0$, hence $A \cap M = 0$. Thus $A = 0$ and so $K \in \mathcal{S}(T)$.

(b) If $J \leq R_r$ and $JM \in \mathcal{S}(T)$, then $JM \in \mathcal{S}(R)$ by (a), whence $J \in \mathcal{S}(R)$.

Now consider any $J \in \mathcal{S}(R)$. Inasmuch as $M \in \mathcal{S}(T)$ and $Z_r(T) = 0$, the left annihilator of $M$ in $T$ is zero. In particular, it follows that every nonzero element of $J$ has a nonzero right multiple in $JM$. Thus $JM$ is an essential $R$-submodule of $J$, hence $JM \in \mathcal{S}(R)$, and then $JM \in \mathcal{S}(T)$ by (a).

(c) follows directly from (a) and (b).

(d) According to (c), $Z_R(T) = Z_r(T) = 0$, and then $Z_r(R) = 0$ also.
Let \( Q \) denote the maximal right quotient ring of \( T \). From [3, Theorem 1 + 2, p. 69] we obtain the following information: \( Q_T \) is an injective hull for \( T_T \), \( Q \) is a von Neumann regular ring, and \( Q_Q \) is injective. Note that \( T \cap Z_T(Q) = Z_T(T) = 0 \), from which we obtain \( Z_T(Q) = 0 \).

**Proposition 8.** \( Q \) is also the maximal right quotient ring of \( R \).

*Proof.* We first show that \( Q \) is a right quotient ring of \( R \), i.e., that \( Q_R \) is a rational extension of \( R_R \). (See [3, pp. 58, 64] for the definitions.) Inasmuch as \( Z_T(R) = 0 \), [3, Proposition 5, p. 59] says that it suffices to prove that \( Q_R \) is an essential extension of \( R_R \). Thus consider any \( A \subseteq Q_R \) such that \( A \cap R = 0 \). Then \( AM \cap M = 0 \). Since \( M \) is an essential right ideal of \( T \), it must be an essential \( T \)-submodule of \( Q \), so that we obtain \( AM = 0 \) and \( A \subseteq Z_T(Q) = 0 \). Therefore, \( Q \) is a right quotient ring of \( R \); hence we may assume that \( Q \) is a subring of the maximal right quotient ring \( P \) of \( R \). The injectivity of \( Q_Q \) implies that \( P_Q = Q \oplus B \) for some \( B \). Then from \( R \cap B = 0 \) we infer that \( B = 0 \) and \( P = Q \).

In view of Proposition 8, we may refer to [3, Theorem 1 + 2, p. 69] again and conclude that \( Q_R \) is an injective hull for \( R_R \). Now we obtain from [5, Proposition 1, p. 427] the following alternate description of the singular submodule of a right \( R \)-module \( A \): \( Z_T(A) = \bigcap \{ \ker f \mid f \in \text{Hom}_R(A, Q) \} \). In particular, \( A \) is singular if and only if \( \text{Hom}_R(A, Q) = 0 \), from which we conclude that any extension of a singular module by a singular module is singular.

N.B.—From this point on, the assumption that \( M \) is a semimaximal right ideal of \( T \) satisfying \( TM = T \) will hold.

It follows from Proposition 7 that every nonsingular right \( T \)-module is also a nonsingular right \( R \)-module. A partial converse is provided in the next proposition: Any nonsingular right \( R \)-module can be canonically embedded in a nonsingular right \( T \)-module.

**Proposition 9.** If \( A_R \) is nonsingular, then the natural map \( A \to A \otimes_R T \) is injective and \((A \otimes_R T)_T \) is nonsingular.

*Proof.* In view of the discussion following Proposition 8, the intersection of the kernels of the homomorphisms from \( A \) into \( Q_R \) must be zero. Thus we may assume that \( A \) is a submodule of some direct product \( B \) of copies of \( Q \).

Since \( Q \) is a nonsingular right \( T \)-module, so is \( B \). We now get a natural map \( A \otimes_R T \to B \otimes_R T \to B \), and the composition \( A \to A \otimes_R T \to B \)
is just the inclusion map, whence $A \to A \otimes_R T$ must be injective. Also, we see from the flatness of $\mathcal{E}_B T$ that $A \otimes_R T \to B \otimes_R T$ is injective. Since $B \otimes_R T \to B$ is an isomorphism, we infer that $A \otimes_R T \cong AT$; hence $(A \otimes_R T)_R$ is nonsingular.

We say that $R$ is a splitting ring provided that for any right $R$-module $A$, $Z_R(A)$ is a direct summand of $A$. It is noted in [1, Proposition 1.12] that $R$ is a splitting ring if and only if $\text{Ext}^1_R(A,C) = 0$ for all nonsingular $A_R$ and all singular $C_R$.

**Theorem 10.** $R$ is a splitting ring if and only if $T$ is a splitting ring.

**Proof.** Suppose that $R$ is a splitting ring. Given a nonsingular right $T$-module $A$ and a singular right $T$-module $C$, it follows from Proposition 7 that $A_R$ is nonsingular and $C_R$ is singular. Thus $\text{Ext}^1_R(A,C) = 0$; hence from Proposition 2 we obtain $\text{Ext}^1_T(A,C) = 0$.

Now assume that $T$ is a splitting ring. Given a nonsingular module $A_R$ and a singular module $C_R$, we must show that $\text{Ext}^1_T(A,C) = 0$. It suffices to prove that $\text{Ext}^1_T(A,C/CM) = 0$ and $\text{Ext}^1_T(A,CM) = 0$.

**Case I.** $CM = 0$. We first show that $\text{Tor}_1^T(A,R/M) = 0$.

According to Proposition 9, we may assume that $A$ is an $R$-submodule of a nonsingular right $T$-module $B$. The natural map $T \otimes_R M \to T \otimes_R R \to T$ is injective because $T_R$ is projective; hence in view of the condition $TM = T$ we see that $T \otimes_R M \to T$ is an isomorphism. Thus $AT \otimes_T T \otimes_R M \to AT \otimes_T T$ is an isomorphism; equivalently, $AT \otimes_R M \to AT$ is an isomorphism.

Inasmuch as the natural map $R \otimes_R M \to T \otimes_R M \to T$ is injective, $R \otimes_R M \to T \otimes_R M$ must be injective. In light of the projectivity of $T_R$, we obtain from this that $\text{Tor}_1^R(T/R, M) = 0$. Now since $AT/A$ is isomorphic to a summand of a direct sum of copies of $T/R$, we must have $\text{Tor}_1^T(AT/A,M) = 0$. Therefore, the map $A \otimes_R M \to AT \otimes_R M \to AT$ is injective, hence $A \otimes_R M \to A \otimes_R R$ is injective. Thus $\text{Tor}_1^T(A,R/M) = 0$.

Now consider any short exact sequence $E: 0 \to C \to B \to A \to 0$. Since $\text{Tor}_1^T(A,R/M) = 0$, we obtain another exact sequence $E^*: 0 \to C \to B/BM \to A/AM \to 0$. The sequence $E^*$ splits because $R/M$ is a semisimple ring, hence $E$ splits.

**Case II.** $CM = C$. Here $C \cong P/J$ for some direct sum $P$ of copies of $M$ and some $R$-submodule $J$ of $P$. To prove that $\text{Ext}^1_T(A,C) = 0$, it suffices to show that $\text{Ext}^1_T(A,P/JM) = 0$ and $\text{Ext}^1_T(A,J/JM) = 0$.

Inasmuch as $M \in \mathcal{S}(R)$, $J/JM$ is a singular right $R$-module. Choos-
ing an exact sequence $0 \to K \to F \to A \to 0$ with $F_R$ free, we have $\text{Ext}^2_R(A, J/JM) \cong \text{Ext}^1_R(K, J/JM)$. Since $Z_*(R) = 0$, $F$ and thus $K$ are nonsingular; hence $\text{Ext}^1_R(K, J/JM) = 0$ by Case I. Therefore, $\text{Ext}^1_R(A, J/JM) = 0$.

All that remains is to show that $\text{Ext}^1_R(A, D) = 0$, where $D = P/JM$. Inasmuch as $P$ is a right $T$-module and $JM$ is a $T$-submodule of $P$, $D$ is a right $T$-module. Since $P/J$ and $J/JM$ are both singular $R$-modules, it follows from the discussion after Proposition 8 that $D_R$ must be singular. Thus from Propositions 7 and 9 we obtain that $D_T$ is singular and $(A \otimes_R T)_T$ is nonsingular.

Given any exact sequence $0 \to D \to B \to A \to 0$, we get a commutative diagram with exact rows as follows:

$$
\begin{array}{c}
0 \\
\downarrow \cong \\
0 \\
\end{array}
\begin{array}{cccccc}
0 & \longrightarrow & D & \longrightarrow & B & \longrightarrow & A & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D \otimes_R T & \longrightarrow & B \otimes_R T & \longrightarrow & A \otimes_R T & \longrightarrow & 0.
\end{array}
$$

The bottom row splits because $T$ is a splitting ring; hence the top row splits. Therefore, $\text{Ext}^1_R(A, D) = 0$.

One special case of Theorem 10 has been proved in [4]. The authors start with a left and right principal ideal domain $C$ such that $C$ is a simple ring but not a division ring, and such that every simple right $C$-module is injective. (Examples of such rings are constructed in [2].) Then they choose a maximal right ideal $M$ of $C$ and prove that the idealizer $I$ of $M$ in $C$ is a splitting ring [Lemma 2].

It is not hard to prove that every singular right $C$-module is semisimple, and hence that every singular right $C$-module is injective. (Details may be found in [6, Chapter 3].) Thus $C$ is certainly a splitting ring. The right ideal $M$ is nonzero because $C$ is not a division ring; hence from the simplicity of $C$ we obtain $CM = C$. Also, $C$ is a right Ore domain, from which it follows easily that $M$ is an essential right ideal of $C$. Thus it now also follows from Theorem 10 that $I$ is a splitting ring.

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Received February 11, 1972 and in revised form March 23, 1973.

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The Pacific Journal of Mathematics is issued monthly as of January 1966. Regular subscription rate: $48.00 a year (6 Vols., 12 issues). Special rate: $24.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Kokusai Bunken Insatsuisha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan

* C. R. DePrima California Institute of Technology, Pasadena, CA 91109, will replace J. Dugundji until August 1974.

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